A DECOMPOSITION THEOREM IN MATHEMATICAL MORPHOLOGY

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Abstract

In this short paper, we present two constructive decompositions for any mapping between lattices in terms of the elementary mappings of Mathematical Morphology: erosions, dilations, anti–erosions and anti–dilations. This decomposition result extends the ones already known for translation invariant mappings.

1. INTRODUCTION

Nowadays, Computer Vision is object of intense researches because of its large number of potential applications. Mathematical Morphology (MM) seems to be well adapted to solve a large class of problems in this field.

Until the publication of volume 2 of Serra's book (1988), MM has been, essentially, restricted to translation invariant transformations. The field of applications of MM became larger with the introduction of a more general definition of erosion and dilation that uses the lattice notion.

In this paper, we introduce two constructive decompositions for any transformation or mapping between complete lattices that generalize the already known decompositions for translation invariant mappings (Matheron, 1975; Maragos, 1985; Banon and Barrera, 1991).

In Section 3, we recall the definitions of the four elementary mappings of the MM: erosions, dilations, anti–erosions and anti–dilations. In Section 4, we give the decomposition theorem and therefore we answer the question: does the MM elementary mappings can be used to represent any mapping between complete lattices and in what way this is possible?

2. MAPPINGS BETWEEN LATTICES

Let $(\mathcal{L}, \leq)$, or simply $\mathcal{L}$, be a complete lattice (Birkhoff, 1967). The infimum and the supremum of a subset $\mathcal{X}$ of $\mathcal{L}$ will be denoted, respectively, $\bigwedge\mathcal{X}$ and $\bigvee\mathcal{X}$. The least and greatest element of $\mathcal{L}$ will be denoted, respectively, O and I.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two complete lattices. We denote by ${\mathcal{L}_2}^{\mathcal{L}_1}$ the set of all functions from $\mathcal{L}_1$ to $\mathcal{L}_2$. The elements in ${\mathcal{L}_2}^{\mathcal{L}_1}$ (and ${\mathcal{L}_1}^{\mathcal{L}_2}$) will be called *mappings*. A generic element of $\mathcal{L}_1$ will be denoted by X and one of $\mathcal{L}_2$ by Y. A generic mapping in ${\mathcal{L}_2}^{\mathcal{L}_1}$ (or ${\mathcal{L}_1}^{\mathcal{L}_2}$) will be denoted by lower case Greek letters α, β, etc. Finally, the two constant mappings in ${\mathcal{L}_1}^{\mathcal{L}_2}$ that assume the values O and I will be denoted, respectively, by **O** and **I**.

In image processing, the most usual transformations on binary images can be represented by the translation invariant set mappings from $\mathcal{P}(E)$ to itself, where the set E is an Abelian group and $\mathcal{P}(E)$ is the collection of all parts of E. The collection $\mathcal{P}(E)$ is an example of complete lattice.

3. EROSIONS AND DILATIONS

Let $\psi \in {\mathcal{L}_2}^{\mathcal{L}_1}$ and let $\mathcal{X}$ be a subset of $\mathcal{L}_1$. We will denote by $\psi(\mathcal{X})$ the image of $\mathcal{X}$ through ψ.

A mapping $\psi \in {\mathcal{L}_2}^{\mathcal{L}_1}$ is *increasing* (or *isotone*) iff

$$X \leqslant X' \Rightarrow \psi(X) \leqslant \psi(X') \quad (X \text{ and } X' \in \mathcal{L}_1).$$

Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are complete lattices, then the above axiom is equivalent to any one of the following statements:

$$\psi(\bigwedge\mathcal{X}) \leqslant \bigwedge\psi(\mathcal{X}) \quad (\mathcal{X} \subset \mathcal{L}_1); \tag{1}$$

$$\bigvee\psi(\mathcal{X}) \leqslant \psi(\bigvee\mathcal{X}) \quad (\mathcal{X} \subset \mathcal{L}_1); \tag{2}$$

Following Serra (1988), if the equality holds in (1), then ψ is called an *erosion*. If it holds in (2), then ψ is called a *dilation*.

Let $\oplus$ and $\ominus$ be, respectively, the Minkowski addition and subtraction (Hadwiger, 1950) between subsets of an Abelian group E. The translation invariant set mappings from $\mathcal{P}(E)$ to itself, $X \mapsto X \oplus B$ and $X \mapsto X \ominus B$, where B is a subset of E, are, respectively, examples of a dilation and an erosion (Heijmans and Ronse, 1990).

