Characterization of the integer-valued translation-invariant regular metrics on the discrete plane

G. J. F. Banon

National Institute for Space Research, INPE, Brazil

Abstract

We say that a metric space is regular if a straight-line (in the metric space sense) passing through the center of a sphere and any other point has at least two diametrically opposite points. Normed vector spaces have this property. Nevertheless, this property might not be satisfied in some metric spaces. In this work, we give a characterization of an integer-valued translation-invariant regular metric defined on the discrete plane, in terms of a symmetric subset B that induces through, what we call, the Minkowski product, a chain of subsets that are morphologically closed with respect to B.

Keywords: Mathematical Morphology, symmetric subset, ball, regular metric space, integer-valued metric, translation-invariant metric, triangle inequality, Minkowski product, morphologial closed subset, discrete plane, computational geometry, discrete geometry, digital geometry.

1. Introduction

The continuous plane or, more precisely, the two-dimensional Euclidean vector space, has good geometrical properties. For example, in such space, a closed ball is included in another one only if the radius of the latter is greater than or equal to the sum of the distance between their centers and the radius of the former. Furthermore, in such space, two closed balls intersect each other if the sum of their radii is greater than or equal to the distance between their centers. Nevertheless, not all metric spaces have these properties.

In the first part of this work, we introduce the concept of *regular* metric space in which the above two geometrical properties are satisfied.

We say that a metric space is regular if its metric satisfies three regularity axioms or equivalently if a straight-line (in the metric space sense) passing through the center of a sphere and any other point has at least two diametrically opposite points. Minkowski spaces (i.e., finite dimensional normed vector spaces) have this property.

This regularity is generally lost when a metric on the continuous plane is restricted to the discrete plane, as it is the case of the Euclidean metric.

In the second part of this work, we study the characterization of the integer-valued translation-invariant *regular* metrics on the discrete plane in terms of some appropriate symmetric subsets.

We show that every such metric can be characterized in terms of a symmetric subset B that induces through, what we call, the Minkowski product, a chain of subsets that are morphologically closed with respect to B.

Our characterization shows the unique way to construct integer-valued translation-invariant regular metrics on the discrete plane.

This is an important issue in digital image analysis since the image domains are then discrete. In the sixties, Rosenfeld and Pfaltz [8] have already introduced a metric property and have used it to describe algorithms for computing some distance functions by performing repeated local operations. It appears that their property is precisely a necessary condition for a metric to be regular.

Actually, we came across the regularity property for a metric because we tried to find a class of metric whose metric dilations satisfy a semigroup relation (relation (9.19) of [3]) and whose balls are morphologically closed with respect to the unit ball. This class contains, for example, the chessboard distance.

In one dimension, we observed that the (discrete) convexity is not a necessary condition to have the morphological closure property, so it was useless to solve our problem.

For the sake of simplicity of the presentation, in this work, we limit ourselves to the class of integer-valued metrics. This is not a serious limitation because on the discrete plane the metrics assume only a countable number of values.

In Section 2, we give an axiomatic definition of regular metric space and we relate it to the Kiselman's properties of upper and lower regularity for the triangle inequality. In particular, we show that of the three axioms only two are sufficient to define the metric regularity. Independently of the definition of metric, in Section 3, we give a definition of ball based on the notions of set translation and set transposition. We introduce the Minkowski product in Section 4, and use it in Section 5 to define the notions of generated balls and radius of a ball. In Section 6, we study the properties of the balls of a regular metric space. Conversely, in Section 7, we study the properties of the metric spaces constructed from symmetric balls having a morphological closure property. Finally, in Section 8, we show the existence of a bijection between the set of integer-valued translation-invariant regular metrics defined on the discrete plane and the set of symmetric balls that satisfy the morphological closure property.

2. Regular metric space

We first recall the definitions of distance, metric and metric space.

Definition 1 (metric space) – Let E be a nonempty set. A *distance d on E* is a mapping from $E \times E$ to $\mathbf{R}$ (the set of real numbers) satisfying, for any x and $y \in E$:

(i) $d(x, y) \geq 0$, (positiveness)

(ii) $d(x, y) = 0 \Leftrightarrow x = y$,

(iii) $d(x, y) = d(y, x)$. (symmetry)

Furthermore, a distance d on E is a *metric* if in addtion it satisfies, for any x, y and $z \in E$:

(iv) $d(x, y) \leq d(x, z) + d(z, y)$. (triangle inequality)

A *metric space* (E, d) is a set E provided with a metric d on E. ♦

The mapping d_0 from $E \times E$ to $\mathbf{R}$ defined by, for any x and $y \in E$:

$$d_0(x, y) = \begin{cases} 0 & \text{if} \quad x = y \\ 1 & \text{otherwise} \end{cases},$$

is an example of metric, it is called the *discrete metric*.

In a metric space we can define the concepts of straight-line and sphere. Furthermore, based on these concepts, we can define what we call a regular metric and a regular metric space.

Let (E, d) be a metric space. For any x and $y \in E$, and any $i \in d(\{x\} \times E)$ (i.e., the image of $\{x\} \times E$ through d), let us define the following subsets of E:

$$L_1(x, y) = \{z \in E\colon d(x, z) = d(x, y) + d(y, z)\},$$
$$L_2(x, y) = \{z \in E\colon d(x, y) = d(x, z) + d(z, y)\},$$
$$L_3(x, y) = \{z \in E\colon d(z, y) = d(z, x) + d(x, y)\}, \text{ and}$$
$$S(x, i) = \{z \in E\colon d(x, z) = i\}.$$

The subsets $L(x, y) = L_1(x, y) \cup L_2(x, y) \cup L_3(x, y)$ and $S(x, i)$ are, respectively, the *straight-line* passing through the points x and y, and the *sphere* with center at x and of radius i.

Figure 1 and Figure 2 in this section show examples of spheres and straight-lines. We must be aware that the above definition of straight-line is based on the concept of metric and the resulting object is generally different from the usual straight-line defined in the framework of linear vector space.

Because of the symmetry property of the distances, the straight-lines have some kind of symmetry as well.

Proposition 2 (straight-line symmetry) – Let (E, d) be a metric space. For any x and $y \in E$,

(i) $L_1(x, y) = L_3(y, x)$

(ii) $L_2(x, y) = L_2(y, x)$

(iii) $L(x, y) = L(y, x)$. ♦

Proof

(a) Lut us prove (i). For any x, y and $z \in E$,

$z \in L_1(x, y) \Leftrightarrow d(x, z) = d(x, y) + d(y, z)$ (definition of L_1)

$\Leftrightarrow d(z, x) = d(y, x) + d(z, y)$ (symmetry of d)

$\Leftrightarrow d(z, x) = d(z, y) + d(y, x)$ (commutativity of +)

$\Leftrightarrow z \in L_3(y, x)$, (definition of L_3)

that is, $L_1(x, y) = L_3(y, x)$.

(b) Let us prove (ii). For any x, y and $z \in E$,

$z \in L_2(x, y) \Leftrightarrow d(x, y) = d(x, z) + d(z, y)$ (definition of L_2)

$\Leftrightarrow d(y, x) = d(z, x) + d(y, z)$ (symmetry of d)

$\Leftrightarrow d(y, x) = d(y, z) + d(z, x)$ (commutativity of +)

$\Leftrightarrow z \in L_2(y, x)$, (definition of L_2)

that is, $L_2(x, y) = L_2(y, x)$.

(c) Let us prove (iii). For any x and $y \in E$,

$L(x, y) = L_1(x, y) \cup L_2(x, y) \cup L_3(x, y)$ (definition of L)

$= L_3(y, x) \cup L_2(y, x) \cup L_1(y, x)$ (from (i) and (ii))

$= L_1(y, x) \cup L_2(y, x) \cup L_3(y, x)$ (commutativity of $\cup$)

$= L(y, x)$, (definition of L)

that is, $L(x, y) = L(y, x)$. ♦

We first define the regular metric space from three axioms.

Definition 3 (regular metric space) – Let (E, d) be a metric space. The metric d on E is

(i) *lower regular of type 1* if $S(x, i) \cap L_1(x, y) \neq \varnothing$, for any x and $y \in E$, and any $i \in d(\{x\} \times E)$, such that $d(x, y) \leq i$;

(ii) *lower regular of type 2* if $S(x, i) \cap L_2(x, y) \neq \varnothing$, for any x and $y \in E$, and any $i \in d(\{x\} \times E)$, such that $i \leq d(x, y)$;

(iii) *upper regular* if $S(x, i) \cap L_3(x, y) \neq \varnothing$, for any x and $y \in E$, and any $i \in d(\{x\} \times E)$;

(iv) *regular* if it is lower regular (of type 1 and 2) and upper regular. A metric space (E, d) is *regular* if its metric is regular. ♦

Actually, the three regularity axioms are not independent each other as we show in the next proposition.

Proposition 4 (axiom dependence) – Let (E, d) be a metric space. The metric d on E is lower regular of type 1 if and only if (iff) it is upper regular. ♦

Proof

(a) Let us prove that the lower regularity of type 1 implies the upper regularity. For any x and $y \in E$, and any $i \in d(\{x\} \times E)$,

(d is lower regular of type 1) $\Rightarrow S(y, i + d(x, y)) \cap L_1(y, x) \neq \varnothing$

(definition of lower regularity of type 1)

$\Leftrightarrow S(y, i + d(x, y)) \cap L_3(x, y) \neq \varnothing$

(Property (i) of Proposition 2 - straight-line symmetry)

$\Leftrightarrow \exists\, z \in E\colon d(y, z) = i + d(x, y)$ and $d(z, y) = d(z, x) + d(x, y)$

(definitions of S and L_3)

$\Leftrightarrow \exists\, z \in E\colon d(z, y) = i + d(x, y)$ and $d(z, y) = d(z, x) + d(x, y)$

(symmetry of d)

$\Leftrightarrow \exists\, z \in E\colon d(z, x) = i$ and $d(z, y) = d(z, x) + d(x, y)$

(transitivity of = and regularity of +)

$\Leftrightarrow \exists\, z \in E\colon d(x, z) = i$ and $d(z, y) = d(z, x) + d(x, y)$

(symmetry of d)

$\Leftrightarrow S(x, i) \cap L_3(x, y) \neq \varnothing$,

(definitions of S and L_3)

that is, d is upper regular.

(b) Let us prove that the upper regularity implies the lower regularity of type 1. For any x and $y \in E$, and any $i \in d(\{x\} \times E)$ such that $d(x, y) \leq i$,

(d is upper regular) $\Rightarrow S(y, i - d(x, y)) \cap L_3(y, x) \neq \varnothing$

(definition of upper regularity)

$\Leftrightarrow S(y, i - d(x, y)) \cap L_1(x, y) \neq \varnothing$

(Property (ii) of Proposition 2)

$\Leftrightarrow \exists z \in E$: $d(y, z) = i - d(x, y)$ and $d(x, z) = d(x, y) + d(y, z)$

(definitions of S and L_1)

$\Leftrightarrow \exists z \in E$: $d(x, y) + d(y, z) = i$ and $d(x, z) = d(x, y) + d(y, z)$

(+ versus –)

$\Leftrightarrow \exists z \in E$: $d(x, z) = i$ and $d(x, z) = d(x, y) + d(y, z)$

(transitivity of =)

$\Leftrightarrow S(x, i) \cap L_1(x, y) \neq \varnothing$,

(definitions of S and L_1)

that is, d is lower regular of type 1. ♦

The axiom dependence allows us to make an equivalent definition of regular metric, but simpler with only two axioms, the lower regularity of type 2 being called simply lower regularity.

Corollary 5 (first equivalent definition of regular metric) – Let (E, d) be a metric space. The metric d on E is

(i) *lower regular* if $S(x, i) \cap L_2(x, y) \neq \varnothing$, for any x and $y \in E$, and any $i \in d(\{x\} \times E)$, such that $i \leq d(x, y)$;

(ii) *regular* iff it is lower and upper regular. ♦

Proof

(a) If d is regular in the sense of Definition 3, then d is regular in the sense of the corollary statement since lower regularity means lower regularity of type 2.

(b) Conversely, if d is regular in the sense of the corollary statement, then d is lower regular of type 2 and upper regular, therefore by Proposition 4 (axiom dependence) it is also lower regular of type 1, that is, it is regular in the sense of Definition 3. ♦

In order to prove another equivalent definition of regular metric, we need the following lemma.

Lemma 6 (straight-line and sphere intersection properties) – Let d be a metric on a set E. For any x and $y \in E$, and any $i \in d(\{x\} \times E)$,

(i) $S(x, i) \cap L_1(x, y) \neq \varnothing \Rightarrow d(x, y) \leq i$;

(ii) $S(x, i) \cap L_2(x, y) \neq \varnothing \Rightarrow i \leq d(x, y)$. ♦

Proof

(a) Let us prove (i). For any x and $y \in E$, and any $i \in d(\{x\} \times E)$,

$S(x, i) \cap L_1(x, y) \neq \varnothing$

$\Leftrightarrow \exists u \in E$: $d(x, u) = d(x, y) + d(y, u)$ and $d(x, u) = i$

(definitions of S and L_1)

$\Rightarrow \exists u \in E$: $i = d(x, y) + d(y, u)$

(substitution)

$\Rightarrow d(x, y) \leq i$. (Definition 1 - positiveness of d)

(b) Let us prove (ii). For any x and $y \in E$, and any $i \in d(\{x\} \times E)$,

$S(x, i) \cap L_2(x, y) \neq \varnothing$

$\Leftrightarrow \exists u \in E$: $d(x, y) = d(x, u) + d(u, y)$ and $d(x, u) = i$

(definitions of S and L_2)

$\Rightarrow \exists u \in E$: $d(x, y) = i + d(u, y)$

(substitution)

$\Rightarrow i \leq d(x, y)$. (Definition 1 - positiveness of d)

♦

The next proposition allows a geometrical interpretation for the regular metrics.

Proposition 7 (second equivalent definition of regular metric) – A metric d on E is regular iff for any x and $y \in E$, and any $i \in d(\{x\} \times E)$, the intersection between the straight-line $L(x, y)$ and the sphere $S(x, i)$ have at least two diametrically opposite points in the sense that it exists u and $v \in S(x, i)$ such that $u \in (L_1(x, y) \cup L_2(x, y))$ and $v \in L_3(x, y)$.

♦

Proof

(a) Let us assume that d is a regular metric on E. Since, for any x and $y \in E$, and any $i \in d(\{x\} \times E)$, at least one of the two conditions $d(x, y) \leq i$ and $i \leq d(x, y)$ is satisfied, by Definition 3 (regular metric space), at least one of the two properties $L_1(x, y) \cap S(x, i) \neq \varnothing$ and $L_2(x, y) \cap S(x, i) \neq \varnothing$ is satisfied, in other words, we have $(L_1(x, y) \cup L_2(x, y)) \cap S(x, i) \neq \varnothing$. That is, $\exists u \in S(x, i)$: $u \in (L_1(x, y) \cup L_2(x, y))$. Furthemore, by Definition 3 again, for any x and $y \in E$, and any $i \in d(\{x\} \times E)$, $L_3(x, y) \cap S(x, i) \neq \varnothing$. That is, $\exists v \in S(x, i)$: $v \in L_3(x, y)$.

(b) Conversely, let us assume that for any x and $y \in E$, and any $i \in d(\{x\} \times E)$, $\exists u$ and $v \in S(x, i)$: $u \in (L_1(x, y) \cup L_2(x, y))$ and $v \in L_3(x, y)$.

(b1) If $i < d(x, y)$,

true $\Leftrightarrow S(x, i) \cap (L_1(x, y) \cup L_2(x, y)) \neq \varnothing$ and $i < d(x, y)$ (hypotheses)

$\Rightarrow S(x, i) \cap (L_1(x, y) \cup L_2(x, y)) \neq \varnothing$ and $S(x, i) \cap L_1(x, y) = \varnothing$

(Property (i) of Lemma 6)

$\Leftrightarrow (S(x, i) \cap L_1(x, y)) \cup (S(x, i) \cap L_2(x, y)) \neq \varnothing$ and $S(x, i) \cap L_1(x, y) = \varnothing$ (distributivity of $\cap$ over $\cup$)

$\Leftrightarrow S(x, i) \cap L_2(x, y) \neq \varnothing$ ($\varnothing$ is unity for $\cup$)

$\Rightarrow d$ is lower regular.

(Corollary 5 - first equivalent definition of regular metric)

If $d(x, y) = i$,

true $\Leftrightarrow \exists u \in E$: $d(x, y) = d(x, u) + d(u, y)$ and $d(x, u) = i$

(namely $u = y$ and Property (ii) of Definition 1)

$\Leftrightarrow S(x, i) \cap L_2(x, y) \neq \varnothing$ (definitions of S and L_2)

$\Rightarrow d$ is lower regular. (Corollary 5)

(b2)

true $\Leftrightarrow S(x, i) \cap L_3(x, y) \neq \varnothing$ (hypothesis)

$\Rightarrow d$ is upper regular. (Corollary 5)

That is, by Corollary 5, d is regular. ♦

The Euclidean distance d on $\mathbf{R}^2$, given by, for any $x = (x_1, x_2)$ and any $y = (y_1, y_2) \in \mathbf{Z}^2$,

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}\ ,$$

is regular, nevertheless, when restricted to the discrete plane $\mathbf{Z}^2$ it is not regular. For example, the straight-line $L(x, y)$ containing $x = (0, 0)$ and $y = (1, 1)$ consists of the points (i, i) with $i \in \mathbf{Z}$, and the sphere $S(x, 1)$ of radius 1 with center at x consists of the points (0, 1), (1, 0), (0, –1) and (–1, 0). We observe that this line and this sphere have no intersection. The left-hand side of Figure 1 illustrates this point. Another example of non-regular metric in the discrete plane can be built from the elliptic distance d given by, for any $x = (x_1, x_2)$ and any $y = (y_1, y_2) \in \mathbf{Z}^2$,

$$d(x, y) = \sqrt{(x_1 - y_1)^2 / a^2 + (x_2 - y_2)^2 / b^2}\ ,$$

with $a = 5/3$ and $b = 5/4$. For example, the straight-line $L(x, y)$ containing $x = (0, 0)$ and $y = (2, 1)$ consists of the points $(2i, i)$ with $i \in \mathbf{Z}$, and the sphere $S(x, 1)$ of radius 1 with center at x consists of the points (1, 1), (1, – 1), (– 1, –1) and (–1, 1). Again, both have no intersection. The right-hand side of Figure 1 illustrates this point.

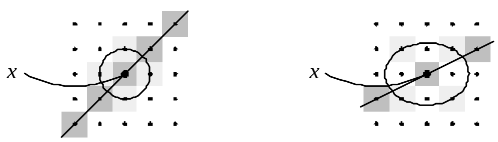

Figure 1 - Examples of non-regular metrics

The discrete metric is not regular despite the fact that for any x and $y \in E$, $x \neq y$ and any $i \in d_0(\{x\} \times E) = \{0, 1\}$, $S(x, i) \cap L(x, y) \neq \varnothing$ (since $L(x, y) = \{x, y\}$, $S(x, 0) = \{x\}$ and $S(x, 1) = \{y\}$). For the discrete metric, we cannot find two diametrically opposite points since $L_3(x, y) = \{x\}$ and $x \notin S(x, 1)$.

From now on, we restrict ourself to translation-invariant metrics on an Abelian group. Such metric property suits most image analysis problems.

Definition 8 (translation-invariant metric sapce) – A distance or metric d on an Abelian group $(E, +)$ is *translation-invariant* (t.i.) if, for any u, x and $y \in E$: $d(x + u, y + u) = d(x, y)$. A *translation-invariant metric space* (E, d) is a set E provided with a t.i. metric d on E. ♦

As it is well known, every translation-invariant metric on E can be characterized in terms of a mapping from E to $\mathbf{R}$ as stated in the next proposition.

Let $(E, +, o)$ be an Abelian group where o is the unit element of +. The subtraction on E, denoted –, is the mapping $E \times E \ni (x, y) \mapsto x + (-y) \in E$, where $-y$ is the inverse of y.

For any mapping d from $E \times E$ to $\mathbf{R}$ and any $x \in E$, let

$$f_d(x) = d(x, o).$$

For any mapping f from E to $\mathbf{R}$ and any x and $y \in E$ let

$$d_f(x, y) = f(x - y).$$

Proposition 9 (characterization of translation-invariant metric) – The mapping $d \mapsto f_d$ from the set of translation-invariant distances on an Abelian group $(E, +, o)$ to the set of mappings f from E to $\mathbf{R}$ satisfying the properties, for any $x \in E$:

(i) $f(x) \geq 0$, (positiveness)
(ii) $f(o) = 0$,
(iii) $f(-x) = f(x)$, (symmetry)

is a bijection and its inverse is $f \mapsto d_f$. Furthermore, $d \mapsto f_d$ is, as well, a bijection from the set of translation-invariant metrics on the Abelian group $(E, +, o)$ to the set of mappings f from E to $\mathbf{R}$ satisfying the properties (i) - (iii) and the property, for any x and $y \in E$:

(iv) $f(x + y) = f(x) + f(y)$ (subadditivity). ♦

Proof

Let us divide the proof into four parts.

(a) Let d be a mapping from $E \times E$ to $\mathbf{R}$.

(a1) Let us assume that d satisfies Property (i) of Definition 1. For any $x \in E$,

$f_d(x) = d(x, o)$ (definition of f_d)
$\quad \geq 0$, (hypothesis)

that is f_d satisfies Property (i).

(a2) Let us assume that d satisfies Property (ii) of Definition 1.

$f_d(o) = d(o, o)$ (definition of f_d)
$\quad = 0$, (hypothesis)

that is f_d satisfies Property (ii).

(a3) Let us assume that d satisfies Property (iii) of Definition 1. For any $x \in E$,

$f_d(-x) = d(-x, o)$ (definition of f_d)
$\quad = d(o, x)$ (d is t.i.)
$\quad = d(x, o)$ (hypothesis)
$\quad = f_d(x)$, (definition of f_d)

that is f_d satisfies Property (iii).

(a4) Let us assume that d satisfies Property (iv) of Definition 1. For any x and $y \in E$,

$f_d(x + y) = d(x + y, o)$ (definition of f_d)
$\quad \leq d(x + y, y) + d(y, o)$ (hypothesis)
$\quad = d(x, o) + d(y, o)$ (d is t.i.)
$\quad = f_d(x) + f_d(y)$, (definition of f_d)

that is f_d satisfies Property (iv).

