CONSTRUCTIVE DECOMPOSITION OF FUZZY MEASURES IN TERMS OF POSSIBILITY OR NECESSITY MEASURES

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Content

INTRODUCTION

The search for uncertainty models culminates at the end of the seventies with the paper by Zadeh on the *Theory of Possibility*.

Among the popular models are the one based on *probability measures*, *possibility measures* and its dual version based on *necessity measures*.

These models are interesting because, as pointed out by Nguyen, they can be characterized by *distributions*.

In this paper, it is shown that one can *decompose* ***any*** fuzzy measure as a conjunctive combination of possibility measures *or* as a disjunctive combination of necessity measures.

These decompositions are a *special case* of the decompositions of mappings between complete lattices presented by Banon & Barrera in the framework of *Mathematical Morphology*.

The combination of weak sources of information modeled by possibility measures may represent *exactly* a stronger source of information modeled by a probability. This is interesting as part of the debate between probability and fuzzy sets.

AXIOMATIC DEFINITION OF POSSIBILITY AND NECESSITY MEASURES

Let Ω be a nonempty set and I the interval $[0,1]$ of real numbers.

A *fuzzy measure* μ on $\mathcal{P}(\Omega)$ is an increasing mapping from $(\mathcal{P}, \subset)$ to $(I, \leq)$ such that $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$

A fuzzy measure μ on $\mathcal{P}(\Omega)$ is an *possiblity measure* (or a *dilation*) iff

$$\max \mu(\mathcal{X}) = \mu(\bigcup \mathcal{X}) \quad (\mathcal{X} \subset \mathcal{P})$$

A fuzzy measure μ on $\mathcal{P}(\Omega)$ is an *necessity measure (or an* erosion*)* iff

$$\mu(\bigcap \mathcal{X}) = \min \mu(\mathcal{X}) \quad (\mathcal{X} \subset \mathcal{P})$$

where $\mu(\mathcal{X})$ denotes the image of $\mathcal{X}$ through μ.

We denote by Π the set of possibility measures and by N the set of necessity measures.

EXAMPLES OF POSSIBILITY AND NECESSITY MEASURES

$$\pi_{U,v}(X) = \begin{cases} 0 \text{ if } X = \emptyset \\ v \text{ if } X \subset U \text{ and } X \neq \emptyset \\ 1 \text{ otherwise} \end{cases} \quad (X \in \mathcal{P})$$

and

$$\nu_{U,v}(X) = \begin{cases} 1 \text{ if } X = \Omega \\ v \text{ if } U \subset X \text{ and } X \neq \Omega \\ 0 \text{ otherwise,} \end{cases} \quad (X \in \mathcal{P})$$

the latter is called *simple support function* by Shafer.

CHARACTERIZATION OF POSSIBILITY MEASURES

Proposition 1 (characterization of possibilities) – The mapping $f \mapsto (\pi,\beta)$ from the complete lattice I^{Ω} of possibility distributions on Ω to the complete lattice of G.c. between $(\mathcal{P}, \subset)$ and $(I, \geq)$, defined by

$$\pi(X) = \max f(X) \quad (X \in \mathcal{P})$$

$$\beta(t) = \{\omega \in \Omega : f(\omega) \leq t\} \quad (t \in I),$$

is a dual lattice–isomorphism. Its inverse $(\pi,\beta) \mapsto f$ is given by

$$f(\omega) = \pi(\{\omega\}) \quad (\omega \in \Omega)$$

or equivalently

$$f(\omega) = f_{\beta}(\omega) \triangleq \min\{t \in I : \omega \in \beta(t)\} \quad (\omega \in \Omega).$$

Using the fact that (π,β) is a G.c.; then, for any $X \in \mathcal{P}$,

$$\pi(X) = \pi_{\beta}(X) \triangleq \min\{t \in I : X \subset \beta(t)\},$$

that is, any π can be expressed in terms of an erosion in $\mathrm{E}(I, \mathcal{P})$.

CHARACTERIZATION OF NECESSITY MEASURES

Proposition 2 (characterization of necessities) – The mapping $f \mapsto (\nu, \alpha)$ from the complete lattice I^{Ω} of necessity distributions on Ω to the complete lattice of G.c. between $(\mathcal{P}, \supset)$ and $(I, \leq)$, defined by

$$\nu(X) = \min f(X) \quad (X \in \mathcal{P})$$

$$\alpha(t) = \{\omega \in \Omega :\ t \leq f(\omega)\}^{c} \quad (t \in I),$$

is a lattice–isomorphism. Its inverse $(\nu, \alpha) \mapsto f$ is given by

$$f(\omega) = \nu(\{\omega\}^{c}) \quad (\omega \in \Omega)$$

or equivalently

$$f(\omega) = f_{\alpha}(\omega) \triangleq \max\{t \in I :\ \omega \notin \alpha(t)\} \quad (\omega \in \Omega).$$

Using the fact that (ν, α) is a G.c.; then, for any $X \in \mathcal{P}$,

$$\nu(X) = \nu_{\alpha}(X) \triangleq \max\{t \in I :\ \alpha(t) \subset X\}).$$

that is, any ν can be expressed in terms of a dilation in $\Delta(I, \mathcal{P})$.