A mapping $\psi \in {\mathcal{L}_2}^{\mathcal{L}_1}$ is *decreasing* iff

$$X \leqslant X' \Rightarrow \psi(X') \leqslant \psi(X) \quad (X \text{ and } X' \in \mathcal{L}_1).$$

Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are complete lattices, then the above axiom is equivalent to any one of the following statements:

$$\bigvee\psi(\mathcal{X}) \leqslant \psi(\bigwedge\mathcal{X}) \quad (\mathcal{X} \subset \mathcal{L}_1); \tag{3}$$

$$\psi(\bigvee\mathcal{X}) \leqslant \bigwedge\psi(\mathcal{X}) \quad (\mathcal{X} \subset \mathcal{L}_1); \tag{4}$$

Following Serra (1987), if the equality holds in (3), then ψ is called an *anti–erosion*. If its holds in (4), then ψ is called an *anti–dilation*.

The translation invariant set mappings from $\mathcal{P}(E)$ to itself, $X \mapsto (X \oplus B)^c$ and $X \mapsto (X \ominus B)^c$, where ^{c} denotes the set complementation, are, respectively, examples of an anti–dilation and an anti–erosion.

We will denote by E, Δ, E^a and Δ^a, respectively, the set of erosions, dilations, anti–erosions and anti–dilations in ${\mathcal{L}_1}^{\mathcal{L}_2}$.

4. MAPPING DECOMPOSITION

Let $\alpha, \beta \in {\mathcal{L}_1}^{\mathcal{L}_2}$. We define the mapping $\overline{\alpha\beta}$ and $\underline{\alpha\beta} \in {\mathcal{L}_2}^{\mathcal{L}_1}$ by setting, for any $X \in \mathcal{L}_1$,

$$\overline{\alpha\beta}(X) = \bigvee\{Y \in \mathcal{L}_2 \colon \alpha(Y) \leqslant X \leqslant \beta(Y)\},$$

$$\underline{\alpha\beta}(X) = \bigwedge\{Y \in \mathcal{L}_2 \colon \alpha(Y) \leqslant X \leqslant \beta(Y)\}.$$

Let $\mathcal{P}(\mathcal{L}_1)^{\mathcal{L}_2}$ be the set of the functions from $\mathcal{L}_2$ to $\mathcal{P}(\mathcal{L}_1)$. We extend the partial order relation $\subset$ on $\mathcal{P}(\mathcal{L}_1)$ to a partial order relation $\leqslant$ on $\mathcal{P}(\mathcal{L}_1)^{\mathcal{L}_2}$ by setting, for any $\mathcal{F}$ and $\mathcal{F}' \in \mathcal{P}(\mathcal{L}_1)^{\mathcal{L}_2}$,

$$\mathcal{F} \leqslant \mathcal{F}' \Leftrightarrow \mathcal{F}(Y) \subset \mathcal{F}'(Y) \quad (Y \in \mathcal{L}_2).$$

For any $\psi \in {\mathcal{L}_2}^{\mathcal{L}_1}$, let $\bullet\mathcal{K}(\psi)$ and $\mathcal{K}\bullet(\psi)$ be the two functions in $\mathcal{P}(\mathcal{L}_1)^{\mathcal{L}_2}$ defined by, for any $Y \in \mathcal{L}_2$,

$$\bullet\mathcal{K}(\psi)(Y) = \{X \in \mathcal{L}_1 \colon Y \leqslant \psi(X)\},$$

$$\mathcal{K}\bullet(\psi)(Y) = \{X \in \mathcal{L}_1 \colon \psi(X) \leqslant Y\}.$$

The function $\bullet\mathcal{K}(\psi)$ is called the *left kernel* (or, simply, *kernel*) of ψ and the function $\mathcal{K}\bullet(\psi)$ is called the *right kernel* of ψ.

These kernel definitions generalize the one given by Matheron (1975) for translation invariant set mappings.

A subset $\mathcal{X}$ of $\mathcal{L}$ is a *closed interval* of $\mathcal{L}$ (Birkhoff, 1967, p. 7) iff there exist two elements A and B in $\mathcal{L}$ such that, for any $X \in \mathcal{L}$, $A \leqslant X \leqslant B \Leftrightarrow X \in \mathcal{X}$. We denote by $[A, B]$ such closed interval.