In other words, from (a1) - (a3), if d is a t.i. distance then f_d satisfies Properties (i) - (iii), and, from (a1) - (a4), if d is a t.i. metric then f_d satisfies Properties (i) - (iv).

(b) Let f be a mapping from E to $\mathbf{R}$.

(b1) Let us assume that f satisfies Property (i). For any x and $y \in E$,

$d_f(x, y) = f(x - y)$ (definition of d_f)
$\quad \geq 0$, (hypothesis)

that is d_f satisfies Property (i) of Definition 1.

(b2) Let us assume that f satisfies Property (ii). For any $x \in E$,

$d_f(x, x) = f(x - x)$ (definition of d_f)
$\quad = f(o)$ (group porperty)
$\quad = 0$, (hypothesis)

that is d_f satisfies Property (ii) of Definition 1.

(b3) Let us assume that f satisfies Property (iii). For any x and $y \in E$,

$d_f(x, y) = f(x - y)$ (definition of d_f)
$= f(-(x - y))$ (hypothesis)
$= f(y - x)$ (group property)
$= d_f(y, x)$ (definition of d_f)

that is d_f satisfies Property (iii) of Definition 1.

(b4) Let us assume that f satisfies Property (iv). For any x, y and $z \in E$,

$d_f(x, y) = f(x - y)$ (definition of d_f)
$= f((x - z) + (z - y))$ (group property)
$\leq f(x - z) + f(z - y)$ (hypothesis)
$= d_f(x, z) + d_f(z, y)$, (definition of d_f)

that is d_f satisfies Property (iv) of Definition 1.

In other words, from (b1) - (b3), if f satisfies Properties (i) - (iii) then d_f is a t.i. distance and, from (b1) - (b4), if f satisfies Properties (i) - (iv) then d_f is a t.i. metric.

(c) Let d be a t.i. distance on an Abelian group $(E, +, o)$. For any x and $y \in E$,

$d_{f_d}(x, y) = f_d(x - y)$ (definition of d_f)
$= d(x - y, o)$ (definition of f_d)
$= d(x, y)$, (d is t.i.)

that is, $f \mapsto d_f$ is the left inverse of $d \mapsto f_d$.

(d) Let f be a mapping from E to $\mathbf{R}$. For any $x \in E$,

$f_{d_f}(x) = d_f(x, o)$ (definition of f_d)
$= f(x - o)$ (definition of d_f)
$= f(x)$, (group properties)

that is, $f \mapsto d_f$ is the right inverse of $d \mapsto f_d$.

In other words, from (a) - (d), both mappings $d \mapsto f_d$ from the set of distances and the set of metrics are bijections. ♦

In the case of t.i. metrics, we can make an explicit relationship between the above concept of regular metric and the concepts of upper and lower regularity proposed by Kiselman [5] in order to compare balls with different centers. Let us recall the Kiselman's definitions.

Definition 10 (first definition of Kilselman's regularity for a metric) – Let $(E, +, o)$ be an Abelian Group and let f a mapping from E to $\mathbf{R}$ satisfying Properties (i) - (iv) of Proposition 9 (characterization of t.i. metric), then the translation-invariant metric d_f on $(E, +, o)$ is

(i) *lower regular* (*of type 1*) *for the triangle inequality* if, for any x and $y \in E$ such that $f(y) \leq f(x)$, there exists a point $\tilde{x} \in E$ such that $f(\tilde{x}) = f(x)$ and $f(\tilde{x}) = f(\tilde{x} - y) + f(y)$;

(ii) *lower regular* (*of type 2*) *for the triangle inequality* if, for any x and $y \in E$ such that $f(x) \leq f(y)$, there exists a point $\tilde{x} \in E$ such that $f(\tilde{x}) = f(x)$ and $f(y) = f(\tilde{x}) + f(y - \tilde{x})$;

(iii) *upper regular for the triangle inequality* if, for any x and $y \in E$, there exists a point $\tilde{y} \in E$ such that $f(\tilde{y}) = f(y)$ and $f(x + \tilde{y}) = f(x) + f(\tilde{y})$. ♦

In [5], Kiselman has defined Axioms (i) and (iii). For the sake of compleness, we have added here Axiom (ii) and the expressions "type 1" and "type 2".

Actually, we can rewrite Definition 10 as shown below.

Definition 11 (second definition of Kilseman's regularity for a metric) – A translation-invariant metric d on an Abelian group $(E, +, o)$ is

(i) *lower regular* (*of type 1*) *for the triangle inequality* if, for any x and $y \in E$ such that $f_d(y) \leq f_d(x)$, there exists $\tilde{x} \in E$ such that $f_d(\tilde{x}) = f_d(x)$ and $f_d(\tilde{x}) = f_d(\tilde{x} - y) + f_d(y)$;

(ii) *lower regular* (*of type 2*) *for the triangle inequality* if, for any x and $y \in E$ such that $f_d(x) \leq f_d(y)$, there exists $\tilde{x} \in E$ such that $f_d(\tilde{x}) = f_d(x)$ and $f_d(y) = f_d(\tilde{x}) + f_d(y - \tilde{x})$;

(iii) *upper regular for the triangle inequality* if, for any x and $y \in E$, there exists $\tilde{y} \in E$ such that $f_d(\tilde{y}) = f_d(y)$ and $f_d(x + \tilde{y}) = f_d(x) + f_d(\tilde{y})$. ♦

Because of Proposition 9 (characterization of t.i. metric), both definitions correspond to exactly the same class of metrics.

Proposition 12 (definition equivalence) – Let A be the subset of mappings f from E to $\mathbf{R}$ satisfying Properties (i) - (iv) of Proposition 9 (characterization of t.i. metric) and let B be the subset of mappings f in A satisfying Conditions (i) - (iii) of Definition 10, then the set $(f \mapsto d_f)(B)$ (the image of B through the inverse of $d \mapsto f_d$) is equal to the set $(d \mapsto f_d)^{-1}(B)$ (the inverse image of B through $d \mapsto f_d$). ♦

Proof

By Proposition 9 (characterization of t.i. metric), $f \mapsto d_f$ is a left and right inverse for the mapping $d \mapsto f_d$ from the set of translation-invariant metrics to the set A, therefore, for any subset X of A, $(f \mapsto d_f)(X) = (d \mapsto f_d)^{-1}(X)$. So, the equality is also true for B. ♦

By using the second definition of Kiselman's regularity for a metric, we show now that the t.i. regular metrics satisfy the Kiselman's regularity axioms and conversely.

Proposition 13 (equivalent definition of translation-invariant regular metric) – Let d be a translation-invariant metric on an Abelian group $(E, +, o)$, then,

(i) d is lower regular of type 1 in the sense of Definition 3 (regular metric space) iff d is lower regular of type 1 for the triangle inequality;

(ii) d is lower regular of type 2 in the sense of Definition 3 iff d is lower regular of type 2 for the triangle inequality;

(iii) d is upper regular in the sense of Definition 3 iff d is upper regular for the triangle inequality;

(iv) d is regular iff d is lower regular of type 2 and upper regular for the triangle inequality. ♦

Proof

Let us divide the proof into three parts.

(a) Let us prove (i).

(a1) Let x and $y \in E$, then $f_d(x)$ and $f_d(y) \in d(\{o\} \times E)$. Let us assume that $f_d(y) \le f_d(x)$,
(d is lower regular of type 1 in the sense of Definition 3)
$\Rightarrow S(o, f_d(x)) \cap L_1(o, y) \ne \emptyset$ and $d(o, y) \le f_d(x)$
(Definition 3 - regular metric space)
$\Leftrightarrow \exists\ \tilde{x} \in E$: $d(o, \tilde{x}) = d(o, y) + d(y, \tilde{x})$ and $d(o, \tilde{x}) = f_d(x)$
(definitions of S and L_1)
$\Leftrightarrow \exists\ \tilde{x} \in E$: $d(\tilde{x}, o) = d(y, o) + d(\tilde{x}, y)$ and $d(\tilde{x}, o) = f_d(x)$
(Definition 1 - symmetry of d)
$\Leftrightarrow \exists\ \tilde{x} \in E$: $d(\tilde{x}, o) = d(y, o) + d(\tilde{x} - y, o)$ and
$d(\tilde{x}, o) = f_d(x)$ (d is t.i. and group properties)
$\Leftrightarrow \exists\ \tilde{x} \in E$: $f_d(\tilde{x}) = f_d(y) + f_d(\tilde{x} - y)$ and $f_d(\tilde{x}) = f_d(x)$
(definition of f_d)
$\Leftrightarrow$ (d is lower regular of type 1 for the triangle inequality).
(Definition 11)
(a2) Conversely, let x and $y \in E$, and $i \in d(\{x\} \times E)$ such that $d(x, y) \le i$, then $f_d(y - x) \le i$ (since d is t.i. and symmetric); furthermore,
(d is lower regular of type 1 for the triangle inequality)
$\Rightarrow \exists\ \tilde{x} \in E$: $f_d(\tilde{x}) = f_d(y - x) + f_d(\tilde{x} - (y - x))$ and $f_d(\tilde{x}) = i$
(Definition 11)
$\Leftrightarrow \exists\ \tilde{x} \in E$: $d(\tilde{x}, o) = d(y - x, o) + d(\tilde{x} - (y - x), o)$ and
$d(\tilde{x}, o) = i$
(definition of f_d)
$\Leftrightarrow \exists\ \tilde{x} \in E$: $d(\tilde{x}, o) = d(y - x, o) + d(\tilde{x} + x - y, o)$ and
$d(\tilde{x}, o) = i$ (group properties)
$\Leftrightarrow \exists\ \tilde{x} \in E$: $d(\tilde{x} + x, x) = d(y, x) + d(\tilde{x} + x, y)$ and
$d(\tilde{x} + x, x) = i$ (d is t.i. and group properties)
$\Leftrightarrow \exists\ u \in E$: $d(u, x) = d(y, x) + d(u, y)$ and $d(u, x) = i$
(namely $u = \tilde{x} + x$)
$\Leftrightarrow \exists\ u \in E$: $d(x, u) = d(x, y) + d(y, u)$ and $d(x, u) = i$
(d is symmetric)
$\Leftrightarrow S(x, i) \cap L_1(x, y) \ne \emptyset$ (definitions of S and L_1)
$\Leftrightarrow$ (d is lower regular of type 1 in the sense of Definition 3).
(Definition 3)
(b) Let us prove (ii).
(b1) Let x and $y \in E$, then $f_d(x)$ and $f_d(y) \in d(\{o\} \times E)$. Let us assume that such that $f_d(x) \le f_d(y)$,
(d is lower regular of type 2 in the sense of Definition 3)
$\Rightarrow S(o, f_d(x)) \cap L_2(o, y) \ne \emptyset$ and $f_d(x) \le d(o, y)$
(Definition 3)
$\Leftrightarrow \exists\ \tilde{x} \in E$: $d(o, y) = d(o, \tilde{x}) + d(\tilde{x}, y)$ and $d(o, \tilde{x}) = f_d(x)$
(definitions of S and L_2)
$\Leftrightarrow \exists\ \tilde{x} \in E$: $d(o, y) = d(o, \tilde{x}) + d(o, y - \tilde{x})$ and
$d(o, \tilde{x}) = f_d(x)$ (d is t.i. and group properties)
$\Leftrightarrow \exists\ \tilde{x} \in E$: $d(y, o) = d(\tilde{x}, o) + d(y - \tilde{x}, o)$ and
$d(\tilde{x}, o) = f_d(x)$ (Definition 1 - symmetry of d)
$\Leftrightarrow \exists\ \tilde{x} \in E$: $f_d(y) = f_d(\tilde{x}) + f_d(y - \tilde{x})$ and $f_d(\tilde{x}) = f_d(x)$
(definition of f_d)
$\Leftrightarrow$ (d is lower regular of type 2 for the triangle inequality).
(Definition 11)
(b2) Conversely, let x and $y \in E$, and $i \in d(\{x\} \times E)$ such that $i \le d(x, y)$, then $i \le f_d(y - x)$ (since d is t.i. and symmetric); furthermore,
(d is lower regular of type 2 for the triangle inequality)
$\Rightarrow \exists\ \tilde{x} \in E$: $f_d(y - x) = f_d(\tilde{x}) + f_d((y - x) - \tilde{x})$ and $f_d(\tilde{x}) = i$
(Definition 11)
$\Leftrightarrow \exists\ \tilde{x} \in E$: $d(y - x, o) = d(\tilde{x}, o) + d((y - x) - \tilde{x}, o)$ and
$d(\tilde{x}, o) = i$
(definition of f_d)
$\Leftrightarrow \exists\ \tilde{x} \in E$: $d(y - x, o) = d(\tilde{x}, o) + d(y - (\tilde{x} + x), o)$ and
$d(\tilde{x}, o) = i$ (group properties)
$\Leftrightarrow \exists\ \tilde{x} \in E$: $d(y, x) = d(\tilde{x} + x, x) + d(y, \tilde{x} + x)$ and
$d(\tilde{x} + x, x) = i$ (d is t.i. and group properties)
$\Leftrightarrow \exists\ u \in E$: $d(y, x) = d(u, x) + d(y, u)$ and $d(u, x) = i$
(namely $u = \tilde{x} + x$)
$\Leftrightarrow \exists\ u \in E$: $d(x, y) = d(x, u) + d(u, y)$ and $d(x, u) = i$
(d is symmetric)
$\Leftrightarrow S(x, i) \cap L_2(x, y) \ne \emptyset$ (definitions of S and L_2)
$\Leftrightarrow$ (d is lower regular of type 2 in the sense of Definition 3).
(Definition 3)
(c) Let us prove (iii).
(c1) Let x and $y \in E$, then $f_d(x)$ and $f_d(y) \in d(\{o\} \times E)$; furthermore,
(d is upper regular in the sense of Definition 3)
$\Rightarrow S(o, f_d(y)) \cap L_3(o, x) \ne \emptyset$
(Definition 3)
$\Leftrightarrow \exists\ \tilde{y} \in E$: $d(-\tilde{y}, x) = d(-\tilde{y}, o) + d(o, x)$ and
$d(o, \tilde{y}) = f_d(y)$ (definitions of S and L_3)
$\Leftrightarrow \exists\ \tilde{y} \in E$: $d(o, x + \tilde{y}) = d(o, \tilde{y}) + d(o, x)$ and
$d(o, \tilde{y}) = f_d(y)$ (d is t.i. and group properties)
$\Leftrightarrow \exists\ \tilde{y} \in E$: $d(x + \tilde{y}, o) = d(\tilde{y}, o) + d(x, o)$ and
$d(\tilde{y}, o) = f_d(y)$ (Definition 1 - symmetry of d)
$\Leftrightarrow \exists\ \tilde{y} \in E$: $f_d(x + \tilde{y}) = f_d(x) + f_d(\tilde{y})$ and $f_d(\tilde{y}) = f_d(y)$
(definition of f_d)
$\Leftrightarrow$ (d is upper regular for the triangle inequality).
(Definition 11)
(c2) Conversely, let x and $y \in E$, and $i \in d(\{x\} \times E)$, then,
(d is upper regular for the triangle inequality)
$\Rightarrow \exists\ \tilde{y} \in E$: $f_d((y - x) + \tilde{y}) = f_d(\tilde{y}) + f_d(y - x)$ and
$f_d(\tilde{y}) = i$
(Definition 11)
$\Leftrightarrow \exists\ \tilde{y} \in E$: $d((y - x) + \tilde{y}, o) = d(\tilde{y}, o) + d(y - x, o)$ and
$d(\tilde{y}, o) = i$
(definition of f_d)
$\Leftrightarrow \exists\ \tilde{y} \in E$: $d(y - (x - \tilde{y}), o) = d(\tilde{y}, o) + d(y - x, o)$ and
$d(\tilde{y}, o) = i$ (group properties)
$\Leftrightarrow \exists\ \tilde{y} \in E$: $d(y, x - \tilde{y}) = d(x, x - \tilde{y}) + d(y, x)$ and
$d(x, x - \tilde{y}) = i$ (d is t.i. and group properties)
$\Leftrightarrow \exists\ v \in E$: $d(y, v) = d(x, v) + d(y, x)$ and $d(x, v) = i$
(namely $v = x - \tilde{y}$)
$\Leftrightarrow \exists\ v \in E$: $d(v, y) = d(v, x) + d(x, y)$ and $d(x, v) = i$
(d is symmetric)
$\Leftrightarrow S(x, i) \cap L_3(x, y) \ne \emptyset$ (definitions of S and L_3)
$\Leftrightarrow$ (d is upper regular in the sense of Definition 3).
(Definition 3)
(d) Property (iv) is a consequence of Corollary 5 (first equivalent defintion of regular metric) and Properties (ii) - (iii). ♦

Actually, we have reused in this work the word "regular" of the Kiselman's definitions for the metric satisfying Definition 3 because of the previous proposition.

As a consequence of Property (iv), we observe that the important Kiselman's axioms are the lower regularity of type 2 and the upper regularity for the triangle inequality and not the lower regularity of type 1 for the triangle inequality.

Proposition 13 may be a short cut for the proof that some t.i. metrics are regular as we show now for the chessboard distance which is of particular interest in image processing.

The set $\mathbf{Z}^2$ equipped with the addition $((x_1, x_2), (y_1, y_2)) \mapsto (x_1 + y_1, x_2 + y_2)$, denoted +, forms an Abelian group $(\mathbf{Z}^2, +, o)$ with $o = (0, 0)$ as unitary element.

Definition 14 (chessboard distance) – The *chessboard distance* $d_8(x, y)$ *between* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$ in $\mathbf{Z}^2$ is the natural number defined by

$d_8(x, y) = \max\{| x_1 - y_1|, | x_2 - y_2|\}$.

The mapping d_8: $(x, y) \mapsto d_8(x, y)$ from $\mathbf{Z}^2 \times \mathbf{Z}^2$ to $\mathbf{R}$, is called the *chessboard distance*. ♦

Actually, the chessboard distance d_8 is a metric. Furthermore, with respect to the Abelian group $(\mathbf{Z}^2, +, o)$, d_8 is by construction tanslation-invariant.

Let $f_8 = f_{d_8}$, and let $z = (z_1, z_2)$ be a point in $\mathbf{Z}^2$ then,

$$\begin{aligned}
f_8(z) &= d_8(z, o) && \text{(definition of } f_d) \\
&= \max\{| z_1 - 0 |, | z_2 - 0 |\} && \text{(Definition 14- def. of } d_8) \\
&= \max\{| z_1 |, | z_2 |\}. && \text{(properties of + on } \mathbf{Z})
\end{aligned}$$

In other words, f_8 is given by, for any $z = (z_1, z_2) \in \mathbf{Z}^2$,

$f_8(z) = \max\{| z_1 |, | z_2 |\}$.

Let us show that d_8 is regular.

Proposition 15 (example of regular metric space) – The metric space $(\mathbf{Z}^2, d_8)$ is regular. ♦

Proof

(a) Let us show that d_8 is lower regular (of type 2) for the triangle inequality. Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbf{Z}^2$ such that $f_8(x) \le f_8(y)$, let $i = f_8(x)$ and let $\tilde{x} = (\text{sign}(y_1).\min\{| y_1|, i\}, \text{sign}(y_2).\min\{| y_2 |, i\})$, then

$$\begin{aligned}
f_8(\tilde{x}) &= \max\{| \text{sign}(y_1).\min\{| y_1|, i\} |, | \text{sign}(y_2).\min\{| y_2 |, i\} |\} \\
& && \text{(expressions of } f_8 \text{ and } \tilde{x}) \\
&= \max\{\min\{| y_1|, i\}, \min\{| y_2 |, i\}\} && (i \ge 0) \\
&= i && \text{(by hypothesis, } i \le \max\{| y_1|, | y_2 |\}) \\
&= f_8(x). && \text{(hypothesis)}
\end{aligned}$$

Furthermore, for $k = 1, 2$,

$$\begin{aligned}
| y_k - \tilde{x}_k | &= | y_k - \text{sign}(y_k).\min\{| y_k |, i\} | && \text{(expression of } \tilde{x}_k) \\
&= | \text{sign}(y_k).y_k - \min\{| y_k |, i\} | \\
& && \text{(properties of sign and } |\bullet|) \\
&= | | y_k | - \min\{| y_k |, i\} |. \\
& && \text{(definition of } |\bullet|)
\end{aligned}$$

If $| y_1| \le | y_2 |$, then

$$\begin{aligned}
| y_1 - \tilde{x}_1 | &= | | y_1| - \min\{| y_1|, i\} | && \text{(see above, } k = 1) \\
&= \begin{cases} 0 & \text{if } |y_1| \le i \\ |y_1| - i & \text{otherwise} \end{cases} && \text{(definition of min)}
\end{aligned}$$

and

$$\begin{aligned}
| y_2 - \tilde{x}_2 | &= | | y_2 | - \min\{| y_2|, i\} | && \text{(see above, } k = 2) \\
&= | y_2 | - i && \text{(by hypothesis, } i \le | y_2 |)
\end{aligned}$$

therefore, $| y_1 - \tilde{x}_1 | \le | y_2 - \tilde{x}_2 |$.

In other words, if $| y_1| \le | y_2 |$, then

$$\begin{aligned}
\max\{| y_1 - \tilde{x}_1 |, | y_2 - \tilde{x}_2 |\} &= | y_2 - \tilde{x}_2 | && \text{(see above)} \\
&= | y_2 | - i && \text{(see above)} \\
&= \max\{| y_1 |, | y_2 |\} - i. && \text{(hypothesis)}
\end{aligned}$$

If $| y_2 | \le | y_1|$, then, under the same arguments, $\max\{| y_1 - \tilde{x}_1 |, | y_2 - \tilde{x}_2 |\} = \max\{| y_1 |, | y_2 |\} - i$. Hence, in any case, $\max\{| y_1 - \tilde{x}_1 |, | y_2 - \tilde{x}_2 |\} = \max\{| y_1 |, | y_2 |\} - i$.

Finally,

$$\begin{aligned}
f_8(\tilde{x}) + f_8(y - \tilde{x}) &= i + f_8(y - \tilde{x}) && \text{(see above)} \\
&= i + \max\{| y_1 - \tilde{x}_1 |, | y_2 - \tilde{x}_2 |\} && \text{(expression of } f_8) \\
&= i + \max\{| y_1 |, | y_2 |\} - i && \text{(see above)} \\
&= \max\{| y_1 |, | y_2 |\} && \text{(cancelation)} \\
&= f_8(y) && \text{(expression of } f_8)
\end{aligned}$$

That is, for any x and $y \in \mathbf{Z}^2$ such that $f_8(x) \le f_8(y)$, there exists a point $\tilde{x} \in \mathbf{Z}^2$ such that $f_8(\tilde{x}) = f_8(x)$ and $f_8(y) = f_8(\tilde{x}) + f_8(y - \tilde{x})$. In other words, d_8 is lower regular (of type 2) for the triangle inequality.

