A STOCHASTIC APPROACH TO THE PROBLEM OF SPACECRAFT OPTIMAL MANEUVERS

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Abstract. In this paper the problem of spacecraft orbit transfer with minimum fuel consumption is considered. A new version of the suboptimal and hybrid control approach of numerically treating the problem, where one can take into account the accuracy in the satisfaction of constraints is developed. To solve the nonlinear programming problem in each iteration, a stochastic version of the projection of the gradient method is used together with the well-known hybrid approach to find the optimal control in this kind of dynamic problem. For the maneuvers considered, the spacecraft is supposed to be in Keplerian motion perturbed by the thrusts whenever they are active. These thrusts...
are assumed to be of fixed magnitude (either low or high) and operating in an on-off mode. The solution is given in terms of the location of the "burning arcs", "time"-histories of thrust' attitude (pitch and yaw), final orbit acquired and fuel consumed. Numerical results are presented.

Key Words: Optimal Transfer, Stochastic Approximation, Optimal Control, Spacecraft Orbit Maneuver.

1 - INTRODUCTION

This paper comes in sequence of a previous one, where the authors (Prado and Rios-Neto, 1989) numerically solved the problem of spacecraft maneuvers with minimum fuel expenditure using suboptimal and optimal control associated with a nonlinear programming projection of the gradient method. Here, the same problem is considered with the difference that the constraints do not need to be exactly satisfied and a stochastic version of the projection of the gradient is used (Rios-Neto and Pinto, 1987). This is done to realistically treat the numerical inaccuracies and/or flexibility in
terms of tolerance in mission requirements, leading to situations where the final state is constrained to lie inside a given region, instead of being an exact value. The results of numerical simulations are compared to assess the fuel savings given by the stochastic approach. A more general study of the problem of orbit transfer maneuvers is available in Prado and Rios-Neto (1993).

2 - DEFINITION OF THE PROBLEM

The basic problem discussed in this paper is the problem of orbit transfer maneuvers. The objective of this problem is to modify the orbit of a given spacecraft. In the case considered in this paper, an initial and a final orbit around the Earth is completely specified. The problem is to find how to transfer the spacecraft between those two orbits in a such way that the fuel consumed is minimum. There is no time restriction involved here and the spacecraft can leave and arrive at any point in the given initial and final orbits. The maneuver is performed with the use of an engine that is able to deliver a thrust with constant magnitude and variable direction. The mechanism, time and fuel consumption to change the direction of the thrust is not considered in this paper.

3 - MODEL USED

The spacecraft is supposed to be in Keplerian motion controlled only by the thrusts, whenever they are active. This means that there are two types of motion:

i) A Keplerian orbit, that is an orbit obtained by assuming that the Earth's gravity (assumed to be a point of mass) is the only force acting on the spacecraft. This motion occurs when the thrusts are not firing;

ii) The motion governed by two forces: the Earth's gravity field (also assumed to be a point of mass) and the force delivered by the thrusts. This motion occurs during the time that the thrusts are firing.

Figure 1 shows this situation. \( F_E \) is the gravitational force of the Earth (assumed to be a point of mass) and \( F_t \) is the force given by the thrusts.
The thrusts are assumed to have the following characteristics:

i) Fixed magnitude: The force generated by them is always of constant magnitude during the maneuver. The value of this constant is a free parameter (an input for the algorithm developed here) that can be high or low;

ii) Constant Ejection Velocity: Meaning that the velocity of the gases ejected from the thrusts is constant. The importance of this fact can be better understood by examining Prado (1989);

iii) Either free or constrained angular motion: This means that the direction of the force given by the thrusts can be modified during the transfer. This direction can be specified by the angles $\vartheta$ and $\varphi$, called pitch (the angle between the direction of the thrust and the perpendicular to the line Earth-spacecraft) and yaw (the angle with respect to the orbital plane). The motion of those angles can be free or constrained (constant, linear variations, forbidden regions for firing the thrusts, etc.);
iv) Operation in on-off mode: It means that intermediate states are not allowed. The thrusts are either at zero or maximum level all the time.