We say that a function $\mathcal{I}$ is an *interval function* from $\mathcal{L}_2$ to $\mathcal{P}(\mathcal{L}_1)$ iff, for any $Y \in \mathcal{L}_2$, $\mathcal{I}(Y)$ is the empty set of $\mathcal{P}(\mathcal{L}_1)$ or a closed interval of $\mathcal{P}(\mathcal{L}_1)$.

To each pair (α, β), such that $\forall\, Y \in \mathcal{L}_2$, $(\alpha(Y) \leqslant \beta(Y))$ or (exclusive) $(\alpha(Y) = I$ and $\beta(Y) = O))$, we can associate a unique interval function $[\alpha, \beta] \in \mathcal{P}(\mathcal{L}_1)^{\mathcal{L}_2}$ given by, for any $Y \in \mathcal{L}_2$,

$$[\alpha, \beta](Y) \;=\; \begin{cases} [\alpha(Y), \beta(Y)] & \text{if } \alpha(Y) \leq \beta(\mathrm{Y}) \\ \emptyset & \text{otherwise.} \end{cases}$$

We call the mapping α and β the *extremities of the interval function* $[\alpha, \beta]$.

Finally, let $\Delta\Delta^{a}$ and $E^{a}E$ be the sets of pairs (α, β), respectively, in $\Delta \times \Delta^{a}$ and $E \times E^{a}$, such that $\forall\, Y \in \mathcal{L}_2$, $(\alpha(Y) \leqslant \beta(Y))$ or (exclusive) $(\alpha(Y) = I$ and $\beta(Y) = O)$.

We are now ready to state the following decomposition theorem.

Theorem 1 – *Any mapping ψ from $\mathcal{L}_1$ to $\mathcal{L}_2$ has the following sup–inf constructive decomposition*

$$\psi = \bigvee\{\overline{\alpha\mathbf{I}} \wedge \overline{\mathbf{O}\beta} \colon (\alpha, \beta) \in \Delta\Delta^{a} \text{ and } [\alpha, \beta] \leqslant \bullet\mathcal{K}(\psi)\}.$$

The proof of Theorem 1 is given in (Banon and Barrera, 1993) and it lies upon very nice properties of what we have called a morphological connection.

Let $\psi \in \mathcal{L}_2{}^{\mathcal{L}_1}$ and $\alpha, \beta \in \mathcal{L}_1{}^{\mathcal{L}_2}$. The pair $(\psi, (\alpha, \beta))$ is a *morphological connection between $\mathcal{L}_1$ and $\mathcal{L}_2$* iff

$$\alpha(Y) \leqslant X \leqslant \beta(Y) \Leftrightarrow Y \leqslant \psi(X) \quad ((X, Y) \in \mathcal{L}_1 \times \mathcal{L}_2).$$

Figure 1 illustrate for a given pair (X, Y) the above morphological connection property.

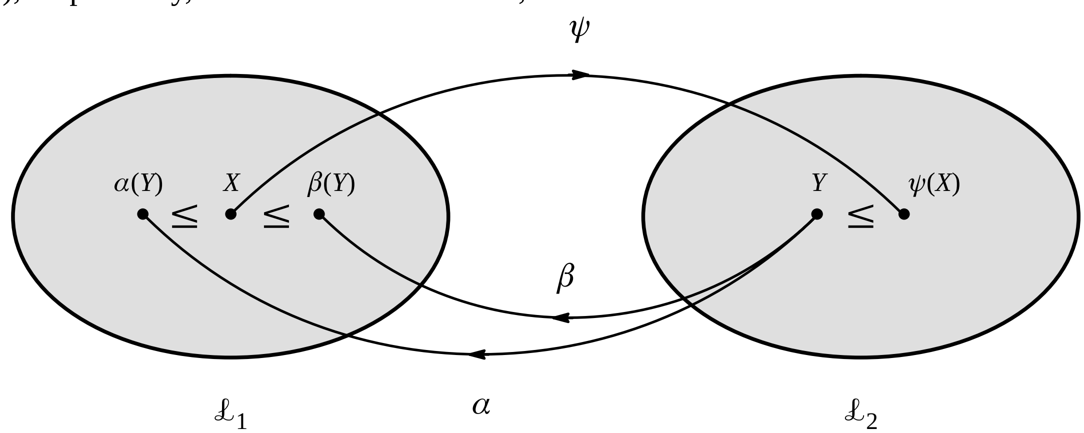

Figure 1. Morphological connection $(\psi, (\alpha, \beta))$.