(b) Let us show that d_8 is upper regular for the triangle inequality. Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbf{Z}^2$, let $i = f_8(y)$ and let $\tilde{y} = (\text{sign}(x_1).i, \text{sign}(x_2).i)$, then

$$\begin{aligned}
f_8(\tilde{y}) &= \max\{| \text{sign}(x_1).i |, | \text{sign}(x_2).i |\} && \text{(expression of } \tilde{y}) \\
&= \max\{i, i\} && (i \ge 0) \\
&= i && \text{(max is idempotent)} \\
&= f_8(y). && \text{(hypothesis)}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
f_8(x + \tilde{y}) &= \max\{| x_1 + \tilde{y}_1|, | x_2 + \tilde{y}_2 |\} && \text{(expression of } f_8) \\
&= \max\{| x_1 + \text{sign}(x_1).i |, | x_2 + \text{sign}(x_2).i |\} \\
& && \text{(expression of } \tilde{y}) \\
&= \max\{| \text{sign}(x_1).x_1 + i |, | \text{sign}(x_2).x_2 + i |\} \\
& && \text{(properties of sign and } |\bullet|) \\
&= \max\{| | x_1| + i |, | | x_2 | + i |\} \\
& && \text{(definition of } |\bullet|) \\
&= \max\{| x_1| + i, | x_2 | + i\} \\
& && (i \ge 0) \\
&= \max\{| x_1 |, | x_2|\} + i && \text{(distribuitvity of + over max)} \\
&= f_8(x) + i && \text{(expression of } f_8) \\
&= f_8(x) + f_8(\tilde{y}). && \text{(see above)}
\end{aligned}$$

That is, for any x and $y \in \mathbf{Z}^2$, there exists a point $\tilde{y} \in \mathbf{Z}^2$ such that $f_8(\tilde{y}) = f_8(y)$ and $f_8(x + \tilde{y}) = f_8(x) + f_8(\tilde{y})$. In other words, d_8 is upper regular for the triangle inequality.

Therefore, by applying Property (iv) of Proposition 13 (equivalent definition of translation-invariant regular metric), we may conclude that d_8 is regular. ♦

The upper part of Figure 2 shows the "straight-line" $L(x, y)$ relative to the chessboard distance d_8 passing through the point x and y and the "sphere" $S(x, 2)$ of center x and radius 2. As it was expected from Proposition 7, we observe that they have a nonempty intersection (see the lower part of the figure). Furthermore, this intersection have at least two diametrically opposite points, for instance, the points u and v. In this figure, $u \in L_2(x, y)$ and $v \in L_3(x, y)$.

Rosenfeld and Pfaltz [8] have shown in their Proposition 6 that d_8 satisfies the following property: for any $y \in \mathbf{Z}^2$ such that $1 \le f_8(y)$, there exists a point $\tilde{x} \in \mathbf{Z}^2$ such that $f_8(\tilde{x}) = 1$ and $f_8(y) = f_8(\tilde{x}) + f_8(y - \tilde{x})$. This result is a consequence of the lower regularity (of type 2) of d_8. More interesting for us is their counter example showing that the octogonal distance $d_{\text{oct}} = \sup\{d_8, g\}$ where g is the distance given by, for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbf{Z}^2$, $g(x, y) = 2(|\, x_1 - y_1| + |\, x_2 - y_2| + 1)/3$, doesn't satisfy this property. In other words, such octagonal distance is an example of non-regular metric.

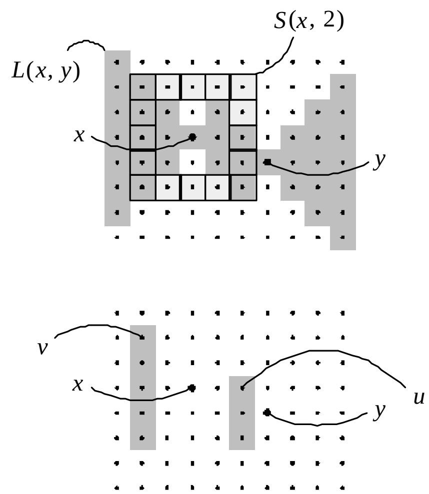

Figure 2 - Intersection of a diameter and a sphere.

Before ending this section, we show that the good geometrical properties of the Euclidean vector space mencioned at the beginning of the introduction are satisfied in a regular metric space.

First, we show that the lower regularity property for a distance is a necessary and sufficient condition to have the usual ball intersection property of the Euclidean vector space.

Proposition 16 (ball intersection in a lower regular metric space) – Let (E, d) be a metric space then,

(i) for any x and $y \in E$, any $i \in d(\{x\} \times E)$ and any $j \in d(E \times \{y\})$,
$\exists\, z \in E$: $d(x, z) \le i$ and $d(z, y) \le j \Rightarrow d(x, y) \le i + j$,

(ii) (E, d) is lower regular iff, for any x and $y \in E$, any $i \in d(\{x\} \times E)$ and any $j \in d(E \times \{y\})$,
$d(x, y) \le i + j \Rightarrow \exists\, z \in E$: $d(x, z) \le i$ and $d(z, y) \le j$. ♦

Proof

(a) Let us prove Property (i).

For any x and $y \in E$, any $i \in d(\{x\} \times E)$ and any $j \in d(E \times \{y\})$,

$\exists\, z \in E$: $d(x, z) \le i$ and $d(z, y) \le j$

$\Leftrightarrow \exists\, z \in E$: $d(x, z) + d(z, y) \le i + d(z, y)$ and $i + d(z, y) \le i + j$ (addition is increasing)

$\Rightarrow \exists\, z \in E$: $d(x, z) + d(z, y) \le i + j$ (transitivity of $\le$)

$\Rightarrow d(x, y) \le i + j$.

(Definition 1 - triangle inequality and transitivity of $\le$)

(b) Let us prove the if part of Property (ii).

For any x and $y \in E$, any $i \in d(\{x\} \times E)$ and any $j \in d(E \times \{y\})$,

$(d(x, y) \le i + j \Rightarrow \exists\, z \in E$: $d(x, z) \le i$ and $d(z, y) \le j)$

$\Rightarrow (d(x, y) = i + j \Rightarrow \exists\, z \in E$: $d(x, z) \le i$ and $d(z, y) \le j)$ (particular case)

$\Rightarrow (d(x, y) = i + j \Rightarrow \exists\, z \in E$: $d(x, z) \le i$ and $d(z, y) \le j$ and $d(x, z) + d(z, y) \le i + j)$ (+ is increasing)

$\Rightarrow (d(x, y) = i + j \Rightarrow \exists\, z \in E$: $d(x, z) \le i$ and $d(z, y) \le j$ and $d(x, y) \le d(x, z) + d(z, y) \le i + j)$ (triangle inequality)

$\Rightarrow (d(x, y) = i + j \Rightarrow \exists\, z \in E$: $d(x, z) \le i$ and $d(z, y) \le j$ and $d(x, y) \le d(x, z) + d(z, y) \le i + j = d(x, y))$ (logical derivation)

$\Rightarrow (d(x, y) = i + j \Rightarrow \exists\, z \in E$: $d(x, z) \le i$ and $d(z, y) \le j$ and $d(x, z) + d(z, y) = i + j)$ (anti-symmetry of $\le$)

$\Rightarrow (d(x, y) = i + j \Rightarrow \exists\, z \in E$: $d(x, z) \le i$ and $d(z, y) \le j$ and $d(x, z) + d(z, y) = i + j$ and $d(x, z) + d(z, y) \le i + d(z, y$ and $d(x, z) + d(z, y) \le d(x, z) + j)$ (+ is increasing)

$\Rightarrow (d(x, y) = i + j \Rightarrow \exists\, z \in E$: $d(x, z) \le i$ and $d(z, y) \le j$ and $d(x, z) + d(z, y) = i + j$ and $i + j \le i + d(z, y)$ and $i + j \le d(x, z) + j)$ (substitution)

$\Rightarrow (d(x, y) = i + j \Rightarrow \exists\, z \in E$: $d(x, z) = i$ and $d(z, y) = j$ and $d(x, z) + d(z, y) = i + j)$ (anti-symmetry of $\le$)

$\Rightarrow (d(x, y) = i + j \Rightarrow \exists\, z \in E$: $d(x, z) = i$ and $d(z, y) = j$ and $d(x, z) + d(z, y) = d(x, y))$ (logical derivation)

$\Rightarrow (d(x, y) = i + j \Rightarrow \exists\, z \in E$: $d(x, z) = i$ and $d(x, z) + d(z, y) = d(x, y))$. (logical derivation)

That is, under the hypothesis, for any x and $y \in E$, any $i \in d(\{x\} \times E)$ and any $j \in d(E \times \{y\})$ such that $d(x, y) = i + j$, we have $\exists\, z \in E$: $d(x, z) = i$ and $d(x, z) + d(z, y) = d(x, y)$, but this implies that, for any x and $y \in E$, and any $i \in d(\{x\} \times E)$, $i \le d(x, y)$, we have $\exists\, z \in E$: $d(x, z) = i$ and $d(x, z) + d(z, y) = d(x, y)$, or equivalently by definitions of S and L_2, for any x and $y \in E$, and any $i \in d(\{x\} \times E)$, $i \le d(x, y)$, we have $S(x, i) \cap L_2(x, y) \neq \varnothing$ which proves, by Corollary 5 (first equivalent definition of regular metric), that d is lower regular.

(c) Let us prove the ony-if part Property (ii).

For any x and $y \in E$, any $i \in d(\{x\} \times E)$ and any $j \in d(E \times \{y\})$, such that $d(x, y) \le i + j$,

(b1) if $i = j = 0$, then $y = x$, therefore $\exists\, z \in E$: $d(x, z) \le i$ and $d(z, y) \le j$, namely $z = y = x$;

(b2) if $i = 0$ and $j \neq 0$, then $d(x, y) \le j$, therefore $\exists\, z \in E$: $d(x, z) \le i$ and $d(z, y) \le j$, namely $z = x$;

(b3) if $i \neq 0$ and $j = 0$, then $d(x, y) \le i$, therefore $\exists\, z \in E$: $d(x, z) \le i$ and $d(z, y) \le j$, namely $z = y$;

(b4) if $i \neq 0$ and $j \neq 0$, and if $d(x, y) < i$ or $d(x, y) < j$ then $\exists\, z \in E$: $d(x, z) \le i$ and $d(z, y) < j$, or $d(x, z) < i$ and $d(z, y) \le j$, namely $z = y$ or $z = x$;

(b5) if $i \le d(x, y)$ and $j \le d(x, y)$,

d is lower regular

$\Leftrightarrow S(x, i) \cap L_2(x, y) \neq \varnothing$ (Corollary 5)

$\Leftrightarrow \exists\, z \in E$: $d(x, z) + d(z, y) = d(x, y)$ and $d(x, z) = i$ (definitions of S and L_2)

$\Leftrightarrow \exists\, z \in E$: $i + d(z, y) = d(x, y)$ and $d(x, z) = i$; (substitution)

furthermore,

$d(x, y) \leq i + j \Rightarrow \exists\, z \in E$: $d(x, z) = i$ and $d(z, y) \leq j$

(lower regularity of d (see above) and + is double-side increasing)

$\Rightarrow \exists\, z \in E$: $d(x, z) \leq i$ and $d(z, y) \leq j$

(reflexivity of $\leq$)

♦

In order to illustrate better the previous proposition, we recall the definition of ball derived from a metric and we give a corollary. We call ball of center $x \in E$ and radius $i \in d(\{x\} \times E)$, derived from a metric d, the subset $B_d(x, i) = \{z \in E$: $d(x, z) \leq i\}$; in the next section we will give another definition of ball.

Corollary 17 (ball intersection in a lower regular metric space) – Let (E, d) be a metric space then,

(i) for any $x, y \in E$, any $i \in d(\{x\} \times E)$ and any $j \in d(E \times \{y\})$,
$B_d(x, i) \cap B_d(y, j) \neq \emptyset \Rightarrow d(x, y) \leq i + j$,

(ii) (E, d) is lower regular iff, for any $x, y \in E$, any $i \in d(\{x\} \times E)$ and any $j \in d(E \times \{y\})$,
$d(x, y) \leq i + j \Rightarrow B_d(x, i) \cap B_d(y, j) \neq \emptyset$. ♦

Proof

Properties (i) and (ii) are, respectively, equivalent to Properties (i) and (ii) of Proposition 16 (ball intersection in a lower regular metric space) since ($\exists\, z \in E$: $d(x, z) \leq i$ and $d(z, y) \leq j$) is equivalent to ($B_d(x, i) \cap B_d(y, j) \neq \emptyset$) by definition of $B_d(x, i)$. ♦

Figure 3 illustrates a counter example showing that the elliptic metric restricted to the discrete plane depicted in Figure 1 doesn't satisfy Property (ii) of Corollary 17. This is another way to conclude that this elliptic metric is not regular.

In Figure 3 the ellipses are the locus of points at distance 1 to their centers of symmetry x and y and the elliptic distance between these centers is $d(x, y) = 9/5$ which is less than 2. In other words, the condition $d(x, y) \leq i + j$ (with $i = j = 1$) is satisfied. Nevertheless, as we can see on this figure, the ball intersection $B_d(x, 1) \cap B_d(y, 1)$ is empty.

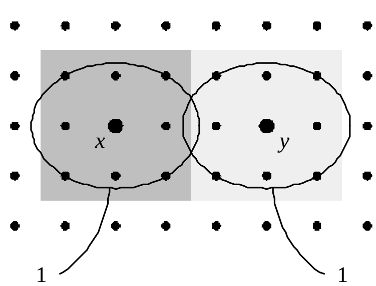

Figure 3 - Lack of intersection (discrete elliptic distance).

Figure 4 shows two examples of lack of intersection in the octagonal metric space $(\mathbf{Z}^2, d_{\text{oct}})$. On the left hand-side the two balls are of radius $i = j = 1$ and their centers x and y, 2 units apart. On the right-hand-side the two balls are of radius $i = 1$ and $j = 3$, and their centers x and y, 4 units apart. On both cases the two balls have no intersections despite the fact that $d_{\text{oct}}(x, y) = i + j$. The right-hand side example is due to Rosenfeld and Pfaltz [8].

Proposition 16 will be a key result to prove Proposition 53 in Section 6 and consequently Proposition 62 of Section 8 showing that the lower regular metrics can be reconstructed from their unit balls using the Minkowski product of Section 4.

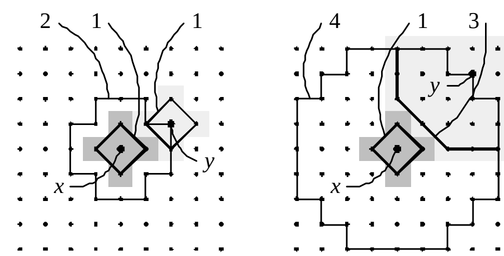

Figure 4 - Lack of intersection (octagonal distance).

Finally, we show that the upper regularity property for a distance is a sufficient condition to have the usual ball inclusion property of the Euclidean vector space, see also [5, Theorem 5.19].

Proposition 18 (ball inclusion in an upper regular metric space) – Let (E, d) be a metric space then, for any $x, y \in E$, any $i \in d(\{x\} \times E)$ and any $j \in d(E \times \{y\})$,

(i) $i + d(x, y) \leq j \Rightarrow B_d(x, i) \subset B_d(y, j)$,

(ii) if (E, d) is upper regular then
$B_d(x, i) \subset B_d(y, j) \Rightarrow i + d(x, y) \leq j$. ♦

Proof

(a) Let us prove Property (i).

For any x, y and $z \in E$, any $i \in d(\{x\} \times E)$ and any $j \in d(E \times \{y\})$,

$z \in B_d(x, i) \Leftrightarrow d(x, z) \leq i$ (def. of B_d)

$\Leftrightarrow d(x, y) + d(x, z) \leq d(x, y) + i$ (positivness of d and + is increasing)

$\Leftrightarrow d(y, x) + d(x, z) \leq d(x, y) + i$ (symmetry of d)

$\Leftrightarrow d(y, z) \leq d(y, x) + d(x, z) \leq d(x, y) + i$ (triangle inequality)

$\Rightarrow d(y, z) \leq d(x, y) + i$ (transitivity of $\leq$)

$\Rightarrow d(y, z) \leq i + d(x, y)$ (commutativity of +)

$\Rightarrow d(y, z) \leq d(x, y) + i \leq j$ (hypothesis)

$\Rightarrow d(y, z) \leq j$ (transitivity of $\leq$)

$\Leftrightarrow z \in B_d(y, j)$, (def. of B_d)

that is, by inclusion definition, under the hypothesis, $B_d(x, i) \subset B_d(y, j)$.

(b) Let us prove Property (ii).

Let d be an upper regular metric, let x and $y \in E$, let $i \in d(\{x\} \times E)$ and let $j \in d(E \times \{y\})$. We divide the proof in five parts.

(b1) Let us prove that $B_d(x, i) \subset B_d(y, j) \Rightarrow d(y, x) \leq j$:

$B_d(x, i) \subset B_d(y, j) \Rightarrow x \in B_d(y, j)$

$(d(x, x) = 0 \Rightarrow x \in B_d(x, i))$

$\Leftrightarrow d(y, x) \leq j$. (def. of B_d)

(b2) By Proposition 4 (axiom dependence), if d is upper regular, then d is lower regular of type 1. Furthermore,

true $\Leftrightarrow S(y, j) \cap L_1(y, x) \neq \varnothing$ and $S(x, i) \cap L_3(x, y) \neq \varnothing$

(d is lower regular of type 1 and by (b1) $d(y, x) \leq j$, and d is upper regular)

$\Leftrightarrow \exists\, u \in E$: $d(y, u) = j$ and $d(y, u) = d(y, x) + d(x, u)$ and $\exists\, v \in E$: $d(x, v) = i$ and $d(v, y) = d(v, x) + d(x, y)$

(definitions of S, L_1 and L_3)

(b3) For any $v \in E$,

$$
\begin{aligned}
d(x, v) = i &\Rightarrow v \in B_d(x, i) && \text{(def. of } B_d\text{)}\\
&\Rightarrow v \in B_d(y, j) && (B_d(x, i) \subset B_d(y, j))\\
&\Leftrightarrow d(y, v) \leq j. && \text{(def. of } B_d\text{)}
\end{aligned}
$$

(b4) For some u and $v \in E$,

$$
\begin{aligned}
d(x, v) + d(x, y) &= d(v, x) + d(x, y) && \text{(symmetry of } d\text{)}\\
&= d(v, y) && \text{((b2))}\\
&= d(y, v) && \text{(symmetry of } d\text{)}\\
&\leq j && \text{((b3))}\\
&= d(y, u) && \text{((b2))}\\
&= d(y, x) + d(x, u) && \text{((b2))}\\
&= d(x, y) + d(x, u), && \text{(symmetry of } d\text{)}
\end{aligned}
$$

that is, for some u and $v \in E$, by regularity of +, $d(x, v) \leq d(x, u)$.

(b5) For some u and $v \in E$,

$$
\begin{aligned}
i + d(x, y) &= d(x, v) + d(x, y) && \text{((b2))}\\
&\leq d(x, u) + d(x, y) && \text{((b4))}\\
&= d(x, y) + d(x, u) && \text{(commutativity of +)}\\
&= d(y, x) + d(x, u) && \text{(symmetry of } d\text{)}\\
&= d(y, u) && \text{((b2))}\\
&= j, && \text{((b2))}
\end{aligned}
$$

that is, if d is upper regular, then $B_d(x, i) \subset B_d(y, j) \Rightarrow i + d(x, y) \leq j$. ♦

3. Balls

The ball definition in this section is independent of the concept of distance seen in the previous section. Instead, it is based on the concepts of set translation and transposition. Because of our interest in digital image processing, the balls will be considered as subsets of the discrete plane $\mathbf{Z}^2$ (the Cartesian product of the set of integers by itself).

We denote by $\boldsymbol{P}(\mathbf{Z}^2)$ the collection of all the subsets of $\mathbf{Z}^2$.

Let $\mathbf{N}$ be the set of natural numbers: 0, 1, 2, … A subset X of $\mathbf{Z}^2$ is finite if there exists a natural number n and a bijection between X and the subset of natural numbers $\{1, 2, \dots, n\}$. We denote by $\#X$ the natural number n, and by $\#X < \infty$ the finiteness of X.

The Abelian group $(\mathbf{Z}^2, +, o)$ equipped with the scalar multiplication $(j, (x_1, x_2)) \mapsto j(x_1, x_2) = (jx_1, jx_2)$ with operand j in $\mathbf{Z}$ forms a module over the ring $\mathbf{Z}$ [6, p. 166].

We begin recalling the definition of translated version of a subset.

Definition 19 (translated version) – The *translated version of a subset* X of $\mathbf{Z}^2$ *by a point* u in $\mathbf{Z}^2$ is the subset $X_u = \{y \in \mathbf{Z}^2 : y - u \in X\}$. ♦

The next proposition shows a regularity property of the translated version: if $X_u = X_v$ then $u = v$. This property will be important to prove the uniqueness of the concept of radius of a ball in Section 5. The proof of this property relies on a basic poset theorem.

Let X and Y be two subsets of $\mathbf{Z}^2$. The *complement of* X in $\mathbf{Z}^2$ is denoted X^c, that is, for any $x \in \mathbf{Z}^2$, $x \in X^c \Leftrightarrow x \notin X$, and the *difference between* X *and* Y is denoted $X - Y$, that is, $X - Y = X \cap Y^c$.

Proposition 20 (translated version property) – If the subset X of $\mathbf{Z}^2$ is finite, then the mapping $u \mapsto X_u$ from $\mathbf{Z}^2$ to $\boldsymbol{P}(\mathbf{Z}^2)$ is injective. Furthermore, for any u and v in $\mathbf{Z}^2$ such that $u \neq v$, $X_v - X_u \neq \varnothing$ and $X_u - X_v \neq \varnothing$ or, equivalently, $X_u \cap X_v \neq X_v$ and $X_u \cap X_v \neq X_u$. ♦

Proof

Let u be a point in $\mathbf{Z}^2$ and let X be a subset of $\mathbf{Z}^2$, by the regularity of the addition on $\mathbf{Z}^2$, the mapping $x \mapsto x + u$ from X to X_u is a bijection, therefore, if X is finite then X_u is finite as well.