FUZZY MEASURE DECOMPOSITION

(1/2)

Special case of Banon & Barrera's lemma:

Lemma 3 (fuzzy measure decomposition) – Any fuzzy measure μ on $\mathcal{P}(\Omega)$ can be written

$$\mu = \inf\{\pi \in \Pi : \mu \leq \pi\}$$

$$= \sup\{\nu \in \mathrm{N} : \nu \leq \mu\}.$$

Due to Serra:

Theorem 4 (constructive decomposition of fuzzy measures) – Any fuzzy measure μ on $\mathcal{P}(\Omega)$ can be written

$$\mu = \wedge(\pi_{U, \mu(U)} : U \in \mathcal{P})$$

$$= \vee(\nu_{U, \mu(U)} : U \in \mathcal{P}).$$

FUZZY MEASURE DECOMPOSITION

(2/2)

Special case of Banon & Barrera's theorem:

Theorem 5 (general constructive decomposition of fuzzy measures) – Any fuzzy measure μ on $\mathcal{P}(\Omega)$ can be written

$$\mu = \wedge(\pi_\beta : \beta \in \mathrm{E}(I, \mathcal{P}) \text{ and } \mu\beta \leq \iota)$$

$$= \vee(\nu_\alpha : \alpha \in \Delta(I, \mathcal{P}) \text{ and } \iota \leq \mu\alpha),$$

where ι is the identity mapping from I to I.

In terms of possibility and necessity distributions, we have, for any $X \in \mathcal{P}$,

$$\mu(X) = \wedge(\max f_\beta(X) : \beta \in \mathrm{E}(I, \mathcal{P}) \text{ and } \mu\beta \leq \iota)$$

$$= \vee(\min f_\alpha(X) : \alpha \in \Delta(I, \mathcal{P}) \text{ and } \iota \leq \mu\alpha).$$

MINIMAL FUZZY MEASURE DECOMPOSITION

If a subset E' of E, such that, for any $\beta' \in E'$, $\mu\beta' \leq \iota$ is such that for any $\beta \in E$ such that $\mu\beta \leq \iota$, there exists $\beta' \in E'$ such that $\beta \leq \beta'$, then

$$\mu = \wedge (\pi_\beta : \beta \in E').$$

If a subset Δ' of Δ, such that, for any $\alpha' \in \Delta'$, $\iota \leq \mu\alpha'$, is such that for any $\alpha \in \Delta$ such that $\iota \leq \mu\alpha$, there exists $\alpha' \in \Delta'$ such that $\alpha' \leq \alpha$, then

$$\mu = \vee (\nu_\alpha : \alpha \in \Delta').$$

When the elements of E' (resp., Δ') are all *maximal* (resp., *minimal*) then the above decompositions are said to be *minimal*.

EXAMPLE OF A PROBABILITY MEASURE DECOMPOSITION IN TERMS OF POSSIBILITY MEASURES

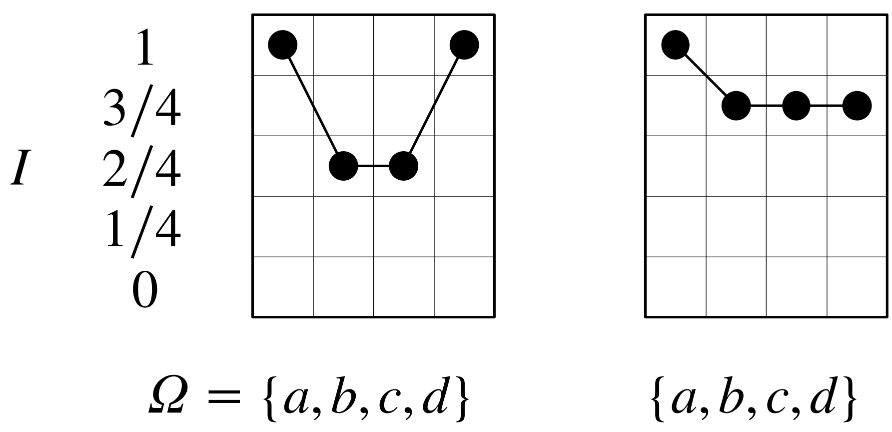
(1/3)

We now assume that Ω is finite and $n = \#\Omega$.

Let μ be the probability on $\mathcal{P}(\Omega)$ given by

$$\mu(X) = (\#X)/n \quad (X \in \mathcal{P}(\Omega)).$$

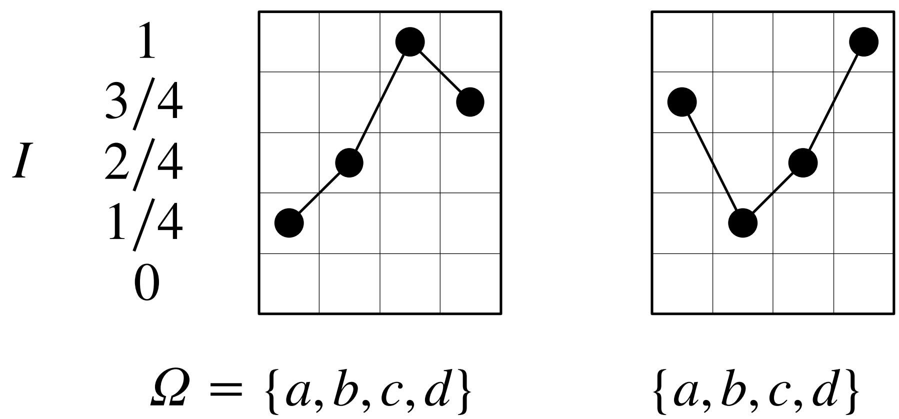
The decomposition derived from Th. 4 has 2^n terms.



EXAMPLE OF A PROBABILITY MEASURE DECOMPOSITION IN TERMS OF POSSIBILITY MEASURES

(2/3)

The minimal decomposition derived from Th. 5 has $n!$ terms.



EXAMPLE OF A PROBABILITY MEASURE DECOMPOSITION IN TERMS OF POSSIBILITY MEASURES

(3/3)

There are *simpler* decompositions.

For example, for $n = 4$ we can use 6 distributions.