The solution is given in terms of the time-histories of the thrusts (pitch and yaw angles) and fuel consumed. Several numbers of "thrusting arcs" (arcs with the thrusts active) are tested for each maneuver.

Instead of time, the "range angle" (the angle between the radius vector of the spacecraft and an arbitrary reference line in the orbital plane) is used as the independent variable.

4 - OPTIMAL CONTROL PROBLEM FORMULATION

The minimum fuel spacecraft maneuver can be treated as a typical optimal control problem, formulated as follows:

Objective Function: Let $M_f$, the final mass of the vehicle, to be maximized with respect to the control $u(\cdot)$;

Subject to:

\[ \dot{x} = f(x,u,s); \quad (1) \]
\[ C_e(x,u,s) = E_e; \quad (2) \]
\[ C_d(x,u,s) \leq E_d; \quad (3) \]
\[ h(x(t_f),t_f) = E_h, \quad t_0 \text{ and } x(t_0) \text{ given} \quad (4) \]

where $x$ is the state vector, $f(\cdot)$ is the right hand side of equations of motion, as in Biggs (1979) and Prado (1989); $s$ is the independent variable ($s_0 \leq s \leq s_f$), $C_e(\cdot)$ and $C_d(\cdot)$ are the algebraic dynamic constraints on state and control of dimensions $m_e$ and $m_d$; $h(\cdot)$ are the boundary constraints of dimension $m_h$; and $E_e$, $E_d$, $E_h$ error vectors satisfying:

\[ |E_{e,i}| \leq E_{e,i}^T, \quad i = 1, 2, 3, ..., m_e \quad (5) \]
\[ |E_{d,i}| \leq E_{d,i}^T, \quad i = 1, 2, 3, ..., m_d \quad (6) \]
\[ |E_{h,i}| \leq E_{h,i}^T, \quad i = 1, 2, 3, ..., m_h \quad (7) \]

where the fixed given values $E_{e,i}^T$, $E_{d,i}^T$, $E_{h,i}^T$, characterizes the region around zero within which errors are considered tolerable.

5 - SUBOPTIMAL METHOD
In this approach (Biggs, 1978; Rios-Neto and Ceballos, 1979; Rios-Neto and Bambace, 1981; Ceballos and Rios-Neto, 1981; Prado and Rios-Neto, 1989), parametrization is used for an approximation of the control (angles of pitch (\(u_1\)) and yaw (\(u_2\))), and for the problem at hand:

\[
u_1 = p_1 + p_2 * (s - s_i) \tag{8}\]

\[
u_2 = p_3 + p_4 * (s - s_i) \tag{9}\]

where \(p_1, p_2, p_3, p_4\) are the parameters to be found; \(s\) is the instantaneous range angle and \(s_i\) is the range angle of the instant when the motor is turned-on.

With this, for each "burning arc" in the maneuver, there is a set of six variables to be optimized (start and end of thrusting and the four parameters for the angles of pitch and yaw). Note that the number of "burning arcs" is chosen "a priori".

With control parametrization, the problem is reduced to one of nonlinear programming, which will be solved by the stochastic version of the projection of the gradient method.

6 - OPTIMAL METHOD

This approach is based on Optimal Control Theory (Bryson, 1975). First order necessary conditions for a local minimum are used to obtain the adjoint equations and the Pontryagin's Maximum Principle to obtain the control angles at each range angle, leading to a "Two Point Boundary Value Problem" (TPBVP), where the difficulty is to find the initial values of the Lagrange multipliers. The treatment given here (Biggs, 1979) is the hybrid approach of guessing a set of values, integrating numerically all the differential equations and then searching for a new set of values, based on a nonlinear programming algorithm. With this approach, the problem is again reduced to parametric optimization, as in the suboptimal method, with the difference that the angles' parameters are replaced by the initial values of the Lagrange multipliers, as variables to be optimized.

The method by Biggs (1979) was used, where the "adjoint-control" transformation is performed and instead of the initial values of Lagrange multipliers one guesses control angles and their rates at the beginning of thrusting. With this, it is easier to find a good initial guess, and the convergence is faster. This hybrid approach has the advantage that, since the Lagrange multipliers remain constant during the "ballistic arcs", it is necessary to guess values of the control angles and its
rates only for the first "burning arc". This transformation reduces very much the number of variables to be optimized and, in consequence, the time of convergence.