Let $u \in \mathbf{Z}^2$ and let R_u be the binary relation on $\mathbf{Z}^2$ given by, for any x and $y \in \mathbf{Z}^2$, $x\ \mathrm{R}_u\ y \Leftrightarrow \exists\, j \in \mathbf{N}$: $y = x + ju$. Let us first prove that R_u is a partial order. For any $x \in \mathbf{Z}^2$, $x\ \mathrm{R}_u\ x$ is true since for $j = 0$, $x = x + ju$ by group property, that is R_u is reflexive. If $u = o$, for any $x, y \in \mathbf{Z}^2$, $x\ \mathrm{R}_u\ y$ and $y\ \mathrm{R}_u\ x$ imply $x = y$ (and $y = x$). If $u \neq o$, for any x and $y \in \mathbf{Z}^2$,

$x\ \mathrm{R}_u\ y$ and $y\ \mathrm{R}_u\ x \Leftrightarrow \exists\, j, k \in \mathbf{N}$: $y = x + ju$ and $x = y + ku$

(definition of R_u)

$\Leftrightarrow \exists\, j, k \in \mathbf{N}$: $y = x + ju$ and $x = y + ku$ and $y = y + (j + k)u$

(by substitution and module property)

$\Leftrightarrow \exists\, j, k \in \mathbf{N}$: $y = x + ju$ and $x = y + ku$ and $(j + k)u = o$

(o is unitary element of +)

$\Leftrightarrow \exists\, j, k \in \mathbf{N}$: $y = x + ju$ and $x = y + ku$ and $j + k = 0$

($u \neq o$ and $\mathbf{Z}$ has no zero divisor)

$\Leftrightarrow y = x + ju$ and $x = y + ku$ and $j = k = 0$

(natural number property)

$\Leftrightarrow y = x$, (natural number property)

that is, R_u is antisymmetric. For any x, y and $z \in \mathbf{Z}^2$,

$x\ \mathrm{R}_u\ y$ and $y\ \mathrm{R}_u\ z \Leftrightarrow \exists\, j, k \in \mathbf{N}$: $y = x + ju$ and $z = y + ku$

(definition of R_u)

$\Leftrightarrow \exists\, j, k \in \mathbf{N}$: $z = x + (j + k)u$

(by substitution)

$\Leftrightarrow \exists\, i \in \mathbf{N}$: $z = x + iu$ (namely, $i = j + k$)

$\Leftrightarrow x\ \mathrm{R}_u\ z$, (definition of R_u)

that is, R_u is transitive.

Let u and v be two points in $\mathbf{Z}^2$, $u \neq v$ and let X be a finite subset of $\mathbf{Z}^2$, then X_u is finite as well. Therefore, by Theorem 3 of Birkhoff [4] or Theorem 3 of Szász [7], $\exists\, x \in X_u$ such that x is maximal with respect to R_{v-u}.

Let y be the point in $\mathbf{Z}^2$ such that $y = x + v - u$, then $x\ \mathrm{R}_{v-u}\ y$ (doing $j = 1$ in the definition of R_{v-u}), $y \in X_v$ (by translated version definition) and $y \neq x$ (by regularity of the addition on $\mathbf{Z}^2$). Since $u \neq v$ and x is maximal, $x\ \mathrm{R}_{v-u}\ z$ for no z in $X_u - \{x\}$, consequently y doesn't belong to X_u. In other words, there exists a point in X_v (namely y) which doesn't belong to X_u, that is, $X_v - X_u \neq \varnothing$ and $X_u \neq X_v$. Furthermore, for any u, $v \in \mathbf{Z}^2$,

$X_v - X_u \neq \emptyset \Leftrightarrow X_v \not\subset X_u$ $\qquad (B - A) = \emptyset \Leftrightarrow B \subset A)$

$\Leftrightarrow X_u \cap X_v \neq X_v.$ $\qquad (B \subset A \Leftrightarrow (A \cap B) = B)$

Finally, by changing the role of u and v, for any u and $v \in \mathbf{Z}^2$, $u \neq v$, $X_u - X_v \neq \emptyset$ and $X_v \cap X_u \neq X_u$. ♦

We go on recalling the definition of transpose of a subset.

Definition 21 (transpose) – The *transpose* of a subset X of $\mathbf{Z}^2$ is the subset $X^t = \{x \in \mathbf{Z}^2 : -x \in X\}$. ♦

A subset X of $\mathbf{Z}^2$ is *symmetric* (with respect to the origin o) iff it is equal to its transpose, that is $X = X^t$ or equivalently, iff $x \in X \Leftrightarrow -x \in X$. $\mathbf{Z}^2$ is an example of symmetric subset. We denote by $\boldsymbol{S}$ the sub collection of all the finite symmetric subsets of $\mathbf{Z}^2$, i.e., $\boldsymbol{S} = \{X \in \boldsymbol{P}(\mathbf{Z}^2)$: $\#X < \infty$ and $X = X^t\}$ and by $\boldsymbol{S}^+$ the sub collection $\boldsymbol{S} + \{\mathbf{Z}^2\}$.

From the operations of translation and transposition, we can build the sub collection of symmetric finite subsets and their translated versions, that we call balls. The symmetry assumption was made in order to establish, in Sections 6 and 7, the relationship with distance (which is a symmetric mapping).

Definition 22 (ball) – A subset X of $\mathbf{Z}^2$ is a *ball* iff $\exists\, u \in \mathbf{Z}^2$ such that $X_u \in \boldsymbol{S}^+$. ♦

By definition of $\boldsymbol{S}^+$, the set $\mathbf{Z}^2$ is a ball. Symmetric subsets are balls (doing $u = o$).

The 3 by 3 discrete square, denoted by B_8, consisting of the points (0, 0), (0, 1), (1, 1) (1, 0), (1, -1), (0, -1), (-1, -1), (-1, 0) (-1, 1) is ball since it is a symmetric subset.

Another important examples of balls (important from the theoretical point of view) are the singletons (the subsets containing just one point).

Proposition 23 (ball example) – Any singleton of $\mathbf{Z}^2$ is a ball. ♦

Proof

For any $x \in \mathbf{Z}^2$,

$(\{x\}_{-x})^t = \{x - x\}^t$ (Definition 19 - translated version)

$= \{o\}^t$ (group property)

$= \{-o\}$ (Definition 21 - transpose)

$= \{o\}$ (group property)

$= \{x - x\}$ (group property)

$= \{x\}_{-x}$, (Definition 19)

in other words, for any $x \in \mathbf{Z}^2$, $\exists\, u \in \mathbf{Z}^2$ (namely $u = -x$) such $(\{x\}_u)^t = \{x\}_u$, that is, by Definition 22, $\{x\}$ is a ball of $\mathbf{Z}^2$. ♦

We denote by $\boldsymbol{B}^+$ the sub collection of all the balls and by $\boldsymbol{B}$ the sub collection of finite balls, i.e., $\boldsymbol{B} = \boldsymbol{B}^+ - \{\mathbf{Z}^2\}$. As a consequence of these definitions, $\boldsymbol{S} \subset \boldsymbol{B} \subset \boldsymbol{B}^+ \subset \boldsymbol{P}(\mathbf{Z}^2)$.

The opposite $-u$ of the point u appearing in the ball definition is a center of symmetry for the ball, it is called its center.

Definition 24 (ball center) – Let X be a ball ($X \in \boldsymbol{B}^+$), if X is finite, $u \in \mathbf{Z}^2$ is the *center of* X iff $X_{-u} \in \boldsymbol{S}$, and if X is $\mathbf{Z}^2$, its center is the origin o. ♦

Hence, by Definition 22 (ball), a ball of $\mathbf{Z}^2$ has always a center, furthermore, if the ball is finite, this center is unique as shown in the next proposition. If the ball is $\mathbf{Z}^2$, by convention, we assume that its center is the origin o.

Proposition 25 (ball center construction) – Let X be a finite ball (i.e., $X \in \boldsymbol{B}$), u is the center of X iff $u = \sum_{x \in X} x / \# X$. ♦

Proof

Let R be a relation on $\mathbf{Z}^2$ given by $x \,\mathrm{R}\, y \Leftrightarrow y = x$ or $y = -x$. By group properties, R is an equivalent relation.

Let u_1 be a center of X and let $u_2 = \sum_{x \in X} x / \# X$.

Let us prove that $u_1 = u_2$.

$$
\begin{aligned}
u_2 - u_1 &= \left(\sum_{x \in X} x / \# X\right) - u_1 && \text{(def. of } u_2) \\
&= \left(\sum_{x \in X} x - u_1\right) / \# X && \text{(group property)} \\
&= \sum_{v \in X_{-u_1}} v / \# X && (v = x - u_1) \\
&= \sum_{v \in X_{-u_1} / \mathrm{R}} (v + (-v)) / \# X && (u_1 \text{ center of } X \Rightarrow X_{-u_1} \in \boldsymbol{S}) \\
&= o, && \text{(group properties)}
\end{aligned}
$$

that is, $u_1 = u_2$. ♦

We denote by center(X) the center of $X \in \boldsymbol{B}^+$. If the center of X is u, then we say that X is *with center at* u.

The center of a ball is not sufficient to characterize a ball, we need one more parameter that we call here the ball matrix.

Definition 26 (ball matrix) – Let X be a ball ($X \in \boldsymbol{B}^+$), the *matrix of* X, denoted by matrix(X), is the ball $X_{-\mathrm{center}(X)}$, that is, matrix(X) = $X_{-\mathrm{center}(X)}$. ♦

The ball matrix is a symmetric ball.

Proposition 27 (ball matrix property) – Let X be a ball (i.e., $X \in \boldsymbol{B}^+$), then matrix(X) $\in \boldsymbol{S}^+$. ♦

Proof

For any $X \in \boldsymbol{B}^+$,

$$
\begin{aligned}
(\mathrm{matrix}(X))^t &= (X_{-\mathrm{center}(X)})^t && \text{(ball matrix definition)} \\
&= X_{-\mathrm{center}(X)} && \text{(center definition)} \\
&= \mathrm{matrix}(X), && \text{(ball matrix definition)}
\end{aligned}
$$

that is, matrix(X) $\in \boldsymbol{S}^+$. ♦

The balls with center at origin o are symmetric and every finite symmetric subset is a ball with center at origin. This, and other properties of the balls with center at origin are shown in the next proposition.

Proposition 28 (properties of a ball with center at origin) – Let B be a ball (i.e., $B \in \boldsymbol{B}^+$), then the following statements are equivalent:
(i) for any $x, y \in \mathbf{Z}^2$, $x \in B_y \Leftrightarrow y \in B_x$,
(ii) $B \in \boldsymbol{S}^+$,
(iii) $\mathrm{center}(B) = o$,
(iv) $\mathrm{matrix}(B) = B$. ♦

Proof
(a) Let us prove (i) $\Rightarrow$ (ii). For any finite ball B (i.e., $B \in \boldsymbol{B}$) and any x and $y \in \mathbf{Z}^2$,
$x \in B_y \Leftrightarrow y \in B_x$ (hypothesis (i))
$\Leftrightarrow x \in (B^t)_y$, (Property (2) of Prop. 4.10 of [2])
that is, for $y = o$, $B^t = B$, in other words, $B \in \boldsymbol{S}$. Furthermore, for $B = \mathbf{Z}^2$, $B \in \boldsymbol{S}^+$ as a consequence of the definition of $\boldsymbol{S}^+$.
(b) Let us prove (i) $\Leftarrow$ (ii). For any symmetric ball B and any x and $y \in \mathbf{Z}^2$,
$x \in B_y \Leftrightarrow x - y \in B$ (Definition 19 - translated version)
$\Leftrightarrow y - x \in B$ (hypothesis (ii))
$\Leftrightarrow y \in B_x$. (Definition 19)
(c) Let us prove (ii) $\Leftrightarrow$ (iii). For any finite ball B,
$B \in \boldsymbol{S} \Leftrightarrow B = B^t$ (definition of $\boldsymbol{S}$)
$\Leftrightarrow B_o = (B_o)^t$ (Property (1) of Prop. 4.6 of [2])
$\Leftrightarrow \mathrm{center}(B) = o$. (Definition 24 - ball center)
For $B = \mathbf{Z}^2$, $B \in \boldsymbol{S}^+$ and $\mathrm{center}(B) = o$ are both true as a consequence of the definitions of $\boldsymbol{S}^+$ and center of a ball.
(d) Let us prove (iii) $\Rightarrow$ (iv). For any ball B,
$\mathrm{matrix}(B) = B_{-\mathrm{center}(B)}$ (Definition 26 - ball matrix)
$= B_{-o}$ (hypothesis (iii))
$= B_o$ (group property)
$= B$ (Property (1) of Prop. 4.6 of [2])
(e) Let us prove (iii) $\Leftarrow$ (iv). For any ball B,
$B_{-\mathrm{center}(B)} = \mathrm{matrix}(B)$ (Definition 26)
$= B$ (hypothesis (iv))
$= B_o$, (Property (1) of Prop. 4.6 of [2])
$= B_{-o}$ (group property)
that is, if B is finite, by Proposition 20 (translated version property), $\mathrm{center}(B) = o$, if $B = \mathbf{Z}^2$, by convention, $\mathrm{center}(B) = o$ as well. ♦

The sub collection of balls is closed under translation as we show in the next proposition.

Proposition 29 (ball translation) – Let X be a finite ball (i.e., $X \in \boldsymbol{B}$) and let $v \in \mathbf{Z}^2$, then X_v is a finite ball (i.e., $X_v \in \boldsymbol{B}$). Furthermore, $\mathrm{center}(X_v) = \mathrm{center}(X) + v$ and $\mathrm{matrix}(X_v) = \mathrm{matrix}(X)$. ♦

Proof
Let X be a finite ball and let $u = \mathrm{center}(X)$. For any $v \in \mathbf{Z}^2$,
$((X_v)_{-(u+v)})^t = (X_{-u})^t$ (Prop. 4.6 of [2])
$= X_{-u}$ (X is a ball and u is its center)
$= (X_v)_{-(u+v)}$, (Prop. 4.6 of [2])
in other words, for any $X \in \boldsymbol{B}$ and any $v \in \mathbf{Z}^2$, $\exists\, u' \in \mathbf{Z}^2$: $((X_v)_{u'})^t = (X_v)_{u'}$, namely $u' = -(\mathrm{center}(X) + v)$, that is,
$X_v \in \boldsymbol{B}$ and by Definition 24 (ball center), $\mathrm{center}(X_v) = \mathrm{center}(X) + v$.
Let us prove the last statement. For any $X \in B$ and any $v \in \mathbf{Z}^2$,
$\mathrm{matrix}(X_v) = (X_v)_{-\mathrm{center}(X_v)}$ (ball matrix definition)
$= X_{v - \mathrm{center}(X_v)}$ (Property (2) of Prop. 4.6 of [2])
$= X_{v - \mathrm{center}(X) - v}$ (first statement)
$= X_{-\mathrm{center}(X)}$ (group property)
$= \mathrm{matrix}(X)$. (ball matrix definition)
♦

Their center and matrix characterize the finite balls.

Proposition 30 (ball characterization) – The mapping from $\boldsymbol{B}$ to $\mathbf{Z}^2 \times \boldsymbol{S}$, $X \mapsto (\mathrm{center}(X), \mathrm{matrix}(X))$ is a bijection, its inverse is $(u, B) \mapsto B_u$. ♦

Proof
Let us divide the proof in four parts.
(a) By definition of center, for any $X \in \boldsymbol{B}$, $\mathrm{center}(X) \in \mathbf{Z}^2$. Furthermore, by Proposition 27 (ball matrix property), for any $X \in \boldsymbol{B}$, $\mathrm{matrix}(X) \in \boldsymbol{S}$.
(b) Let us prove that for any $u \in \mathbf{Z}^2$ and any $B \in \boldsymbol{S}$, $B_u \in \boldsymbol{B}$. For any $u \in \mathbf{Z}^2$ and any $B \in \boldsymbol{S}$, let $v = -u$, then,
$((B_u)_v)^t = (B_{u+v})^t$ (Property (2) of Prop. 4.6 of [2])
$= (B_o)^t$ ($v = -u$ and inverse definition)
$= B^t$ (Property (1) of Prop. 4.6 of [2])
$= B$ ($B \in \boldsymbol{S}$)
$= (B_u)_v$, (using the same arguments as above)
that is, there exists a point v in $\mathbf{Z}^2$ such that $((B_u)_v)^t = (B_u)_v$.
(c) Let us prove that $(u, B) \mapsto B_u$ is a left inverse for $X \mapsto (\mathrm{center}(X), \mathrm{matrix}(X))$. For any $X \in \boldsymbol{B}$,
$\mathrm{matrix}(X)_{\mathrm{center}(X)} = (X_{-\mathrm{center}(X)})_{\mathrm{center}(X)}$
(Definition 26 - ball matrix)
$= X_{(-\mathrm{center}(X)) + \mathrm{center}(X)}$
(Property (2) of Prop. 4.6 of [2])
$= X_o$ (inverse definition)
$= X$. (Property (1) of Prop. 4.6 of [2])
(d) Let us prove that $(u, B) \mapsto B_u$ is a right inverse for $X \mapsto (\mathrm{center}(X), \mathrm{matrix}(X))$. For any $u \in \mathbf{Z}^2$ and any $B \in \boldsymbol{S}$,
$(\mathrm{center}(B_u), \mathrm{matrix}(B_u)) = (\mathrm{center}(B) + u, \mathrm{matrix}(B))$
(Proposition 29 - ball translation)
$= (o + u, B)$
($B \in \boldsymbol{S}$ implies (iii) and (iv) of Proposition 28 - properties of a ball with center at origin)
$= (u, B)$. (group property)
Therefore, from (a) - (d), $X \mapsto (\mathrm{center}(X), \mathrm{matrix}(X))$ has an inverse (which is $(u, B) \mapsto B_u$), consequently it is a bijection.
♦

In order to combine geometrically the balls, we recall the Minkowski addition and subtraction definitions [2], [3], [6].

Definition 31 (Minkowski addition and subtraction) – Let A and B be two subsets of $\mathbf{Z}^2$, their Minkowski sum is the subset $A \oplus B = \{a + b: a \in A, b \in B\}$ and their Minkowski difference is the subset $A \ominus B = \{x \in \mathbf{Z}^2: \forall b \in B, \exists a \in A: x = a - b\}$. $(A, B) \mapsto A \oplus B$ is the Minkowski addition and $(A, B) \mapsto A \ominus B$ is the Minkowski subtraction from $\boldsymbol{P}(\mathbf{Z}^2) \times \boldsymbol{P}(\mathbf{Z}^2)$ to $\boldsymbol{P}(\mathbf{Z}^2)$. ♦

The sub collection of balls with center at origin is closed under Minkowski's addition as we show in the next proposition.

Proposition 32 (symmetric ball addition) – Let X and Y be two symmetric balls (i.e., $X, Y \in \boldsymbol{S}^+$), then $X \oplus Y$ is a symmetric ball (i.e., $X \oplus Y \in \boldsymbol{S}^+$). ♦

Proof

For any X and $Y \in \boldsymbol{S}^+$,

$(X \oplus Y)^t = X^t \oplus Y^t$ (Minkowski's addition property)

$= X \oplus Y$, (X and $Y \in \boldsymbol{S}$)

that is, $X \oplus Y \in \boldsymbol{S}^+$. ♦

As a consequence of the above symmetric ball addition result, in the next proposition we show that the sub collection of balls is closed under Minkowski's addition as well.

Proposition 33 (ball addition) – Let X and Y be two finite balls (i.e., X and $Y \in \boldsymbol{B}$), then $X \oplus Y$ is a ball (i.e., $X \oplus Y \in \boldsymbol{B}$), furthermore, center$(X \oplus Y)$ = center(X) + center(Y) and matrix$(X \oplus Y)$ = matrix$(X) \oplus$ matrix(Y). ♦

Proof

Let divide the proof into four parts.

(a) For any X and $Y \in \boldsymbol{B}$,

$X \oplus Y = \text{matrix}(X)_{\text{center}(X)} \oplus \text{matrix}(Y)_{\text{center}(Y)}$

(Proposition 30 - ball characterization)

$= (\text{matrix}(X)_{\text{center}(X)} \oplus \text{matrix}(Y))_{\text{center}(Y)}$

(Property (5) of Prop. 4.12 of [2])

$= ((\text{matrix}(X) \oplus \text{matrix}(Y))_{\text{center}(X)})_{\text{center}(Y)}$

(Property (5) of Prop. 4.12 of [2])

$= (\text{matrix}(X) \oplus \text{matrix}(Y))_{\text{center}(X)+\text{center}(Y)}$.

(Property (2) of Prop. 4.6 of [2])

(b) For any $X, Y \in \boldsymbol{B}$, by Proposition 27 (ball matrix property) and by Proposition 32 (symmetric ball addition) matrix$(X) \oplus$ matrix$(Y) \in \boldsymbol{S}$, therefore, by Part (a) of the proof, it exists $u \in \mathbf{Z}^2$ (namely, $u = -$ (center(X) + center(Y)) such that $(X \oplus Y)_u \in \boldsymbol{S}$. That is, by Definition 22 (ball definition), $X \oplus Y \in \boldsymbol{B}$.

(c) For any $X, Y \in \boldsymbol{B}$,

center$(X \oplus Y)$

$= \text{center}((\text{matrix}(X) \oplus \text{matrix}(Y))_{\text{center}(X)+\text{center}(Y)})$

(Part (a) of the proof)

$= \text{center}(\text{matrix}(X) \oplus \text{matrix}(Y)) + \text{center}(X) + \text{center}(Y)$

(Proposition 29 - ball translation)

$= o + \text{center}(X) + \text{center}(Y)$

(matrix$(X) \oplus$ matrix$(Y) \in \boldsymbol{S}$, therefore, by Proposition 28 (properties of a ball with center at origin), $\text{center}(\text{matrix}(X) \oplus \text{matrix}(Y)) = o$)

$= \text{center}(X) + \text{center}(Y)$. (group property)

(d) For any X and $Y \in \boldsymbol{B}$,

matrix$(X \oplus Y)$

$= \text{matrix}((\text{matrix}(X) \oplus \text{matrix}(Y))_{\text{center}(X)+\text{center}(Y)})$

(Part (a) of the proof)

$= \text{matrix}(\text{matrix}(X) \oplus \text{matrix}(Y))$

(Proposition 29 - ball translation)

$= \text{matrix}(X) \oplus \text{matrix}(Y)$.

(as in (c) above, by Proposition 28)

♦

Before continuing our study about the ball collection, we need to introduce the Minkowski product.