For complete lattices, the notion of morphological connection can be seen as a generalization of the notion of Galois connection (Birkhoff, 1967, p. 124).

In Theorem 1, the mappings $\overline{\alpha\mathbf{I}}$ and $\overline{\mathbf{O}\beta}$ are, respectively, erosions and anti–dilations. We say that $\overline{\alpha\mathbf{I}}$ and $\overline{\mathbf{O}\beta}$ are derived from the extremities of the interval function $[\alpha, \beta]$. Actually, the pair $(\overline{\alpha\mathbf{I}} \wedge \overline{\mathbf{O}\beta}, (\alpha, \beta))$ is a morphological connection between $\mathcal{L}_1$ and $\mathcal{L}_2$.

From Theorem 1, we see that any mapping ψ is the *supremum* of a set of mappings that are the *infimum* of an *erosion* and an *anti–dilation*. Such erosion and anti–dilation are derived from the extremities of an interval function that is less than or equal to the kernel of ψ and whose extremities are a dilation and an anti–dilation.

For a translation invariant set mapping ψ, the sup–inf decomposition of Theorem 1 can be written as

$$\psi = \bigvee\{(\bullet \ominus A) \wedge (\bullet \oplus B^{\mathrm{t\,c}})^{\mathrm{c}}: \\ [A, B] \subset \{X: \mathrm{o} \in \psi(X)\}\},$$

where B^{t} is the transpose of B relatively to the origin o. For a direct proof, see (Banon and Barrera, 1991).

In the same way, any mapping ψ from $\mathcal{L}_1$ to $\mathcal{L}_2$ has the following inf–sup constructive decomposition

$$\psi = \bigwedge\{\underline{\alpha\mathbf{I}} \vee \underline{\mathbf{O}\beta}: (\alpha, \beta) \in E\,{}^{\mathrm{a}}E \text{ and} \\ [\alpha, \beta] \leqslant \mathcal{K}\bullet(\psi)\}.$$

In the above decomposition, the mappings $\underline{\alpha\mathbf{I}}$ and $\underline{\mathbf{O}\beta}$ are, respectively, anti–erosions and dilations.

For a translation invariant set mapping ψ, the above inf–sup decomposition can be written as

$$\psi = \bigwedge\{(\bullet \oplus A^{\mathrm{t}}) \vee (\bullet \ominus B^{\mathrm{c}})^{\mathrm{c}}: \\ [A, B] \subset \{X: \mathrm{o} \notin \psi(X)\}\},$$

or, equivalently, by using the definition of dual mapping ψ^* ($\psi^*(X)=(\psi(X^{\mathrm{c}}))^{\mathrm{c}}$) and changing, respectively, A and B into B^{c} and A^{c},

$$\psi = \bigwedge\{(\bullet \oplus A^{\mathrm{t}}) \vee (\bullet \ominus B^{\mathrm{c}})^{\mathrm{c}}: \\ [A, B] \subset \{X: \mathrm{o} \in \psi^*(X)\}\},$$

For an increasing translation invariant set mapping ψ, we get the well known Matheron's decompositions (1975),

$$\psi = \bigvee\{\bullet \ominus A: \mathrm{o} \in \psi(A)\} \\ = \bigwedge\{\bullet \oplus A^{\mathrm{t}}: \mathrm{o} \in \psi^*(A)\}.$$

5. CONCLUSION

In the previous sections, we have presented two constructive decompositions for any mapping between lattices in terms of the elementary mappings of MM: erosions, dilations, anti–erosions and anti–dilations. Actually, the proposed decompositions are redundant in the sense that smaller families of such elementary mappings can be involved in the decompositions. The problem of a minimal decomposition (or minimal decompositions) have been studied in (Banon and Barrera, 1991, 1993) and can be related to the well known problem of Boolean function simplification.

The proposed decompositions apply to mappings between different lattices and so can be used to decompose many kinds of image transformations (not only translation invariant set mappings). For example, they apply to transformations between grayscale and binary images.

By choosing a partial order relation between the pixel positions, the proposed decomposition can be used, as well, to decompose the digital images themselves.

ACKNOWLEDGMENTS

The authors thank Prof. Jean Serra who has encouraged the present study.

This work has been supported by FAPESP (Fundação de Amparo a Pesquisa do Estado de São Paulo) under contract 91/3532–2.

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