7 - NUMERICAL METHOD

To solve the nonlinear programming problem, the stochastic version of the projection of the gradient method (Rios-Neto and Pinto, 1987) was used.

Its general scheme is resumed in what follows:

Given a value \( \mathbf{p} \) of the searched vector of parameters, from an initial guess or from an immediately previous iteration, a first order, direct search approach is adopted in a typical iteration to determine an approximate solution for the increment \( \Delta \mathbf{p} \) in the problem:

Minimize: \[ J(\mathbf{p} + \Delta \mathbf{p}) \] (10)

Subject to: \[ \mathbf{c}(\mathbf{p} + \Delta \mathbf{p}) = \alpha \mathbf{c}(\mathbf{p}) + \mathbf{e} \] (11)
\[ \mathbf{d}(\mathbf{p} + \Delta \mathbf{p}) = \beta \mathbf{d}(\mathbf{p}) + \mathbf{d} \] (12)

where \( J(\mathbf{p}) \) is the objective function; \( \mathbf{c}(\mathbf{p}) \) the equality constraints; \( \mathbf{d}(\mathbf{p}) \) the active inequality constraints at \( \mathbf{p} \); and \( 0 \leq \alpha < 1, 0 \leq \beta < 1 \) are chosen close enough to one to lead to increments \( \Delta \mathbf{p} \) of a first order of magnitude.

Linearized approximations are taken for the left hand sides of Equations (11) and (12) together with a stochastic interpretation for the errors \( \mathbf{e} \) and \( \mathbf{d} \), resulting in:

\[ (\alpha - 1) \mathbf{c}(\mathbf{p}) = (\mathbf{d}[\mathbf{c}(\mathbf{p})]/d_{\mathbf{p}}) \Delta \mathbf{p} + \mathbf{e} \] (13)
\[ (\beta - 1) \mathbf{d}(\mathbf{p}) = (\mathbf{d}[\mathbf{d}(\mathbf{p})]/d_{\mathbf{p}}) \Delta \mathbf{p} + \mathbf{d} \] (14)

where \( \mathbf{e} \) and \( \mathbf{d} \) are now assumed to be zero mean uniformly distributed errors, modeled as:

\( \mathbb{E}[\mathbf{e} \mathbf{e}^T] = \text{diag } [ e_i, i = 1,2,\ldots,m_e ] \)
\( \mathbb{E}[\mathbf{d} \mathbf{d}^T] = \text{diag } [ d_i, i = 1,2,\ldots,m_d ] \)

where \( \mathbb{E}[.] \) indicate the expected value of its argument.

The condition of Equation (10) is approximated by the following "a priori information":

\[ -\mathbf{g} \cdot \nabla J(\mathbf{p}) = \Delta \mathbf{p} + \mathbf{n} \] (15)

where \( \mathbf{g} \geq 0 \) is to be adjusted to guarantee a first order of magnitude for the increment, that is, such that \( \Delta \mathbf{p} \) is small enough to permit the use of a linearized representation of \( J(\mathbf{p} + \Delta \mathbf{p}) \); and \( \mathbf{n} \) is taken
as a zero mean uniformly distributed random vector, modelling the a priori searching error in the direction of the gradient $\nabla J(p)$, with:

$$E[mT] = \bar{P}$$

as its diagonal covariance matrix. The values of the variances in $\bar{P}$ are chosen such as to characterize an "adequate order of magnitude" for the dispersion of $n$. The diagonal form adopted is to model the assumption that it is not imposed any a priori correlation between the errors in the gradient components.