4. Minkowski product

Within the collection of balls we can identify sub collections in which the balls are mutually related through an external binary operation between a natural number and a subset, that we call the Minkowski product.

We denote by $\mathbf{N}^+$ the set of extended natural numbers (i.e., the natural numbers plus an element denoted ∞) with the usual addition extended in such a way that, for any $j \in \mathbf{N}^+$, $j + \infty = \infty + j = \infty$ and with the usual order extended in such a way that, for any $j \in \mathbf{N}^+$, $j \leq \infty$.

Definition 34 (Minkowski product) – Let $B \in P(\mathbf{Z}^2)$ such that $B \neq \varnothing$, and let $j \in \mathbf{N}^+$, the *Minkowski product of B by j* is the subset jB of $\mathbf{Z}^2$ given by

$$jB = \begin{cases} \{o\} & \text{if} \quad j = 0 \\ B & \text{if} \quad j = 1 \\ ((j-1)B) \oplus B & \text{if} \quad 1 < j < \infty \\ \mathbf{Z}^2 & \text{if} \quad j = \infty \end{cases}.$$ ♦

The Minkowski product jB should be distinguished from the result of the usual scaling operation of B by j used in the continuous case.

The scaling transform of a subset B by a real number j is the subset denoted scaling(j, B) and given by scaling$(j, B) = \{jx: x \in B\}$.

In the discrete case, both results, jB and scaling(j, B), may be different as it can be verified when B is the 3 by 3 discrete square B_8. The difference goes on even if we try to preserve the "interior" of the square using a slightly modified version of the scaling transform given by scaling2$(j, B) = \{ix: x \in B$ and $i \in [0, j]\}$.

Figure 5 shows (from left to right and top to bottom) the B_8 square, its transformation when applying the above two scaling transformations and at last the Minkowski product, with $j = 2$. We observe that these expansion definitions lead to different results.

In the discrete domain $\mathbf{Z}^2$, the Minkowski product is more appropriate for our purpose than the scaling operation (see Section 7).

We now verify the mix distributivity of this product and then some other elementary properties. Based on the mix distributivity, we could establish in Section 5 Proposition 44 and Proposition 48 about the intersection and inclusion of generated balls.

Proposition 35 (mix distributivity of the Minkowski product) – Let $B \in \boldsymbol{P}(\mathbf{Z}^2)$ such that $B \neq \emptyset$, for any i and $j \in \mathbf{N}^+$, $(i + j)B = (iB) \oplus (jB)$. ♦

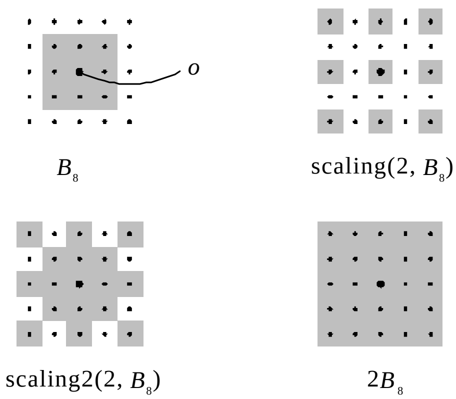

Figure 5 - Minkowski product versus scaling

Proof

Let us divide the proof in three parts.

(a) Let us prove that for any $i \in \mathbf{N}$, $(i + 1)B = (iB) \oplus B$. If $i = 0$,

$(i + 1)B = 1B$ (hypothesis)

$= B$ (Definition 34 - Minkowski's)

$= \{o\} \oplus B$ (Property (4) of Prop. 4.12 of [2])

$= (0B) \oplus B$ (Definition 34)

$= (iB) \oplus B$ (hypothesis)

If $i > 0$,

$(i + 1)B = (((i + 1) - 1)B) \oplus B$ (product definition)

$= (iB) \oplus B.$ (natural number property)

(b) Let us prove the proposition statement for any i and $j \in \mathbf{N}$. If $j = 0$,

$(i + j)B = iB$ (hypothesis)

$= (iB) \oplus \{o\}$ (Property (4) of Prop. 4.12 of [2])

$= (iB) \oplus (jB)$; (hypothesis)

if $j = 1$,

$(i + j)B = (i + 1)B$ (hypothesis)

$= (iB) \oplus B$ (Part (a) of the proof)

$= (iB) \oplus (1B)$ (Definition 34)

$= (iB) \oplus (jB)$; (hypothesis)

if $j > 1$,

$(i + j)B = ((i + j - 1) + 1)B$ (natural number property)

$= ((i + j - 1)B) \oplus B$ (Part (a) of the proof)

$= ((iB) \oplus ((j - 1)B)) \oplus B$ (is true for $j - 1$)

$= (iB) \oplus (((j - 1)B) \oplus B)$ (associativity of $\oplus$)

$= (iB) \oplus (jB).$ (Part (a) of the proof)

(c) Let us prove the proposition statement for any $i \in \mathbf{N}^+$ and $j = \infty$:

$(i + j)B = (i + \infty)B$ (hypothesis)

$= \infty B$ (assumption on the extended addition)

$= \mathbf{Z}^2$ (Definition 34)

$= iB \oplus \mathbf{Z}^2$ ($iB \neq \emptyset$,)

$= (iB) \oplus (\infty B)$ (Definition 34)

$= (iB) \oplus (jB).$ (hypothesis)

The same result can be obtained by changing the hypotheses on i and j.

That is, from (b) and (c), the product is mix distributive. ♦

For convenience, let us assume that $\# \mathbf{Z}^2 = \infty$.

Proposition 36 (elementary properties of the Minkowski product) – Let $B \in \boldsymbol{P}(\mathbf{Z}^2)$ such that $B \neq \emptyset$, for any $i \in \mathbf{N}$ and any $j \in \mathbf{N}^+$,

(i) if $o \in B$ then $o \in jB$;

(ii) if $o \in B$ then the mapping $j \mapsto jB$ from $\mathbf{N}^+$ to $\boldsymbol{P}(\mathbf{Z}^2)$ is increasing;

(iii) if $j \neq 0$ and $\#B > 1$ then $\#(jB) > 1$;

(iv) $\#(iB) \leq (\#B)^i$;

(v) if $B \in \boldsymbol{S}$ then $jB \in \boldsymbol{S}^+$. ♦

Proof

(a) Let us prove Property (i).

By Definition 34 (Minkowski product), $o \in jB$ is true for $j = 0$, 1 and ∞. Let us assume that it was true for $j - 1$ ($1 < j < \infty$),

$o \in (j - 1)B$ (hypothesis)

$\subset ((j - 1)B) \oplus B$

($o \in B$ and Property (6) of Prop. 4.12 of [2])

$= jB,$ (Definition 34)

that is, by inclusion definition, $o \in jB$.

(b) Let us prove Property (ii).

Let i and $j \in \mathbf{N}$, such that $i \leq j$,

$iB \subset iB \oplus (j - i)B$

($o \in (j - i)B$ (Property. (i)) and Property (6) of Prop. 4.12 of [2])

$= jB.$ ($j = i + (j - i)$ and Proposition 35)

Let $i \in \mathbf{N}^+$ and let $j = \infty$, then $i \leq j$, furthermore,

$iB \subset \mathbf{Z}^2$ (Definition 34)

$= \infty B$ (Definition 34)

$= jB.$ (hypothesis)

That is the mapping $j \mapsto jB$ from $\mathbf{N}^+$ to $\boldsymbol{P}(\mathbf{Z}^2)$ is increasing.

(c) Let us prove Property (iii).

By hypothesis, $\#(jB) > 1$ is true for $j = 1$ and it is always true for $j = \infty$. Let us assume that it was true for $j - 1$ ($1 < j < \infty$), and let $x \in (j - 1)B$,

$\#(jB) = \#(((j - 1)B) \oplus B)$ (Definition 34)

$\geq \#(\{x\} \oplus B)$

(Minkowski's addition is increasing (Property (6) of Exerc. 4.9 of [2]) and cardinality property)

$= \#(B_x)$ (Property (1) of Prop. 4.12 of [2])

$= \#B$ ($X \mapsto X_x$ is a bijection)

$> 1.$ (hypothesis)

(d) Let us prove Property (iv).

$\#(iB) \le (\#B)^i$ is true for $i = 1$. Let us assume that it was true for $i - 1$ $(1 < i)$,

$\#(iB) = \#(((i-1)B) \oplus B)$ (Definition 34)

$= \# \bigcup_{u \in B} ((i-1)B)_u$

(equivalent definition of $\oplus$ (Property (1) of Prop. 4.12 of [2]))

$\le (\#B)(\#(i-1)B)$ (union property)

$= (\#B)(\#(B))^{i-1}$ (hypothesis)

$= (\#(B))^i$. (arithmetic property)

(e) Let us prove Property (v).

Let $B = B^t$, by Minkowski product definition, $jB = (jB)^t$ is true for $j = 0$ (since $\{o\}$ is symmetric), 1 and ∞ (since $\mathbf{Z}^2$ is symmetric as well). Let assume that it was true for $j - 1$ $(1 < j < \infty)$,

$(jB)^t = (((j-1)B) \oplus B)^t$ (Definition 34)

$= ((j-1)B) \oplus B$

(hypotheses and Proposition 32 (symmetric ball addition))

$= jB$. (Definition 34)

♦

Property (v) says that if a subset is symmetric then its product by any natural number is still symmetric.

Next proposition about the regularity of the Minkowski product will be useful for the definition of radius of a ball in the next section.

Proposition 37 (regularity of the Minkowski product) – Let $B \in \boldsymbol{P}(\mathbf{Z}^2)$, such that B is finite and $\#B > 1$, then the mapping $j \mapsto \#(jB)$ from $\mathbf{N}^+$ to $\mathbf{N}^+$, is injective and increasing. Furthermore, under the above assumption on B, the mapping $j \mapsto jB$ from $\mathbf{N}^+$ to $\boldsymbol{P}(\mathbf{Z}^2)$ is injective (i.e. $iB = jB \Rightarrow i = j$). ♦

Proof

Let i and $j \in \mathbf{N}$, such that $i < j$,

$jB = iB \oplus (j-i)B$

($j = i + (j - i)$ and Proposition 35 - mix distributivity of the Minkowski product)

$= \bigcup_{u \in (j-i)B} (iB)_u$.

(equivalent definition of $\oplus$ (Property (1) of Prop. 4.12 of [2]))

By Property (iii) of Proposition 36, if $i < j$, $\#((j-i)B) > 1$.

Let u_1 and $u_2 \in (j-i)B$,

$(iB)_{u_1} + ((iB)_{u_2} - (iB)_{u_1}) = (iB)_{u_1} \cup (iB)_{u_2}$

(subset addition definition)

$\subset jB$. (see above equality)

Hence, if $i < j$,

$\#(iB) < \#(iB) + \#((iB)_{u_2} - (iB)_{u_1})$

$(iB)_{u_2} - (iB)_{u_1} \neq \varnothing$ by Proposition 20 - translated version property)

$= \#(iB)_{u_1} + \#((iB)_{u_2} - (iB)_{u_1})$

($X \mapsto X_u$ is a bijection)

$= \#((iB)_{u_1} + ((iB)_{u_2} - (iB)_{u_1}))$

(property of the union of disjoint subsets)

$\le \#(jB)$. (see above inclusion)

Let $i \in \mathbf{N}$ and let $j = \infty$, then $i < j$, furthermore,

$\#(iB) \le (\#B)^i$ (by Property (iv) of Proposition 36)

$< \infty$ (B is finite)

$= \# \mathbf{Z}^2$ (by convention)

$= \#(\infty B)$ (Definition 34 - Minkowski product)

$= \#(jB)$. (hypothesis)

Furthermore, for any i and $j \in \mathbf{N}^+$

$i \neq j \Leftrightarrow i < j$ or $j < i$ ($\mathbf{N}^+$ is a chain)

$\Rightarrow \#(iB) < \#(jB)$ or $\#(jB) < \#(iB)$

(the previous conclusion)

$\Leftrightarrow \#(iB) \neq \#(jB)$ ($\mathbf{N}^+$ is a chain)

$\Rightarrow iB \neq jB$. (iB and jB have not the same elements)

That is, the mapping $j \mapsto \#(jB)$ from $\mathbf{N}^+$ to $\mathbf{N}^+$, is injective and increasing, and the mapping $j \mapsto jB$ from $\mathbf{N}^+$ to $\boldsymbol{P}(\mathbf{Z}^2)$, is injective. ♦

Based on the Minkowski product, we define, in the next section, the notions of generated balls and of ball radius. In Section 7, the ball radius will be used to derive a metric from a symmetric ball.

5. Generated balls

Let A and B be two subsets of $\mathbf{Z}^2$. If $\#B > 1$ and B is finite, then we say that A is a *multiple of* B iff there exists an element $j \in \mathbf{N}^+$ such that $A = jB$. In this case, we say that the set B *divide* A and we call the natural number j (which is unique under the restrictions on B) the *quotient* of A by B and we denote it $\frac{A}{B}$. The uniqueness of quotient is a consequence of Proposition 37 (regularity of the Minkowski product): let i and j be two quotients of A by B, by definition of the quotient, $iB = A$ and $jB = A$, that is, $iB = jB$, therefore, since $j \mapsto jB$ is injective (Proposition 37), $i = j$.

With a finite symmetric ball B we can associate, through the Minkowski product, a sub collection of balls. We say that B induces a sub collection or that the sub collection is generated by B.

Definition 38 (sub collection generated by a ball) – Let B be a finite symmetric ball (i.e., $B \in \boldsymbol{S}$), such that $B \neq \{o\}$. The *sub collection generated by* B, denoted by $\boldsymbol{B}_B$, is the set of all the balls whose matrices are multiple of B, that is, $\boldsymbol{B}_B = \{X \in \boldsymbol{B}^+ : \exists j \in \mathbf{N}^+, \text{matrix}(X) = jB\}$. ♦

For any $j \in \mathbf{N}$ and any $B \in \boldsymbol{S}$, by Property (v) of Proposition 36 (elementary properties of the Minkowski product) and by Proposition 28 (property of a ball with center at origin), matrix$(jB) = jB$, that is $jB \in \boldsymbol{B}_B$, in other words, $\boldsymbol{B}_B$ is never empty, moreover, it always contains $\mathbf{Z}^2$. We call the elements of $\boldsymbol{B}_B$ *generated balls* and we say that they are generated from the *prototype ball B*.

The assumption that $\mathbf{Z}^2$ was a ball is convenient in the sense that, in this way, every point in $\mathbf{Z}^2$ is contained in at least one generated ball.

Proposition 39 (example of generated balls) – Let B be a finite symmetric ball (i.e., $B \in \boldsymbol{S}$), such that $B \neq \{o\}$. Any singleton of $\mathbf{Z}^2$ belongs to $\boldsymbol{B}_B$ (the sub collection generated by B). ♦

Proof

For any $x \in \mathbf{Z}^2$, by Proposition 23 (ball example) $\{x\} \in \boldsymbol{B}^+$. Furthermore, for any $x \in \mathbf{Z}^2$,

$\text{matrix}(\{x\}) = \text{matrix}(\{o\}_x)$ (Definition 19 - translated version)

$= \text{matrix}(\{o\})$ (Proposition 29 - ball translation)

$= \{o\}$ (Proposition 28 (property of a ball with center at origin), since $\{o\} \in \boldsymbol{S}^+$)

$= 0B$, (Definition 34 - Minkowski product)

that is, $\exists\, j \in \mathbf{N}^+$, $\text{matrix}(\{x\}) = jB$ (namely, $j = 0$).

Consequently, by Definition 38 (sub collection generated by a ball), for any $x \in \mathbf{Z}^2$, $\{x\} \in \boldsymbol{B}_B$. ♦

As a direct consequence of the definition of a generated ball, we can parameterize its matrix by a natural number that we call its radius.

Definition 40 (radius of a generated ball) – Let B be a finite symmetric ball (i.e., $B \in \boldsymbol{S}$), such that $B \neq \{o\}$. The *radius* of $X \in \boldsymbol{B}_B$, denoted $\text{radius}_B(X)$ is the quotient of $\text{matrix}(X)$ by B, that is, $\text{radius}_B(X) = \dfrac{\text{matrix}(X)}{B}$. ♦

The uniqueness of the quotient guarantees the uniqueness of the radius.

We observe that the same ball generated from different ball prototypes may have different radius. A ball prototype plays the role of a unit ball.

We are now ready to show that the generated balls can be completely characterized in terms of their centers and their radius.

Proposition 41 (characterization of the generated balls) – Let B be a finite symmetric ball (i.e., $B \in \boldsymbol{S}$), such that $B \neq \{o\}$. The mapping from $\boldsymbol{B}_B - \{\mathbf{Z}^2\}$ to $\mathbf{Z}^2 \times \mathbf{N}$ $X \mapsto (\text{center}(X), \text{radius}_B(X))$ is a bijection and its inverse is $(x, j) \mapsto (jB)_x$. Furthermore, for any $X \in \boldsymbol{B}_B$, $X = (\text{radius}_B(X)B)_{\text{center(X)}}$, and for any $(x, j) \in \mathbf{Z}^2 \times \mathbf{N}$, $(x, j) = (\text{center}((jB)_x), \text{radius}_B((jB)_x))$. ♦

Proof

Let us divide the proof in four parts.

(a) For any $X \in \boldsymbol{B}_B - \{\mathbf{Z}^2\}$, by Definition 24 (ball center), $\text{center}(X) \in \mathbf{Z}^2$ and by Definition 40 (radius of generated ball), $\text{radius}_B(X) \in \mathbf{N}$.

(b) For any $x \in \mathbf{Z}^2$ and any $j \in \mathbf{N}$,

$\text{matrix}((jB)_x) = \text{matrix}(jB)$ (Proposition 29 - ball translation)

$= jB$, (Property (v) of Proposition 36 (elementary properties of the Minkowski product) and Proposition 28 (property of a ball with center at origin))

that is, by Proposition 37 (regularity of the Minkowski product) $\exists\, i \in \mathbf{N}$, $\text{matrix}((jB)_x) = iB$, namely $i = j$, and by Definition 38 (sub collection generated by a ball) $(jB)_x \in \boldsymbol{B}_B - \{\mathbf{Z}^2\}$.

(c) Let us prove that $(x, j) \mapsto (jB)_x$ is a left inverse for $X \mapsto (\text{center}(X), \text{radius}_B(X))$. For any $X \in \boldsymbol{B}_B$ and any $x \in \mathbf{Z}^2$,

$x \in ((\text{radius}_B(X))B)_{\text{center}(X)}$

$\Leftrightarrow x \in \left(\left(\dfrac{\text{matrix}(X)}{B}\right)B\right)_{\text{center}(X)}$ (Definition 40)

$\Leftrightarrow x \in \left(\left(\dfrac{X_{-\text{center}(X)}}{B}\right)B\right)_{\text{center}(X)}$ (Definition 26 - ball matrix)

$\Leftrightarrow x \in (X_{-\text{center}(X)})_{\text{center}(X)}$ (quotient property)

$\Leftrightarrow x \in X_{(-\text{center}(X))+\text{center}(X)}$ (translated version property)

$\Leftrightarrow x \in X$, (group property)

that is, $(x, j) \mapsto (jB)_x$ is a left inverse.

(d) Let us prove that the $(x, j) \mapsto (jB)_x$ is a right inverse for $X \mapsto (\text{center}(X), \text{radius}_B(X))$. For any $x \in \mathbf{Z}^2$ and any $j \in \mathbf{N}$,

$\text{center}((jB)_x) = \text{center}(jB) + x$ (Proposition 29 - ball translation)

$= o + x$ (Property (v) of Proposition 36 and Proposition 28 (properties of a ball with center at origin))

$= x$. (group property)

For any $x \in \mathbf{Z}^2$ and any $j \in \mathbf{N}$,

$\text{radius}_B((jB)_x) = \dfrac{((jB)_x)_{-\text{center}((jB)_x)}}{B}$ ((Definition 40)

$= \dfrac{((jB)_x)_{-x}}{B}$ (above result)

$= \dfrac{(jB)_{x-x}}{B}$ (translated version property)

$= \dfrac{jB}{B}$, (group property)

in other words, by quotient definition, $\text{radius}_B((jB)_x)B = jB$, therefore, by uniqueness of the quotient (consequence of Proposition 37), $\text{radius}_B((jB)_x) = j$. That is, $(x, j) \mapsto (jB)_x$ is a right inverse.

Therefore, from (a) - (d), the mapping $X \mapsto (\text{center}(X), \text{radius}_B(X))$ has an inverse (which is $(x, j) \mapsto (jB)_x$) and consequently is a bijection. Furthermore, from (c), for any $X \in \boldsymbol{B}_B$, $X = (\text{radius}_B(X)B)_{\text{center(X)}}$, and from (d), for any $(x, j) \in \mathbf{Z}^2 \times \mathbf{N}$, $(x, j) = (\text{center}((jB)_x), \text{radius}_B((jB)_x))$. ♦

The radius inherits some properties of the Minkowski's addition.

Proposition 42 (radius properties) – Let B be a finite symmetric ball (i.e., $B \in \boldsymbol{S}$), such that $B \neq \{o\}$, then

(i) $\text{radius}_B(\{o\}) = 0$

(ii) $\text{radius}_B(B) = 1$

(iii) for any $X, Y \in \boldsymbol{B}_B - \{\mathbf{Z}^2\}$, $\text{radius}_B(X \oplus Y) = \text{radius}_B(X) + \text{radius}_B(Y)$. ♦

Proof

(a) Let us prove (i).

$\text{radius}_B(\{o\}) = \text{radius}_B(0B)$ (Definition 34 - Minkowski product)

$= 0.$ (Proposition 41 - characterization of the generated balls)

(b) Let us prove (ii).

$\text{radius}_B(B) = \text{radius}_B(1B)$ (Definition 34)

$= 1.$ (Proposition 41)

(c) Let us prove (iii). For any $X, Y \in \boldsymbol{B}_B - \{\mathbf{Z}^2\}$,

$$\text{radius}_B(X \oplus Y) = \frac{(X \oplus Y)_{-\text{center}(X\oplus Y)}}{B} \quad \text{(Definition 40)}$$

$$= \frac{((\text{radius}_B(X)B)_{\text{center}(X)} \oplus (\text{radius}_B(Y)B)_{\text{center}(Y)})_{-\text{center}(X\oplus Y)}}{B}$$

(Proposition 41)

$$= \frac{\text{radius}_B(X)B \oplus \text{radius}_B(Y)B}{B}$$

(Property (5) of Prop. 4.12 of [2] and center($X \oplus Y$) = center(X) + center(Y) by Proposition 33 - ball addition)

$$= \frac{(\text{radius}_B(X) \oplus \text{radius}_B(Y))B}{B},$$

(Proposition 35 - mix distributivity of the Minkowski product)

in other words, by quotient definition, $\text{radius}_B(X \oplus Y)B = (\text{radius}_B(X) + \text{radius}_B(Y))B$, therefore, by uniqueness of the quotient (consequence of Proposition 37), $\text{radius}_B(X \oplus Y) = \text{radius}_B(X) + \text{radius}_B(Y)$. ♦

Within the poset ($\boldsymbol{B}_B$, $\subset$) generated by a ball B, we can identify some useful chains: the chains of generated balls with center at given points.

Let B be a finite symmetric ball (i.e., $B \in \boldsymbol{S}$), such that $B \neq \{o\}$, and let $\boldsymbol{B}_B(x)$ be the sub collection of $\boldsymbol{B}_B$ consisting of all the balls with center at a given point x of $\mathbf{Z}^2$, that is, $\boldsymbol{B}_B(x) = \{X \in \boldsymbol{B}_B : \text{center}(X) = x\}$.

In particular, $\boldsymbol{B}_B(o) = \{jB\text{: } j \in \mathbf{N}^+\}$ (by Property (iv) of Proposition 28 - properties of the generated balls with center at origin).

Proposition 43 (chain of generated balls with center at a given point) – Let B be a finite symmetric ball (i.e., $B \in \boldsymbol{S}$), such that $o \in B$ and $B \neq \{o\}$. Then the mapping $j \mapsto (jB)_x$ from ($\mathbf{N}^+$, $\leq$) to ($\boldsymbol{B}_B(o)$, $\subset$) when $x = o$, and from ($\mathbf{N}$, $\leq$) to ($\boldsymbol{B}_B(x)$, $\subset$) when $x \neq o$, is a poset isomorphism and the sub collection ($\boldsymbol{B}_B(x)$, $\subset$) is a chain. Its inverse is $X \mapsto \text{radius}_B(X)$. The sub collection $\boldsymbol{B}_B(x)$ has a smaller element which is $\{x\}$ and a greater one which is $\mathbf{Z}^2$ when $x = o$. ♦

Proof

Let B be a finite symmetric ball (i.e., $B \in \boldsymbol{S}$), such that $o \in B$ and $B \neq \{o\}$, and let $x \in \mathbf{Z}^2$. We observe that $o \in B$ and $B \neq \{o\} \Rightarrow \#B > 1$. For any j_1 and $j_2 \in \mathbf{N}^+$,

$j_1 \neq j_2 \Rightarrow j_1B \neq j_2B$

(Proposition 37 – regularity of the Minkowski product)

$\Leftrightarrow (j_1B)_x \neq (j_2B)_x$, (translation is injective)

that is, the mapping $j \mapsto (jB)_x$ from $\mathbf{N}^+$ to $\boldsymbol{B}_B(o)$ when $x = o$, and from $\mathbf{N}$ to $\boldsymbol{B}_B(x)$ when $x \neq o$, is injective. By definition of $\boldsymbol{B}_B(x)$, it is surjective, that is it a bijection. From Proposition 41 (characterization of the generated balls), its inverse is $X \mapsto \text{radius}_B(X)$.

Furthermore, for any $j_1, j_2 \in \mathbf{N}^+$,

$j_1 \leq j_2 \Rightarrow j_1B \subset j_2B$

($j \mapsto jB$ is increasing (Property (ii) of Proposition 36))

$\Rightarrow (j_1B)_x \subset (j_2B)_x$, (translation is increasing)

that is, the mapping $j \mapsto (jB)_x$ from $\mathbf{N}^+$ to $\boldsymbol{B}_B(o)$ when $x = o$, and from $\mathbf{N}$ to $\boldsymbol{B}_B(x)$ when $x \neq o$, is increasing.

Conversely, let B_1 and B_2 be two balls in $\boldsymbol{B}_B$ with center at x, and let j_1 and j_2 their respective radius,

$j_2 \leq j_1$ and $j_1 \neq j_2 \Rightarrow B_2 \subset B_1$ and $B_1 \neq B_2$

($j \mapsto (jB)_x$ is injective and increasing)

$\Leftrightarrow B_2 \subset B_1$ and ($B_1 \not\subset B_2$ or $B_2 \not\subset B_1$)

(anti-symmetry of inclusion)

$\Leftrightarrow B_2 \subset B_1$ and $B_1 \not\subset B_2$ (logical derivation)

$\Rightarrow B_1 \not\subset B_2$, (logical derivation)

in other words,

$B_1 \subset B_2 \Rightarrow j_1 < j_2$ or $j_1 = j_2$ (above result)

$\Leftrightarrow j_1 \leq j_2$, (($\mathbf{N}^+$, $\leq$) is a chain)

this proves that the mapping $j \mapsto (jB)_x$ is two-sided increasing.

Consequently, the mapping $j \mapsto (jB)_x$ is a poset isomorphism, and the sub collection ($\boldsymbol{B}_B(x)$, $\subset$) is isomorphic to the chain ($\mathbf{N}^+$, $\leq$). That is, this sub collection is a chain as well. The smaller element (0) of $\mathbf{N}^+$ maps to $\{x\}$ which is therefore the smaller element of $\boldsymbol{B}_B(x)$, and the greater element (∞) of $\mathbf{N}^+$ maps to $\mathbf{Z}^2$ which is therefore the greater element of $\boldsymbol{B}_B(x)$. ♦

Based on the mixt distributivity of the Minkowski product, we can establish the following geometrical property. This proposition shows that the intersection between generated balls occurs under the same condition as for the balls derived from an Euclidean distance on the continuous plane for example.

Proposition 44 (generated balls versus intersection) – Let B be a finite symmetric ball (i.e., $B \in \boldsymbol{S}$), for any points x and y in $\mathbf{Z}^2$ and any numbers i and j in $\mathbf{N}^+$, $y \in ((i + j)B)_x \Leftrightarrow (iB)_x \cap (jB)_y \neq \varnothing$. ♦

Proof

Let x and $y \in \mathbf{Z}^2$ and let i and $j \in \mathbf{N}^+$,

$y \in ((i + j)B)_x \Leftrightarrow y \in ((iB) \oplus (jB))_x$

(Proposition 35 - mix distributivity of the product)

$\Leftrightarrow y \in (iB)_x \oplus (jB)$

(translation invariance of $\oplus$ (Property (5) of Prop. 4.12 of [2]))

$$\Leftrightarrow y \in \bigcup_{z \in (iB)_x} (jB)_z$$

(equivalent definition of $\oplus$ (Property (1) of Prop. 4.12 of [2]))

$\Leftrightarrow \exists\, z \in (iB)_x : y \in (jB)_z$

(union definition)

$\Leftrightarrow \exists\, z \in (iB)_x : z \in (jB)_y$

(Proposition 28 - properties of a ball with center at origin)

$\Leftrightarrow (iB)_x \cap (jB)_y \neq \emptyset.$

(definition of empty set)

♦

We now recall the definitions of erosion by a structuring element and of B-border. We will use the latter definition in the study of the generated ball border.

Definition 45 (erosion by a structuring element) – Let B be a subset of $\mathbf{Z}^2$. The erosion by B is the mappings from $\boldsymbol{P}(\mathbf{Z}^2)$ to $\boldsymbol{P}(\mathbf{Z}^2)$ $e_B : X \mapsto X \ominus B$. The subset B is the structuring element of e_B. ♦

Definition 46 (B-border) – Let B be a subset of $\mathbf{Z}^2$. The B-border of a subset X of $\mathbf{Z}^2$ is the subset $\partial_B(X) = X - e_B(X)$. ♦

We observe that the B-border is an inner border: $\partial_B(X) \subset X$.

Proposition 47 (B-border properties) – Let B and X be two subsets of $\mathbf{Z}^2$ and let x be a point in $\mathbf{Z}^2$, then

(i) if $\#B > 1$ then $(\partial_B(\{o\}) = \{o\}$ ($\{o\}$ is an invariant)

(ii) $(\partial_B(X))_x = \partial_B(X_x)$ (translation-invariance)

(iii) X and $B \in \boldsymbol{S} \Rightarrow \partial_B(X) \in \boldsymbol{S}$. (symmetry preservation)

♦

Proof

(a) Let us prove (i).

$\partial_B(\{o\}) = \{o\} - e_B(\{o\})$ (Definition 46 - B-border)

$= \{o\} - \{u \in \mathbf{Z}^2 : \forall b \in B, \exists\, x \in \{o\} : u = x - b\}$

(Definition 31- Minkowski's addition and subtraction)

$= \{o\} - \{u \in \mathbf{Z}^2 : \forall b \in B, u = o - b\}$

(logical derivation)

$= \{o\} - \{u \in \mathbf{Z}^2 : \forall b \in B, u = -b\}$ (o is unity of +)

$= \{o\} - \emptyset$ ($\#B > 1$)

$= \{o\}$ (definition of – and $\emptyset$)

(b) Let us prove (ii). For any $x \in \mathbf{Z}^2$,

$(\partial_B(X))_x = (X - e_B(X))_x$ (Definition 46)

$= X_x - (e_B(X))_x$ (translation-invariance of –)

$= X_x - (X \ominus B)_x$ (Definition 45 - erosion by a subset)

$= X_x - X_x \ominus B$

(translation invariance of $\ominus$ (Property (4) of Prop. 4.13 of [2]))

$= X_x - e_B(X_x)$ (Definition 45)

$= \partial_B(X_x).$ (Definition 46)

(c) Let us prove (iii). For any $u \in \mathbf{Z}^2$,

$u \in \partial_B(X) \Leftrightarrow u \in X - e_B(X)$ (Definition 46)

$\Leftrightarrow u \in X - (X \ominus B)$ (Definition 45)

$\Leftrightarrow u \in X$ and $\neg\, u \in (X \ominus B)$ (def. of –)

$\Leftrightarrow u \in X$ and $\neg\, \forall b \in B, \exists\, x \in X : u = x - b$

(Definition 31 - Minkowski's addition and subtraction)

$\Leftrightarrow -u \in X$ and $\neg\, \forall\, b' \in B, \exists\, x' \in X : -u = x' - b'$

(writing $x' = -x$ and $b' = -b$, and hypothesis X and $B \in \boldsymbol{S}$)

$\Leftrightarrow -u \in X$ and $\neg -u \in (X \ominus B)$ (Definition 31)

$\Leftrightarrow -u \in X$ and $\neg -u \in e_B(X)$ (Definition 45)

$\Leftrightarrow -u \in \partial_B(X).$ (Definition 46)

♦

In order to be able to characterize the integer-valued t.i. regular metrics later on, we now introduce the concept of "closed" sub collection of balls. A sub collection of balls $\boldsymbol{B}_B$ is *B-closed* if any members of $\boldsymbol{B}_B$ is morphologically B-closed, that is, for any $X \in \boldsymbol{B}_B$, X satisfies the equation $X = (X \oplus B) \ominus B$. If $\boldsymbol{B}_B$ is B-closed then we say for convenience that *B has the closure property*.

The balls multiple of the 3 by 3 discrete square B_8 of Section 3, are examples of morphologically closed balls with respect to B_8 (see Section 6).

But, for example, the ball $B = (2B_8 - 1B_8) + 0B_8$ is not morphologically closed with respect to itself since, in this case, $(B \oplus B) \ominus B = 2B_8$ and consequently $(B \oplus B) \ominus B \neq B$.

Under the closure property the sufficient condition to have nested balls becomes necessary.

Proposition 48 (generated balls versus inclusion) – Let B be a finite symmetric ball (i.e., $B \in \boldsymbol{S}$). For any points x and y in $\mathbf{Z}^2$ and any numbers i in $\mathbf{N}$ and j in $\mathbf{N}^+$,

(i) $x \in (jB)_y \Rightarrow (iB)_x \subset ((i+j)B)_y$;

(ii) if B has the closure property, then $(iB)_x \subset ((i+j)B)_y \Rightarrow x \in (jB)_y$. ♦

Proof

Let B be a finite symmetric ball (i.e., $B \in \boldsymbol{S}$).

(a) Let us prove Property (i). For any x and $y \in \mathbf{Z}^2$, any $i \in \mathbf{N}$ and $j \in \mathbf{N}^+$,

$(iB)_x \subset \bigcup_{u \in (jB)_y} (iB)_u$ ($x \in (jB)_y$)

$= (jB)_y \oplus iB$

(equivalent definition of $\oplus$ (Property (1) of Prop. 4.12 of [2]))

$= (jB \oplus iB)_y$

(translation invariance of $\oplus$ (Property (5) of Prop. 4.12 of [2]))

$= ((i+j)B)_y$,

(Proposition 35- mix distributivity of the Minkowski product)

that is, $x \in (jB)_y \Rightarrow (iB)_x \subset ((i+j)B)_y$.

(b) Let us prove Property (ii). Let x and $y \in \mathbf{Z}^2$, let $i \in \mathbf{N}$ and let $j \in \mathbf{N}^+$. If $i = 0$,

$x \in \{x\}$ (singleton definition)

$= \{o\}_x$ (Definition 19 - translated version)

$= (0B)_x$ (Definition 34 - Minkowski product)

$\subset ((i+0)B)_y$ (hypothesis)

$\subset (iB)_y$, (natural number property)

that is, $(0B)_x \subset ((0+j)B)_y \Rightarrow x \in (jB)_y$.

If $i \neq 0$,

$x \in \{x\}$ (singleton definition)

$= (iB)_x \ominus iB$ ($\ominus$ property)

$\subset ((i+j)B)_y \ominus iB$ (hypothesis and $\ominus$ is increasing)

$= ((((i-1)+j)+1)B)_y \ominus iB$ (properties of + on $\mathbf{N}^+$)

$= ((((i-1)+j)B) \oplus B)_y \ominus iB$

(Proposition 35)

$= ((((i-1)+j)B)_y \oplus B) \ominus iB$

(Property (5) of Prop. 4.12 of [2])

$= ((((i-1)+j)B)_y \oplus B) \ominus (B \oplus (i-1)B)$ (Proposition 35)

$= (((((i-1)+j)B)_y \oplus B) \ominus B) \ominus (i-1)B$

(Property (2) of Prop. 4.13 of [2])

$= (((i-1)+j)B)_y \ominus (i-1)B$

(B has the closure property and Proposition 41 - characterization of the generated balls)

$= (jB)_y$, (repeating the four previous steps $(i-1)$ times)

that is, $(iB)_x \subset ((i+j)B)_y \Rightarrow x \in (jB)_y$. ♦

We observe that the last implication in Property (ii) of Proposition 48 may be false when B has not the closure property. We can construct the following counter example: let $B = (2B_8 - 1B_8) + 0B_8$, let $x \in 1B_8 - 0B_8$ and let $y = o$, then $(1B)_x \subset (2B)_y$ but $x \notin (1B)_y$.

Figure 6 illustrates this counter example. On the left-hand side we see that the dark gray ball $(1B)_x$ is included in the light gray ball $(2B)_y$ (the 9 by 9 square), nevertheless we see on the right-hand side that x doesn't belong to $(1B)_y$.

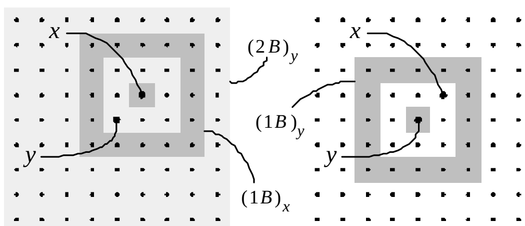

Figure 6 - Counter example for Proposition 48.

The closure property for a propotype ball allows us to "go backward" along a chain of generated balls.

Proposition 49 (erosion of a generated ball) – Let B be a subset of $\mathbf{Z}^2$. For any $x \in \mathbf{Z}^2$ and $j \in \mathbf{N} - \{0\}$,

(i) $e_B((jB)_x) \supset ((j-1)B)_x$

(ii) if B has the closure property, then $e_B((jB)_x) = ((j-1)B)_x$. ♦

Proof

For any subset B of $\mathbf{Z}^2$ having the closure property, any $x \in \mathbf{Z}^2$ and any $j \in \mathbf{N} - \{0\}$,

$e_B((jB)_x) = (jB)_x \ominus B$ (Definition 45 - erosion by a subset)

$= (jB \ominus B)_x$

(translation invariance of $\ominus$ (Property (4) of Prop. 4.13 of [2]))

$= (((j-1)B) \oplus B)_x \ominus B$

(Definition 34 - Minkowski product)

$= (((j-1)B)_x \oplus B) \ominus B$

(translation invariance of $\ominus$ (Property (5) of Prop. 4.12 of [2]))

$\overset{=}{\supset} ((j-1)B)_x$

($=$ since B has the closure property and Proposition 41 (characterization of the generated balls); $\supset$ since closing is extensive)

♦

Next proposition will be useful in Section 7 to show that a metric induced by a ball having the closure property is regular (Proposition 61).

Proposition 50 (generated balls versus B-border) – Let B be a finite symmetric ball (i.e., $B \in \mathbf{S}$) 1 having the closure property and such that $\#B > 1$. For any points x and y in $\mathbf{Z}^2$ and any numbers i in $\mathbf{N}$ and j in $\mathbf{N}^+$,

(i) $x \in \partial_B((jB)_y) \Rightarrow \partial_B(((i+j)B)_y) \cap \partial_B((iB)_x) \neq \emptyset$;

(ii) $x \in \partial_B(((i+j)B)_y) \Rightarrow \partial_B((iB)_x) \cap \partial_B((jB)_y) \neq \emptyset$. ♦

Proof

(a) Let us prove Property (i). Let x and $y \in \mathbf{Z}^2$, let $i \in \mathbf{N}$ and let $j \in \mathbf{N}^+$. If $j = 0$,

$x \in \partial_B((0B)_y) \Leftrightarrow x \in \partial_B(\{o\}_y)$

(Definition 34 - Minkowski product)

$\Leftrightarrow x \in (\partial_B(\{o\}))_y$

(Property (ii) of Proposition 47 - B-border properties)

$\Leftrightarrow x \in \{o\}_y$

($\#B > 1$ and Property (i) of Proposition 47)

$\Leftrightarrow x \in \{y\}$ (Definition 19 - translated version)

$\Leftrightarrow x = y$. (singleton definition)

Furthermore, if $x \in \partial_B((0B)_y)$

$\partial_B(((i+0)B)_y) \cap \partial_B((iB)_x) = \partial_B((iB)_y) \cap \partial_B((iB)_x)$

(natural number property)

$= \partial_B((iB)_x) \cap \partial_B((iB)_x)$

(by hypothesis, as shown above $x = y$)

$= \partial_B((iB)_x)$ ($\cap$ is idempotent)

$= (iB)_x - e_B((iB)_x)$

(Definition 46 - B-border)

$= (iB)_x - ((i-1)B)_x$

(B has the closure porperty and Property (ii) of Proposition 49 - erosion of a generated ball)

$\neq \emptyset$,

(Proposition 37 - regularity of the Minkowski product)

that is, $x \in \partial_B((0B)_y) \Rightarrow \partial_B(((i+0)B)_y) \cap \partial_B((iB)_x) \neq \emptyset$.

If $j \neq 0$,

$x \in \partial_B((jB)_y) \Leftrightarrow x \in (jB)_y - e_B((jB)_y)$ (Definition 46)

$\Rightarrow x \notin e_B((jB)_y)$ (def. of –)

$\Rightarrow x \notin ((j-1)B)_y$

(Property (i) of Proposition 49)

$\Rightarrow (iB)_x \not\subset ((i+j-1)B)_y$

(B has the closure property and Property (ii) of Proposition 48 - generated balls versus inclusion)

$\Leftrightarrow (iB)_x \not\subset e_B(((i+j)B)_y)$

(B has the closure property and Property (ii) of Proposition 49)

$\Leftrightarrow (iB)_x \cap (e_B(((i+j)B)_y))^c \neq \emptyset$;

(inclusion property)

that is, $x \in \partial_B((jB)_y) \Rightarrow X \cap (e_B(Y))^c \neq \emptyset$, where $X = (iB)_x$ and $Y = ((i+j)B)_y$. Furthermore, for any $j \in \mathbf{N}^+$,

$x \in \partial_B((jB)_y) \Leftrightarrow x \in (jB)_y - e_B((jB)_y)$ (Definition 46)

$\Rightarrow x \in (jB)_y$ (def. of –)

$\Rightarrow (iB)_x \subset ((i+j)B)_y$

(Property (i) of Proposition 48)

$\Leftrightarrow X \subset Y$ (definitions of X and Y)

$\Rightarrow e_B(X) \subset e_B(Y)$

(erosion is increasing (Proposition. 3.3 of [2]))

$\Leftrightarrow (e_B(Y))^c \subset (e_B(X))^c$.

(complementation is decreasing)

Hence,

$$x \in \partial_B((jB)_y) \Rightarrow \begin{cases} X \cap (e_B(Y))^c \neq \varnothing \\ X \subset Y \\ (e_B(Y))^c \subset (e_B(X))^c \end{cases} \quad \text{(see above)}$$

$$\Rightarrow$$

$$\begin{cases} X \cap (e_B(Y))^c = (X \cap Y) \cap ((e_B(Y))^c \cap (e_B(X))^c) \\ X \cap (e_B(Y))^c \neq \varnothing \end{cases}$$

(intersection property)

$\Rightarrow (X \cap Y) \cap ((e_B(Y))^c \cap (e_B(X))^c) \neq \varnothing$

(substitution)

$\Leftrightarrow (X \cap (e_B(X))^c) \cap (Y \cap (e_B(Y))^c) \neq \varnothing$

(associativity and commutativity of intersection)

$\Leftrightarrow \partial_B(X) \cap \partial_B(Y) \neq \varnothing$. (Definition 46)

That is, for any $j \in \mathbf{N}^+$, $x \in \partial_B((jB)_y) \Rightarrow \partial_B(((i + j)B)_y) \cap \partial_B((iB)_x) \neq \varnothing$.