The simultaneous consideration of conditions of Equations (13), (14) and (15) characterize a problem of parameter estimation, which in a compact notation can be put as follows:

$$\bar{U} = U + n$$

(16)

$$Y = MU + V$$

(17)

where $\bar{U} \triangleq \Delta g \nabla J^T(p)$ is the "a priori information"; $\bar{U} \triangleq \Lambda p$;

$Y \triangleq [(\alpha - 1)Ce^T(p) : (\beta - 1)Cd^T(p)]$ is the observation vector;

$MT \triangleq [(d(Ce(p))/dp)^T : (d(Cd(p))/dp)^T]; \quad V^T = [Ee^T : Ed^T]$. Adopting a criterion of linear, minimum variance estimation, the optimal search increment can be determined using the classical Gauss-Markov estimator, which in Kalman form (e. g. Jazwinski, 1970) gives:

$$\hat{U} = \bar{U} + K(Y - MU)$$

(18)

$$P = \bar{P} - KM\bar{P}$$

(19)

$$K = \bar{P}MT(M\bar{P}MT + R)^{-1}$$

(20)

where $\bar{P}$ is defined as before; $R \triangleq E[VV^T] = \text{diag}[R_k; k = 1,2,\ldots,m_e+m_b]$; and $P$ has the meaning of being the covariance matrix of the errors in the components estimates of $\bar{U}$, i. e.:

$$P = E[(U - \\hat{U})(U - \\hat{U})^T]$$

(21)

To build a numerical algorithm using the proposed procedure, the following types of iterations are considered:

(i) Initial phase of acquisition of constraints, when starting from a feasible point that satisfies the inequality constraints, the search is done to capture the equality constraints, including those inequality constraints that eventually became active along this phase;
(ii) Search of the minimum, when from a point that satisfies the constraints in the limits of the tolerable errors $V$ in Equation (17), the search is done to take the objective function (Equation (10)) to get nearer to the minimum; this search is conducted relaxing the order of magnitude of the error bounds around the constraints;

(iii) Restoration of the constraints, when from a point that resulted from a type (ii) iteration, the search is done to restore constraints satisfaction, within the limits imposed by the error $V$ in Equation (17).

Rios-Neto and Pinto (1987) suggest how to choose good values for the numerical parameters, that must be different for each type of iteration.

8 - SIMULATIONS AND NUMERICAL TESTS

The algorithm was coded in single precision FORTRAN IV, and the calculations were performed at the National Institute for Space Research (INPE) in a Burroughs 6800 computer.

To verify the algorithm proposed, the maneuver for the initial phase (orbit transfer) of the First Brazilian Remote Sensing Satellite was simulated. These results were compared with the ones obtained by the deterministic version (Prado and Rios-Neto, 1989), without flexibility in constraint's satisfaction.

This transfer phase will occur with the data given in Table 1 (Carrara and Souza, 1988), Table 2 (Prado, 1989) and Table 3 (Prado, 1989).
Table 1 - Data for Transfer Phase of the First Brazilian Remote Sensing Satellite Mission

<table>
<thead>
<tr>
<th>Orbits</th>
<th>Initial</th>
<th>Final</th>
</tr>
</thead>
<tbody>
<tr>
<td>Semi-major axis</td>
<td>6768.14</td>
<td>7017.89</td>
</tr>
<tr>
<td>Eccentricity</td>
<td>0.00591</td>
<td>0.000</td>
</tr>
<tr>
<td>Inclination (degrees)</td>
<td>97.44</td>
<td>97.94</td>
</tr>
<tr>
<td>Ascending Node (degrees)</td>
<td>67.27</td>
<td>Free</td>
</tr>
<tr>
<td>Argument of perigee (degrees)</td>
<td>97.66</td>
<td>Free</td>
</tr>
<tr>
<td>Mean Anomaly (degrees)</td>
<td>270.0</td>
<td>Free</td>
</tr>
</tbody>
</table>

Vehicle data: Initial mass: 170 kg; Thrust: 4.0 N
Fuel used: Hydrazine

Table 2 - Errors Allowed for Final Keplerian Elements

<table>
<thead>
<tr>
<th>Semi-major axis</th>
<th>5.0 Km</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eccentricity</td>
<td>0.001</td>
</tr>
<tr>
<td>Inclination</td>
<td>0.01 deg</td>
</tr>
</tbody>
</table>

Table 3 - Algorithm Parameters

<table>
<thead>
<tr>
<th>---</th>
<th>Constraints Restoration</th>
<th>Minimum Search</th>
</tr>
</thead>
<tbody>
<tr>
<td>α</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>β</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>P</td>
<td>diag [10000:....:10000]</td>
<td>diag [0.01:....:0.01]</td>
</tr>
<tr>
<td>g</td>
<td>0.2</td>
<td>0.2</td>
</tr>
</tbody>
</table>

The choice of the number of "burning arcs" was done after testing different values and concluding for 8 as being a good number (Prado, 1989).