(b) Let us prove Property (ii). For any x and $y \in \mathbf{Z}^2$, any $i \in \mathbf{N}$ and $j \in \mathbf{N}^+$. If $j = 0$,

$x \in \partial_B(((i + 0)B)_y) \Leftrightarrow x \in \partial_B((iB)_y)$

(natural number property)

$\Leftrightarrow x \in (\partial_B((iB))_y$

(Property (ii) of Proposition 47)

$\Leftrightarrow y \in (\partial_B((iB))_x$

(Proposition 28 (properties of a ball with center at origin) and Property (iii) of Proposition 47)

$\Leftrightarrow y \in (\partial_B((iB))_x$ and $y \in \{y\}$

(singleton definition)

$\Leftrightarrow y \in (\partial_B((iB))_x$ and $y \in \{o\}_y$

(Definition 19)

$\Leftrightarrow y \in (\partial_B((iB))_x$ and $y \in (\partial_B(\{o\}))_y$

($\#B > 1$ and Property (i) of Proposition 47)

$\Leftrightarrow y \in (\partial_B((iB))_x$ and $y \in \partial_B(\{o\}_y)$

(Property (ii) of Proposition 47 - B-border properties)

$\Leftrightarrow y \in (\partial_B((iB))_x$ and $y \in \partial_B((0B)_y)$

(Definition 34)

$\Leftrightarrow (\partial_B((iB))_x \cap \partial_B((0B)_y) \neq \varnothing$.

(intersection and empty set definitions)

If $j \neq 0$,

$x \in \partial_B(((i + j)B)_y) \Leftrightarrow x \in ((i + j)B)_y - e_B(((i + j)B)_y)$

(Definition 46)

$\Rightarrow x \notin e_B(((i + j)B)_y)$ (def. of –)

$\Leftrightarrow x \notin ((i + j - 1) B)_y$

(Proprety (i) of Proposition 49)

$\Leftrightarrow (iB)_x \cap ((j - 1) B)_y = \varnothing$,

(Proposition 44 - generated balls versus intersection)

$\Leftrightarrow (iB)_x \subset (((j - 1) B)_y)^c$ (property of $\cap$)

$\Leftrightarrow (iB)_x \subset (e_B((jB)_y))^c$,

(B has the closure porperty and Property (ii) of Proposition 49)

that is, $x \in \partial_B(((i + j)B)_y) \Rightarrow X \subset (e_B(Y))^c$, where $X = (iB)_x$ and $Y = (jB)_y$. Exploring the symmetry role between X and Y,

$x \in \partial_B(((i + j)B)_y) \Leftrightarrow x \in (\partial_B((i + j)B))_y$

(Property (ii) of Proposition 47)

$\Leftrightarrow y \in (\partial_B((i + j)B))_x$

(Proposition 28 (properties of a ball with center at origin) and Property (iii) of Proposition 47)

$\Rightarrow (jB)_y \subset (e_B((iB)_x))^c$, (as above)

that is, $x \in \partial_B(((i + j)B)_y) \Rightarrow Y \subset (e_B(X))^c$. Furthermore,

$x \in \partial_B(((i + j)B)_y) \Leftrightarrow x \in ((i + j)B)_y - e_B(((i + j)B)_y)$

(Definition 46)

$\Rightarrow x \in ((i + j)B)_y$ (def. of –)

$\Leftrightarrow (iB)_x \cap (jB)_y \neq \varnothing$.

(Proposition 44 - generated balls versus intersection)

$\Leftrightarrow X \cap Y \neq \varnothing$. (definitions of X and Y)

Hence,

$$x \in \partial_B(((i + j)B)_y) \Rightarrow \begin{cases} X \cap Y \neq \varnothing \\ X \subset (e_B(Y))^c \\ Y \subset (e_B(X))^c \end{cases} \quad \text{(see above)}$$

$$\Rightarrow$$

$$\begin{cases} X \cap Y = (X \cap (e_B(Y))^c) \cap (Y \cap (e_B(X))^c) \\ X \cap Y \neq \varnothing \end{cases}$$

(intersection property)

$\Rightarrow (X \cap (e_B(Y))^c) \cap (Y \cap (e_B(X))^c) \neq \varnothing$

(substitution)

$\Leftrightarrow (X \cap (e_B(X))^c) \cap (Y \cap (e_B(Y))^c) \neq \varnothing$

(associativity and commutativity of intersection)

$\Leftrightarrow \partial_B(X) \cap \partial_B(Y) \neq \varnothing$.

(Definition 46)

That is, for any $j \in \mathbf{N}^+$, $x \in \partial_B(((i + j)B)_y) \Rightarrow \partial_B((iB)_x) \cap \partial_B((jB)_y) \neq \varnothing$. ♦

We observe that the implication in Property (i) of Proposition 50 may be false when $\boldsymbol{B}_B$ is not B-closed, for example, let $B = (2B_8 - 1B_8) + 0B_8$, let $x \in 1B_8 - 0B_8$ and let $y = o$, then $x \in \partial_B((2B)_y)$ but $\partial_B((3B)_y) \cap \partial_B((1B)_x) = \varnothing$.

Figure 7 illustrates this counter example. On this figure we see that the point x belongs to the dark gray ball border $\partial_B((2B)_y)$, nevertheless we see that the light gray ball border $\partial_B((3B)_y)$ has no intersection with the ball border $\partial_B((1B)_x)$ (depicted as a set of squares).

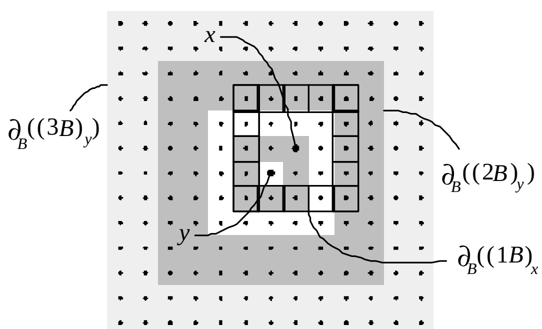

Figure 7 - Counter example for Proposition 50.

In the last proposition of this section we present some properties of the radii of generated balls. Part of these properties will be used in the proof of Proposition 59. Properties (ii) and (iii) will be used in future work.

Proposition 51 (radius properties of nested generated balls) Let B be a finite symmetric ball (i.e., $B \in \boldsymbol{S}$), such that $o \in B$ and $B \neq \{o\}$. For any balls X and Y in $\boldsymbol{B}_B$,

(i) $X \subset Y \Rightarrow \text{radius}_B(X) \leq \text{radius}_B(Y)$;

(ii) $X \neq Y$ and $X \subset Y \Rightarrow \text{radius}_B(X) < \text{radius}_B(Y)$;

(iii) (center(Y) $\in$ $B_{\text{center}(X)}$ and $\text{radius}_B(X) < \text{radius}_B(Y)$) $\Rightarrow$ $X \subset Y$.
(iv) (center(Y) = center(X) and $\text{radius}_B(X) \leq \text{radius}_B(Y)$) $\Rightarrow$ $X \subset Y$. ♦

Proof
For any balls X and Y in $\boldsymbol{B}_B$, let x = center(X), let y = center(Y), let $i = \text{radius}_B(X)$ and let $j = \text{radius}_B(Y)$.
(a) Let us prove (i):
$X \subset Y \Leftrightarrow$ $(\text{radius}_B(X)B)_{\text{center}(X)} \subset (\text{radius}_B(Y)B)_{\text{center}(Y)}$
(Proposition 41 - characterization of the generated balls)
$\Leftrightarrow$ $(iB)_x \subset (jB)_y$ (definitions of x, y, i and j)
$\Rightarrow$ $\#(iB)_x \leq \#(jB)_y$
(property of the cardinality of nested subsets)
$\Leftrightarrow \#(iB) \leq \#(jB)$ ($X \mapsto X_u$ is a bijection)
$\Rightarrow iB = jB$ or $jB \not\subset iB$
(property of the cardinality of nested subsets)
$\Leftrightarrow iB \subset jB$
(Proposition 43 - chain of generated balls with center at a given point (here the point is the origin o))
$\Leftrightarrow i \leq j$
($j \mapsto jB$ is a poset isomorphism (Proposition 43))
$\Leftrightarrow \text{radius}_B(X) \leq \text{radius}_B(Y)$ (definitions of i and j)
(b) Let us prove (ii):
$X \neq Y$ and $X \subset Y \Leftrightarrow$ $(iB)_x \neq (jB)_y$ and $(iB)_x \subset (jB)_y$
(Proposition 41 and definitions of x, y, i and j)
$\Rightarrow$ $\#(iB)_x < \#(jB)_y$
(property of the cardinality of nested subsets)
$\Leftrightarrow \#(iB) < \#(jB)$ ($X \mapsto X_u$ is a bijection)
$\Rightarrow jB \not\subset iB$
(property of the cardinality of nested subsets)
$\Leftrightarrow iB \neq jB$ and $iB \subset jB$ (Proposition 43)
$\Leftrightarrow i < j$
($j \mapsto jB$ is a poset isomorphism (Proposition 43))
$\Leftrightarrow \text{radius}_B(X) < \text{radius}_B(Y)$
(definitions of i and j)
(c) Let us prove (iii):
center(Y) $\in$ $B_{\text{center}(X)} \Leftrightarrow y \in B_x$ (definitions of x and y)
$\Leftrightarrow x \in B_y$
(Proposition 28 - properties of a ball with center at origin)
$\Rightarrow (iB)_x \subset ((i + 1)B)_y$
(Property (i) of Proposition 48 (generated balls versus inclusion) with $j = 1$)
$\Leftrightarrow (\text{radius}_B(X)B)_{\text{center}(X)} \subset ((i + 1)B)_y$
(definitions of x and i)
$\Leftrightarrow X \subset ((i + 1)B)_y$ (Proposition 41)
$\Leftrightarrow X \subset ((\text{radius}_B(X) + 1)B)_y$
(definitions of i)
$\Leftrightarrow X \subset ((\text{radius}_B(Y))B)_y$
($\text{radius}_B(X) + 1 \leq \text{radius}_B(Y)$ and the mapping $j \mapsto (jB)_y$ is a poset isomorphism (Proposition 43))
$\Leftrightarrow X \subset ((\text{radius}_B(Y))B)_{\text{center}(Y)}$
(definitions of y)
$\Leftrightarrow X \subset Y.$ (Proposition 41)
(d) Let us prove (iv):
true $\Leftrightarrow$ center(Y) = center(X) and $\text{radius}_B(X) \leq \text{radius}_B(Y)$
(hypothesis)
$\Leftrightarrow x = y$ and $i \leq j$ (def. of x, y, i and j)
$\Leftrightarrow x = y$ and $(iB)_x \subset (jB)_y$
(the mapping $j \mapsto (jB)_x$ is a poset isomorphism (Proposition 43))
$\Rightarrow (iB)_x \subset (jB)_y$ (logical derivation)
$\Leftrightarrow (\text{radius}_B(X)B)_{\text{center}(X)} \subset (\text{radius}_B(Y)B)_{\text{center}(Y)}$
(def. of x, y, i and j)
$\Leftrightarrow X \subset Y.$ (Proposition 41)
♦

We are now ready to begin the study of the relationship between metrics and symmetric balls.

6. From metric to symmetric ball

With a t.i. metric, we can associate a ball with center at origin. Let $(\mathbf{Z}^2, d)$ be a t.i. metric space, the *unit ball of* $(\mathbf{Z}^2, d)$, denoted by B_d, is the set of all the points at a distance less than or equal to one from the origin, that is,

$$B_d = \{u \in \mathbf{Z}^2 \colon f_d(u) \leq 1\}.$$

An exhaustive inspection of the nine points of B_8 shows that $B_{d_8} = B_8$, in other words, the 3 by 3 discrete square is the unit ball of the chessboard metric space $(\mathbf{Z}^2, d_8)$.

In the next proposition, we show three properties of the unit ball.

Proposition 52 (properties of the unit ball) – For any metric d on $\mathbf{Z}^2$,
(i) $B_d \in \boldsymbol{S}$,
(ii) $o \in B_d$ and
(iii) $B_d \neq \{o\}$. ♦

Proof
(a) Let us prove (i). For any $u \in \mathbf{Z}^2$,
$u \in B_d \Leftrightarrow d(u, o) \leq 1$ (def. of B_d)
$\Leftrightarrow d(-u, o) \leq 1$ (symmetry of d)
$\Leftrightarrow -u \in B_d$, (def. of B_d)
that is, $B_d \in \boldsymbol{S}$.
(b) Let us prove (ii):
true $\Leftrightarrow d(o, o) = 0$
($\Leftarrow$ of Properties (i) of Definition 1 - metric space)
$\Rightarrow d(o, o) \leq 1$, (natural number property)
that is, by definition of B_d, $o \in B_d$.
(c) Let us prove (iii). For any $u \in \mathbf{Z}^2$,
$d(u, o) = 1 \Rightarrow u \neq o$,
($\Rightarrow$ of Properties (i) of Definition 1 - metric space)
that is, $B_d \neq \{o\}$. ♦

Property (i) of the previous proposition shows that the unit ball is really a ball in the sense of Definition 22 and that its center is the origin.

In the next proposition, we show a relationship between a t.i. lower regular metric and the Minkowski product of its unit ball.

Proposition 53 (property of the generated balls in a lower regular metric space) – Let ($\mathbf{Z}^2$, d) be a t.i. metric space, for any x and $y \in \mathbf{Z}^2$ and any $j \in \mathbf{N}$,

(i) $x - y \in jB_d \Rightarrow d(x, y) \leq j$

(ii) if d is lower regular, then
$d(x, y) \leq j \Rightarrow x - y \in jB_d$. ♦

Proof

Let us make a recursive proof. For any x and $y \in \mathbf{Z}^2$,

$x - y \in 1B_d \Leftrightarrow x - y \in B_d$ (Definition 34 - Minkowski product)

$\Leftrightarrow f_d(x - y) \leq 1$ (unit ball def.)

$\Leftrightarrow d(x - y, o) \leq 1$ (f_d def.)

$\Leftrightarrow d(x, y) \leq 1$, (translation-invariance of d)

That is, $x - y \in jB_d \Leftrightarrow d(x, y) \leq j$ is true for $j = 1$.

Let us assume it was true for $j - 1$ ($j > 1$), for any x and $y \in \mathbf{Z}^2$,

$x - y \in jB_d \Leftrightarrow x - y \in ((j - 1)B_d) \oplus B_d$ (Definition 34)

$\Leftrightarrow \exists u \in B_d : x - y \in ((j - 1)B_d)_u$ (equivalent definition of $\oplus$ (Property (1) of Prop. 4.12 of [2]))

$\Leftrightarrow \exists u \in B_d : x - (y + u) \in (j - 1)B_d$ (Definition 19 - translated version)

$\Leftrightarrow \exists u \in \mathbf{Z}^2 : u \in B_d$ and $x - (y + u) \in (j - 1)B_d$ (logical derivation)

$\Leftrightarrow \exists v \in \mathbf{Z}^2 : x - v \in B_d$ and $v - y \in (j - 1)B_d$ ($v = x - u$)

$\Leftrightarrow \exists v \in \mathbf{Z}^2 : x - v \in 1B_d$ and $v - y \in (j - 1)B_d$ (Definition 34)

$\Leftrightarrow \exists v \in \mathbf{Z}^2 : d(x, v) \leq 1$ and $d(v, y) \leq j - 1$ (hypothesis and first part of the proof)

$\Leftrightarrow d(x, y) \leq j$.

(by substitution of i and j of Proposition 16 (ball intersection in a lower regular metric space), by $j - 1$ and 1, respectively; hence, $\Leftarrow$ is true under the hypothesis that d is lower regular)

♦

The next corollary illustrates better the previous proposition. We recall that $B_d(y, j)$ is the ball of center y and radius j, derived from the distance d (see Section 3). Comparing the notations, we have $B_d(o, 1) = B_d$.

Corollary 54 (property of the generated balls in a lower regular metric space) – Let ($\mathbf{Z}^2$, d) be a t.i. metric space, for any $y \in \mathbf{Z}^2$ and any $j \in \mathbf{N}$,

(i) $(jB_d(o, 1))_y \subset B_d(y, j)$

(ii) if d is lower regular, then
$B_d(y, j) \subset (jB_d(o, 1))_y$. ♦

Proof

For any x and $y \in \mathbf{Z}^2$, and any $j \in \mathbf{N}$,

$x \in (jB_d(o, 1))_y \Leftrightarrow x \in (j\{z \in \mathbf{Z}^2: d(o, z) \leq 1\})_y$ (def. of $B_d(y, j)$)

$\Leftrightarrow x - y \in j\{z \in \mathbf{Z}^2: d(o, z) \leq 1\}$ Definition 19 - translated version)

$\Leftrightarrow x - y \in j\{z \in \mathbf{Z}^2: d(z, o) \leq 1\}$ (symmetry of d)

$\Leftrightarrow x - y \in jB_d$ (def. of B_d)

$\Leftrightarrow d(x, y) \leq j$

($\Rightarrow$ by Property (i) of Proposition 53 (property of the generated balls in a lower regular metric space); $\Leftarrow$ by Property (ii) of Proposition 53 and hypothesis)

$\Leftrightarrow d(y, x) \leq j$ (symmetry of d)

$\Leftrightarrow x \in B_d(y, j)$. (def. of $B_d(y, j)$)

♦

We can apply Proposition 53 to the chessboard distance. In this way, we see how to obtain the chessboard balls by using the Minkowski product.

Corollary 55 (property of the Minkowski product of the unit ball of the chessboard metric space) – If d_8 is the chessboard distance, then for any x and $y \in \mathbf{Z}^2$, and any $j \in \mathbf{N}$,
$x - y \in jB_8 \Leftrightarrow d_8(x, y) \leq j$. ♦

Proof

The result is a consequence of Proposition 53 (property of the generated balls in a lower regular metric space) since B_8 is the unit ball of the chessboard metric space and d_8 is regular by Proposition 15 (example of regular metric space).

♦

Next proposition, which is consequence Proposition 53, will be used in Section 8 to characterize the regular metric.

Proposition 56 (closure property of the unit ball of a regular metric space) – Let ($\mathbf{Z}^2$, d) be a t.i. metric space. If d is regular, then B_d has the closure property. ♦

Proof

Let assume that d is regular. We divide the proof in four parts.

(a) Let us prove that for any $j \in \mathbf{N}$ and any $x \in \mathbf{Z}^2$, $(B_d)_x \subset (j + 1)B_d \Rightarrow d(x, o) \leq j$,

$(B_d)_x \subset (j + 1)B_d \Leftrightarrow (1B_d)_x \subset (j + 1)B_d$ (Definition 34 - Minkowski product)

$\Leftrightarrow (1B_d(o, 1))_x \subset (j + 1)B_d(o, 1)$ (definitions of $B_d(y, j)$ and B_d)

$\Leftrightarrow (1B_d(o, 1))_x \subset ((j + 1)B_d(o, 1))_o$ (o is unit element of +)

$\Leftrightarrow B_d(x, 1) \subset B_d(o, j + 1)$

(Corollary 54 (property of the generated balls in a lower regular metric space) , $\Leftarrow$ is true under the hypothesis that d is lower regular)

$\Rightarrow 1 + d(x, o) \leq j + 1$.

(d is upper regular and Property (ii) of Proposition 18 - ball inclusion in an upper regular metric space)

$\Leftrightarrow d(x, o) \leq j$. (+ is double-side increasing)

(b) Let us prove that, for any $j \in \mathbf{N}$, $(((jB_d) \oplus B_d) \ominus B_d) \subset jB_d$. For any $j \in \mathbf{N}$ and any $x \in \mathbf{Z}^2$,

$x \in ((jB_d) \oplus B_d) \ominus B_d \Leftrightarrow x \in ((j + 1)B_d) \ominus B_d$ (product def.)

$\Leftrightarrow (B_d)_x \subset (j + 1)B_d$ (Minkowski's subtraction property)

$\Rightarrow d(x, o) \leq j$ (Part (a))

$\Rightarrow x \in jB_d$,

(hypothesis (d is lower regular) and Property (ii) of Proposition 53)

that is, for any $j \in \mathbf{N}$, $(((jB_d) \oplus B_d) \ominus B_d) \subset jB_d$.

(c) For any $j \in \mathbf{N}$, $((jB_d) \oplus B_d) \ominus B_d$ is the closing of jB_d by B_d, therefore by closing extensivity (Property (3) of Prop. 6.21 of [2]) $jB_d \subset (((jB_d) \oplus B_d) \ominus B_d)$.

Hence, from Parts (b) and (c), if d is regular, then, for any $j \in \mathbf{N}$, $(((jB_d) \oplus B_d) \ominus B_d) = jB_d$, in other words, jB_d is B_d-closed.

(d) For any $X \in \boldsymbol{B}_B$, let $x = \text{center}(X)$ and let $i = \text{radius}_B(X)$,

$(X \oplus B_d) \ominus B_d) = ((\text{radius}_B(X)B_d)_{\text{center}(X)} \oplus B_d) \ominus B_d$

(Proposition 41 - characterization of the generated balls)

$= ((iB_d)_x \oplus B_d) \ominus B_d$ (def. of x and i)

$= ((iB_d) \oplus B_d)_x \ominus B_d$

(Property (5) of Prop. 4.12 of [2])

$= (((iB_d) \oplus B_d) \ominus B_d)_x$

(Property (5) of Prop. 4.12 of [2])

$= (iB_d)_x$ (Part (c))

$= (\text{radius}_B(X)B_d)_{\text{center}(X)}$

$= X$, (Proposition 41)

that is, if d is regular, then $\boldsymbol{B}_{B_d}$ is B_d-closed. ♦

Based on the regularity of the Chessboard distance, we have the following result.

Corollary 57 (closure property of the Minkowski product of the 3 by 3 discrete square) – Let B_8 be the 3 by 3 discrete square with center at origin, for any $j \in \mathbf{N}$, jB_8 is B_8-closed. ♦

Proof

By Proposition 15 (example of geometrical metric space) $(\mathbf{Z}^2, d_8)$ is regular. Furthermore, B_8 is the unit ball of $(\mathbf{Z}^2, d_8)$: $B_{d_8} = B_8$. Therefore, by Proposition 56 (closure property of the unit ball of a regular metric space), for any $j \in \mathbf{N}$, jB_8 is B_8-closed. ♦

Let $\boldsymbol{B}_c$ be the collection of finite symmetric balls (i.e., $B \in \boldsymbol{S}$) having the closure property and such that $o \in B$ and $B \neq \{o\}$, that is, $\boldsymbol{B}_c = \{B \in \boldsymbol{S}$: $\boldsymbol{B}_B$ is B-closed, $o \in B$ and $B \neq \{o\}\}$.

Let $\boldsymbol{M}_r$ be the set of integer-valued t.i. regular metrics, that is, $\boldsymbol{M}_r = \{d \in \mathbf{N}^{\mathbf{Z}^2 \times \mathbf{Z}^2}$: d is t.i. and regular$\}$.