The solutions found ("time"-histories of the thrusting angles) are showed in Table 4 (suboptimal method) and Fig. 2 (optimal method). The comparisons with deterministic methods are showed in Table 5.
Table 4 - Transfer Phase with Suboptimal Scheme

<table>
<thead>
<tr>
<th>Arc</th>
<th>s₁(deg)</th>
<th>s₂(deg)</th>
<th>p₁(deg)</th>
<th>p₁(deg)</th>
<th>p₂(deg)</th>
<th>p₄(deg)</th>
<th>Fuel(kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>525.9</td>
<td>578.6</td>
<td>0.4</td>
<td>-19.2</td>
<td>0.010</td>
<td>-0.126</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>1053.5</td>
<td>1097.2</td>
<td>5.1</td>
<td>32.9</td>
<td>-0.147</td>
<td>-0.085</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>1619.0</td>
<td>1667.4</td>
<td>1.3</td>
<td>-36.8</td>
<td>-0.002</td>
<td>0.533</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>2132.7</td>
<td>2185.6</td>
<td>5.6</td>
<td>33.1</td>
<td>-0.138</td>
<td>-0.071</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>2326.4</td>
<td>2372.1</td>
<td>0.2</td>
<td>-20.5</td>
<td>0.010</td>
<td>-0.129</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>2852.7</td>
<td>2905.5</td>
<td>5.9</td>
<td>33.9</td>
<td>-0.150</td>
<td>-0.096</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>3418.7</td>
<td>3467.1</td>
<td>1.2</td>
<td>-36.9</td>
<td>-0.002</td>
<td>0.532</td>
<td>-</td>
</tr>
<tr>
<td>8</td>
<td>3932.6</td>
<td>3985.3</td>
<td>5.6</td>
<td>33.2</td>
<td>-0.138</td>
<td>-0.080</td>
<td>11.66</td>
</tr>
</tbody>
</table>

Table 5 - Fuel Expenditure Comparisons

<table>
<thead>
<tr>
<th></th>
<th>Fuel Consumed (optimal)</th>
<th>Fuel Consumed (suboptimal)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stochastic approach</td>
<td>11.62 Kg</td>
<td>11.66 Kg</td>
</tr>
<tr>
<td>Deterministic approach</td>
<td>11.87 Kg</td>
<td>11.93 Kg</td>
</tr>
</tbody>
</table>

9 - CONCLUSIONS

The problem of spacecraft maneuvers with minimum fuel consumption and consideration of accuracy tolerance in constraint's satisfaction was treated and solved using the new nonlinear programming algorithm proposed by Rios-Neto and Pinto (1987).

The results showed that some fuel can be saved by exploring tolerable errors allowed for constraint's satisfaction. The amount saved in both examples is not negligible, since it represents half of the amount necessary for one orbit correction (Carrara, 1988), and since it certainly could be higher with more flexibility in constraint's satisfaction.

ACKNOWLEDGMENTS

The authors are grateful to the National Institute for Space Research (INPE) for the support given during the development of this work, to Dr. Kondapalli Rama Rao for the help in the use of the numerical integration subroutines implemented by him (Rama Rao, 1984 and 1986) and to M.Sc. Valdemir Carrara for the help in the use of the graphic subroutines implemented by him (Carrara, 1984).

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Fuel consumed: 11.62 kg

Fig. 2 – “Time-histories” for optimal stochastic maneuvers.
Fig. 2 – “Time-histories” for optimal stochastic maneuvers (Cont.).
Fig. 2 – “Time-histories” for optimal stochastic maneuvers (Cont.).
Fuel Consumed: 11.62 kg

Fig. 2 "Time"-histories for optimal stochastic maneuvers with 8 "burningarcs".