Before closing this section, we verify that the image of $\boldsymbol{M}_r$ through $d \mapsto B_d$ is contained in $\boldsymbol{B}_c$. In other words, the unit ball of a metric space which metric is integer-valued t.i. regular and defined on the discrete plane has the closure property.

Proposition 58 (properties of the Minkowski product of the unit ball of a regular metric space) – $(d \mapsto B_d)(\boldsymbol{M}_r) \subset \boldsymbol{B}_c$. ♦

Proof

By Proposition 52 (properties of the unit ball) $B_d \in \boldsymbol{S}$, $o \in B$ and $B \neq \{o\}$, and by Proposition 56 (closure property of the unit ball of a regular metric space) d being regular, $\boldsymbol{B}_B$ is B_d-closed. Therefore, for any $d \in \boldsymbol{M}_r$, $B_d \in \boldsymbol{B}_c$. ♦

7. From symmetric ball to metric

By using the Minkowski product, with a symmetric ball containing the origin, we can associate a t.i. metric.

Let B be a finite symmetric ball (i.e., $B \in \boldsymbol{S}$), such that $o \in B$ and $B \neq \{o\}$.

For any $x \in \mathbf{Z}^2$, let

$$M^x = \bigcap \{X \in \boldsymbol{B}_B(o) : x \in X\} .$$

By Proposition 43 (chain of generated balls with center at a given point), $M^x \in \boldsymbol{B}_B(o)$ and it is the smallest ball with center at origin that contains the point x.

Let f_B be the mapping from $\mathbf{Z}^2$ to $\mathbf{N}^+$ given by, for any $x \in \mathbf{Z}^2$,

$$f_B(x) = \text{radius}_B(M^x).$$

In the next proposition, we give the relationship between the ball border and its radius. Property (i) will be useful to prove the subadditivity of f_B, and Property (iii) to prove its regularity.

Proposition 59 (ball border versus ball radius) – Let B be a finite symmetric ball (i.e., $B \in \boldsymbol{S}$), such that $o \in B$ and $B \neq \{o\}$. For any $x \in \mathbf{Z}^2$ and any finite $X \in \boldsymbol{B}_B(o)$,

(i) $x \in X \Leftrightarrow f_B(x) \leq \text{radius}_B(X)$,

(ii) $x \in \partial_B X \Rightarrow f_B(x) = \text{radius}_B(X)$,

(iii) if B has the closure property, then $f_B(x) = \text{radius}_B(X) \Rightarrow x \in \partial_B X$. ♦

Proof

Let $x \in \mathbf{Z}^2$, let $M^x = \bigcap \{U \in \boldsymbol{B}_B(o) : x \in U\}$ and let $X \in \boldsymbol{B}_B(o)$.

(a) Let us prove $\Rightarrow$ of (i). First,

$x \in X \Leftrightarrow X \in \{U \in \boldsymbol{B}_B(o)$: $x \in U\}$ (set definition)

$\Rightarrow M^x \subset X$. (definition of M^x and property of $\bigcap$)

Second,

$f_B(x) = \text{radius}_B(M^x)$ (definitions of M^x and $f_B(x)$)

$\leq \text{radius}_B(X)$.

($M^x \subset X$ and Property (i) of Proposition 51 - radius properties of nested generated balls)

(b) Let us prove $\Leftarrow$ of (i). First,

$\text{radius}_B(M^x) = f_B(x)$ (definitions of M^x and $f_B(x)$)

$\leq \text{radius}_B(X)$, (hypothesis)

that is, under the hypothesis, $\text{radius}_B(M^x) \leq \text{radius}_B(X)$.

Second,

$x \in M$ (property of $\bigcap$)

$\subset X$,

($\text{radius}_B(M^x) \leq \text{radius}_B(X)$ and Property (iv) of Proposition 51)

that is, under the hypothesis $x \in X$.

(c) Let us prove Properties (ii) and (iii). Let us assume that B has the closure property and let $j = \text{radius}_B(X)$,

$x \in \partial_B(X) \Leftrightarrow x \in X - e_B(X)$ (Definition 46 - B-border)

$\Leftrightarrow x \in jB - e_B(jB)$

($X = jB$ by Proposition 41 - characterization of the generated balls)

$\Leftrightarrow x \in jB - ((j-1)B)$

($\Rightarrow$ by Property (i) of Proposition 49 (erosion of a generated ball) and $\Leftarrow$ by Property (ii) of Proposition 49 since B has the closure property)

$\Leftrightarrow x \in jB$ and $x \notin (j-1)B$
(def. of the set difference)

$\Leftrightarrow f_B(x) \leq \text{radius}_B(jB)$ and $\text{radius}_B((j-1)B) < f_B(x)$
(Property (i))

$\Leftrightarrow f_B(x) \leq j$ and $(j-1) < f_B(x)$
(Proposition 41)

$\Leftrightarrow j \leq f_B(x) \leq j$ (property of $<$ on $\mathbf{N}$)

$\Leftrightarrow f_B(x) = j$ (anti-symmetry of $\leq$)

$\Leftrightarrow f_B(x) = \text{radius}_B(X)$. (definition of j)

♦

The mapping f_B has almost all the properties of a norm as shown in the next proposition. Nevertheless, it doesn't verify the property: $f_B(jx) = |j| f_B(x)$ for any $j \in \mathbf{Z}$ (for example, if B is a five by five square and $x = (1, 1)$, then $f_B(2x) = 1$ and $2f_B(x) = 2$), instead it has a weaker property: the symmetry.

Proposition 60 (properties of f_B) – Let B be a finite symmetric ball (i.e., $B \in \boldsymbol{S}$), such that $o \in B$ and $B \neq \{o\}$. For any x and $y \in \mathbf{Z}^2$,

(i) $f_B(x) \geq 0$, (positiveness)

(ii) $x = o \Leftrightarrow f_B(x) = 0$,

(iii) $f_B(-x) = f_B(x)$, (symmetry)

(iv) $f_B(x + y) \leq f_B(x) + f_B(y)$. (subadditivity)

Proof

Let us prove (i). Since the radius of a ball is a natural number (see Definition 40 - radius of a generated ball), for any $x \in \mathbf{Z}^2$, $f_B(x) \geq 0$.

Let us prove (ii). For any $x \in \mathbf{Z}^2$,

$x = o \Leftrightarrow f_B(x) = \text{radius}_B(\bigcap\{X \in \boldsymbol{B}_B(o) : o \in X\})$
(def. of $f_B(x)$)

$\Leftrightarrow f_B(x) = \text{radius}_B(\{o\})$
(Proposition 43 - chain of generated balls with center at a given point)

$\Leftrightarrow f_B(x) = 0$.
(Property (i) of Proposition 42 - radius property)

Let us prove (iii). For any $x \in \mathbf{Z}^2$,

$f_B(-x) = \text{radius}_B(\bigcap\{X \in \boldsymbol{B}_B(o) : -x \in X\})$ (def. of $f_B(x)$)

$= \text{radius}_B(\bigcap\{X \in \boldsymbol{B}_B(o) : x \in X\})$ ($\boldsymbol{B}_B(o) \subset \boldsymbol{S}^+$)

$= f_B(x)$. (def. of $f_B(x)$)

Let us prove (iv). Let us divide the proof in three parts.

(a) For any $x \in \mathbf{Z}^2$, let us prove that $x \in f_B(x)B$,

true $\Leftrightarrow f_B(x) \leq f_B(x)$ (reflexivity of $\leq$)

$\Leftrightarrow f_B(x) \leq \text{radius}_B(f_B(x)B)$
($f_B(x)B \in \boldsymbol{B}_B(o)$ (definition of $\boldsymbol{B}_B(o)$) and Proposition 41 - characterization of the generated balls)

$\Leftrightarrow x \in f_B(x)B$.
(Property (i) of Proposition 59 - ball border versus ball radius)

(b) For any $x, y \in \mathbf{Z}^2$, let prove that $x + y \in (f_B(x) + f_B(x))B$:

true $\Leftrightarrow x \in f_B(x)B$ and $y \in f_B(y)B$ (Part (a))

$\Rightarrow x + y \in (f_B(x)B) \oplus (f_B(y)B)$
(Definition 31 - Minkowski's addition)

$\Leftrightarrow x + y \in (f_B(x) + f_B(y))B$
(Proposition 35 - mix distributivity of the product)

(c) For any $x, y \in \mathbf{Z}^2$,

$f_B(x + y) \leq \text{radius}_B((f_B(x) + f_B(y))B)$
(Part (b) and Property (i) of Proposition 59)

$= f_B(x) + f_B(y)$. (Proposition 41)

♦

With a mapping f_B we can associate the mapping $d_B = d_{f_B}$. By Proposition 9 (characterization of translation-invariant metric) and Proposition 60 (properties of f_B) d_B is a t.i. metric; we call it the *metric induced by B* or simply *induced metric*. Let x and $y \in \mathbf{Z}^2$, from the definition of d_f in Section 2, we have

$$d_B(x, y) = f_B(x - y).$$

It is interesting to note that by substituing the Minkowski product by the scaling operations: scaling or scaling2, of Section 4, the above construction doesn't lead to a metric.

As we can see in Figure 8, the triangle inequality is not satisfied when the induced distance is obtained by using the scaling transformations.

Before closing this section, we verify that the image of $\boldsymbol{B}_c$ through $B \mapsto d_B$ is contained in $\boldsymbol{M}_r$.

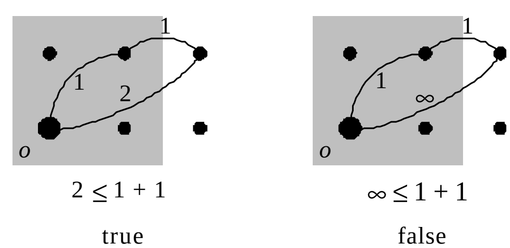

Figure 8 - Induced distances using Minkowski product and scaling.

Proposition 61 (properties of the metric induced by a ball having the closure property) – $(B \mapsto d_B)(\boldsymbol{B}_c) \subset \boldsymbol{M}_r$. ♦

Proof

We must verify four properties.

(a) By Proposition 9 (characterization of translation-invariant metric) and Proposition 60 (properties of f_B) d_B is a t.i. metric.

(b) For any B in $\boldsymbol{B}_c$, by construction, f_B is a mapping from $\mathbf{Z}^2$ to $\mathbf{N}^+$, therefore, by definition of induced metric, $d_B(\mathbf{Z}^2 \times \mathbf{Z}^2) = \mathbf{N}^+$.

(c) Let us prove the lower regularity. Let x and $y \in \mathbf{Z}^2$, let $i \in \mathbf{N}$, and let $j = d_B(x, y)$ such that $i \leq j$,

true $\Leftrightarrow d_B(x, y) = j$ (definition of j)

$\Leftrightarrow f_B(x - y) = j$ (definition of induced metric)

$\Leftrightarrow f_B(x - y) = \text{radius}_B(jB)$

(Proposition 41 - characterization of the generated balls)
$\Rightarrow x - y \in \partial_B(jB)$
(B has the closure property and Property (iii) of Proposition 59 - ball border versus ball radius)
$\Leftrightarrow x \in (\partial_B((jB))_y$ (Definition 19 - translated version)
$\Leftrightarrow x \in \partial_B((jB)_y)$
(Property (ii) of Proposition 47 - B-border properties)
$\Rightarrow \partial_B((iB)_x) \cap \partial_B(((j-i)B)_y) \neq \varnothing$
(Property (ii) of Proposition 50 - generated balls versus B-border)

$\Leftrightarrow \exists\, u \in \mathbf{Z}^2$: $u \in \partial_B((iB)_x)$ and $u \in \partial_B(((j-i)B)_y)$
(intersection and empty set definitions)
$\Leftrightarrow \exists\, u \in \mathbf{Z}^2$: $u \in (\partial_B(iB))_x$ and $u \in (\partial_B((j-i)B))_y$
(Property (ii) of Proposition 47)
$\Leftrightarrow \exists\, u \in \mathbf{Z}^2$: $u - x \in \partial_B(iB)$ and $u - y \in \partial_B((j-i)B)$
(Definition 19)
$\Rightarrow \exists\, u \in \mathbf{Z}^2$: $f_B(u-x) = \text{radius}_B(iB)$ and $f_B(u-y) = \text{radius}_B((j-i)B)$
(Property (ii) of Proposition 59)
$\Leftrightarrow \exists\, u \in \mathbf{Z}^2$: $f_B(u-x) = i$ and $f_B(u-y) = j - i$
(Proposition 41)
$\Leftrightarrow \exists\, u \in \mathbf{Z}^2$: $d_B(u, x) = i$ and $d_B(u, y) = j - i$
(definition of induced metric)
$\Leftrightarrow \exists\, u \in \mathbf{Z}^2$: $d_B(x, u) = i$ and $d_B(u, y) = j - i$
(symmetry of d_B)
$\Leftrightarrow \exists\, u \in \mathbf{Z}^2$: $d_B(x, y) = d_B(x, u) + d_B(u, y)$ and $d_B(x, u) = i$
(definition of j and substitution)
$\Leftrightarrow S(x, i) \cap L_2(x, y) \neq \varnothing$
(definitions of S and L_2 with respect to d_B)
$\Rightarrow d_B$ is lower regular.
(Corollary 5 - first equivalent definition of regular metric)
(d) Let us prove the upper regularity. Let x and $y \in \mathbf{Z}^2$, let $i \in \mathbf{N}$, and let $j = d_{\boldsymbol{B}}(x, y)$,
true $\Leftrightarrow d_B(x, y) = j$ (definition of j)
$\Leftrightarrow x \in \partial_B((jB)_y)$ (as in (b))
$\Rightarrow \partial_B(((i+j)B)_y) \cap \partial_B((iB)_x) \neq \varnothing$
(Property (i) of Proposition 50)
$\Leftrightarrow \exists\, u \in \mathbf{Z}^2$: $d_B(u, y) = i + j$ and $d_B(u, x) = i$
(as in (b))
$\Leftrightarrow \exists\, u \in \mathbf{Z}^2$: $d_B(u, y) = d_B(u, x) + d_B(x, y)$ and $d_B(u, x) = i$
(definition of j and substitution)
$\Leftrightarrow S(x, i) \cap L_3(x, y) \neq \varnothing$
(definitions of S and L_3 with respect to d_B)
$\Rightarrow d_B$ is upper regular. (Definition 3)
Hence, from (a) - (d) and by Corollary 5, for any $B \in \boldsymbol{B}_c$, $d_{\boldsymbol{B}} \in \boldsymbol{M}_r$. ♦

8. Relationship between regular metrics and symmetric balls having the closure property

We now establish two propositions showing that the mapping $B \mapsto d_B$ of the Section 7 is a left and right inverse of the mapping $d \mapsto B_d$ of Section 6.

Let $\boldsymbol{M}_2$ be the set of integer-valued t.i. lower regular metrics, that is, $\boldsymbol{M}_2 = \{d \in \mathbf{N}^{\mathbf{Z}^2 \times \mathbf{Z}^2} : d$ is t.i. and lower regular$\}$.

Proposition 62 (existence of a left inverse) – The mapping $B \mapsto d_B$ is a left inverse for the mapping $d \mapsto B_d$ from $\boldsymbol{M}_2$ to the collection of finite symmetric balls B (i.e., $B \in \boldsymbol{S}$), such that $o \in B$ and $B \neq \{o\}$. ♦

Proof
(a) From Proposition 52 (properties of the unit ball), for any metric d, $B_d \in \boldsymbol{S}$, $o \in B_d$ and $B_d \neq \{o\}$.
(b) For any $d \in \boldsymbol{M}_2$, any x and $y \in \mathbf{Z}^2$ and any $j \in \mathbf{N}$,
$d(x, y) \leq j \Leftrightarrow x - y \in jB_d$
(d is lower regular and Proposition 53 - property of the generated balls in a lower regular metric space)
$\Leftrightarrow f_{B_d}(x-y) \leq \text{radius}_{B_d}(jB_d)$
(Property (i) of Proposition 59 - ball border versus ball radius)
$\Leftrightarrow f_{B_d}(x-y) \leq j$
(Proposition 41 - characterization of the generated balls)
$\Leftrightarrow d_{B_d}(x, y) \leq j$. (definition of induced metric)
That is, by anti-symmetry of $\leq$, for any x and $y \in \mathbf{Z}^2$, $d(x, y) = d_{B_d}(x, y)$. In other words, by mapping equality definition, $d = d_{B_d}$. ♦

Proposition 62 shows that every integer-valued t.i. lower regular metric can be reconstructed from its unit ball using the Minkowski product. Since $\boldsymbol{M}_r \subset \boldsymbol{M}_2$, this is also true for the regular metric. In other words, its unit ball uniquely defines any (lower) regular metric space.

Corollary 63 (chessboard distance as a derived distance) – If d_8 is the chessboard distance, then $d_8 = d_{B_8}$. ♦

Proof
Since B_8 is the unit ball of the chessboard metric space, $B_{d_8} = B_8$. By Proposition 15 (example of regular metric space) $(d_8, \mathbf{Z}^2)$ is a regular metric space, therefore by Proposition 62 (existence of a left inverse) $d_8 = d_{B_8}$. ♦

Proposition 64 (existence of a right inverse) – The mapping $B \mapsto d_B$ is a right inverse for the mapping $d \mapsto B_d$ from $\boldsymbol{M}_r$ to $\boldsymbol{B}_c$. ♦

Proof
(a) By Proposition 61 (properties of the metric induced by a ball having the closure property) for any $B \in \boldsymbol{B}_c$, $d_{\boldsymbol{B}} \in \boldsymbol{M}_r$.
(b) For any $B \in \boldsymbol{B}_c$,
$B_{d_B} = \{u \in \mathbf{Z}^2 : d_B(u, o) \leq 1\}$ (definition of B_d)
$= \{u \in \mathbf{Z}^2 : f_B(u) \leq 1\}$ (definition of induced metric)
$= \{u \in \mathbf{Z}^2 : \text{radius}_B(\bigcap\{X \in \boldsymbol{B}_B(o) : u \in X\}) \leq 1\}$
(definition of f_B)
$= \{u \in \mathbf{Z}^2 : u \in \{o\}$ or $u \in B\}$

(by Proposition 43 (chain of generated balls with center at a given point) $\boldsymbol{B}_B(o)$ is a chain and $\{o\}$ and B are the only two generated balls with radius less than or equal to 1)

$= \{u \in \mathbf{Z}^2 : u \in B\}$ $(o \in B)$

$= B.$ (set definition)

That is, $B_{d_B} = B$. ♦

We are now ready to state our characterization theorem.

Theorem 65 (characterization of integer-valued translation-invariant regular metrics) – The mapping $d \mapsto B_d$ from $\boldsymbol{M}_r$ to $\boldsymbol{B}_c$ is a bijection. Its inverse is the mapping $B \mapsto d_B$. ♦

Proof

Let us divide the proof into two parts.

(a) By Proposition 58 (properties of the Minkowski product of the unit ball of a regular metric space) for any $d \in \boldsymbol{M}_r$, $B_d \in \boldsymbol{B}_c$. By Proposition 62 (existence of a left inverse), $d \mapsto B_d$ from $\boldsymbol{M}_r \subset \boldsymbol{M}_2$ to $\boldsymbol{B}_c$ is one-to-one.

(b) By Proposition 64 (existence of a right inverse), $d \mapsto B_d$ from $\boldsymbol{M}_r$ to $\boldsymbol{B}_c$ is onto.

Hence, from (a) and (b), $d \mapsto B_d$ from $\boldsymbol{M}_r$ to $\boldsymbol{B}_c$ is a bijection and its inverse is $B \mapsto d_B$. ♦

9. Conclusion

In the first part of this work we have introduced a definition of regular metric space and showed its relation to the Kiselman's regularity axioms for translation-invariant metrics. We have shown, in particular, that the lower regularity of type 1 is a redundant axiom in the definition of regular metrics.

In the second part, we have established a one-to-one relationship between the set of integer-valued and translation-invariant regular metrics defined on the discrete plane, and the set of symmetric balls satisfying a special closure property.

From this result we now know how to construct a regular metric on the discrete plane.

To this end, we choose in the discrete plane a symmetric ball B that has the closure property, i.e., that induces, through the Minkowski product, a chain of generated balls that are morphologically closed with respect to B.

Then the distance of a point x to the origin is given by the radius (in the sense of the Minkowski product) of the smallest ball of the chain, that contains x.

This construction shows that to preserve in the discrete plane the regularity property of the Euclidean metric on the continuous plane, we have to reach a compromise between a good approximation of a continuous ball and thin contours. "Closer" B from an Euclidean continuous ball, bigger B and thicker the borders in the discrete plane.

Actually it will be interesting in a future work to give a proof that if B is the intersection of an Euclidean continuous ball with the discrete plane then the generated balls are morphologically closed with respect to B.

The proof should be based on a closure property of the convex subsets of the continuous plane [3, Proposition 9.8].

The regular metric characterization we have proved will also be very useful to derive in future work some important geometrical properties of the skeletons of "expanded" subsets of the discrete plane [1].

Acknowledgments

We wish to thank Arley F. Souza for the compelling discussions about isotropic skeletons. This work was partially supported by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) under contracts 300966/90–3.

References

[1] Banon, G. J. F. New insight on digital topology. In: International Symposium on Mathematical Morphology and its Applications to Image and Signal Processing, 5., 26-28 June 2000, Palo Alto, USA. **Proceedings**... 2000. p. 138-148. Published as: INPE-7884-PRE/3724.

[2] Banon, G. J. F.; Barrera, J. **Bases da morfologia matemática para a análise de imagens binárias**. 2. ed. São José dos Campos: INPE, 1998. Posted at: <http://iris.sid.inpe.br:1912/rep/dpi.inpe.br/banon/1998/06.30.17.56>. Access in: 2003, Apr.17.

[3] Heijmans, H. J. A. M. **Morphological image operators**. Boston: Academic, 1994.

[4] Birkhoff, G. **Lattice theory**. 3. ed. Providence, Rhode Island: American Mathematical Society, 1967.

[5] Kiselman, C. O. **Digital geometry and mathematical morphology**. S.n.t. Lecture Notes, Spring Semester, 2002. Posted at: <http://www.math.uu.se/~kiselman/dgmm2002.pdf >. Access in 2002, Dec. 18.

[6] Serra, J. **Image analysis and mathematical morphology**. London: Academic. 1982.

[7] Szász, G. **Théorie des treillis**. Paris: Dunod, 1971. 227p.

[8] Rosenfeld, A.; Pfaltz, J. L. Digital functions on digital pictures. **Pattern Recognition**, v.1, n. 1, p. 33-61, 1968.