Set Mapping Decompositions by Mathematical Morphology

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1 Introduction

Four well-known elementary set mappings are: erosion, dilation, anti-erosion (the composition of an erosion with the complementation), and anti-dilation (the composition of a dilation with the complementation). These mappings are called here the elementary mappings of mathematical morphology.

Since the sixties, special machines have been developed to efficiently perform the elementary mappings of mathematical morphology. Two well-known examples can be found in Serra (1967) and Sternberg (1982). Nowadays, new architectures have been designed and implemented within chips (Husson et. al., 1988). These machines have shown their adequacy to extract binary image information by solving hundreds of image analysis problems such as edge extraction, separation of overlapping objects, clustering of near objects, closing of holes, etc. (Serra, 1982).

A natural question arises: What class of mappings can be realized by these machines? The answer involves study of the fundamental structure problem of algebra. As Birkhoff (1967*a*) has said:

> The fundamental ***structure problem*** of algebra is that of ***analyzing*** a given algebraic system into simpler ***components***, from which the given system can be reconstructed by synthesis . . . . Such decomposition theorems reveal the structure of a given algebraic system.

In fact, some vector decompositions in vector spaces are well-known, as the function decomposition by a set of sines and cosines or the polynomial decomposition by a set of monomes. In the same way, the decomposition of set mappings in terms of the elementary mappings of mathematical morphology arises.

The first mapping decomposition by a set of erosions is due to Matheron (1975*a*), who introduced, for the translation invariant (t.i.) mappings, the concept of mapping kernel (a subcollection of subsets that characterizes the mapping) and proved that any ***increasing*** set mapping can be decomposed as the supremum of erosions defined from the mapping kernel.

Matheron's result was simplified by Maragos (1985, 1989) and Dougherty and Giardina (1986), who introduced the concept of mapping basis (a subcollection of the mapping kernel). Maragos also proved that any upper semicontinuous (u.s.c.) increasing t.i. mapping can be decomposed as the supremum of erosions defined from the mapping basis.

The hypothesis of growth was removed by Banon and Barrera (1991), who proved that any t.i. set mapping can be decomposed as the supremum of sup-generating mappings (the infimum of an erosion and an anti-dilation) defined from the mapping kernel. They also simplified this result by extending the concept of basis and proving that a u.s.c. t.i. mapping can be decomposed as the supremum of sup-generating mappings defined from the mapping basis.,

A generalization of the concepts of kernel and basis will be presented here in order to prove that any set mapping (not necessarily t.i.) can be decomposed by a set of (non-t.i.) sup-generating mappings.

In Section 2, we recall the general definitions of set erosions and dilations, as well as some of their properties. The representation theorem for set erosions and dilations is proved by using almost directly some results of fuzzy set theory.

In Section 3, the concept of kernel is generalized and the proofs of two decomposition theorems are given, the second one being derived from the first one by a duality principle.

In Section 4, the concept of basis is generalized in order to give the minimal decompositions. Some algebraic and topological aspects are discussed.

In Section 5, we show that Banon and Barrera's decomposition for t.i. mappings is a particular case of the new general setting presented here.

Finally, in Section 6, the decomposition of two simple mappings are given: a morphological opening defined through a graph, and an adaptive shape recognizer.

2 Review of Erosions and Dilations

Let E be a nonempty set and let $\mathcal{P}(E)$, or simply $\mathcal{P}$, be the collection of all subsets of E. Let $\subset$ be the usual inclusion relation between sets. Let X^c be the complementary set of a subset X of E. We know that $(\mathcal{P}, \subset)$ is a complete (Boolean) lattice (Birkhoff, 1967*b*). Let $\mathcal{X} \subset \mathcal{P}$. Then $\cap\mathcal{X}$, the intersection of the subsets of E in $\mathcal{X}$, is the infimum of $\mathcal{X}$ in $(\mathcal{P}, \subset)$ and $\cup\mathcal{X}$, the union of the subsets of E in $\mathcal{X}$, is the supremum of $\mathcal{X}$ in $(\mathcal{P}, \subset)$.

Let Ψ be the collection of set mappings from $\mathcal{P}$ to $\mathcal{P}$. The mappings in Ψ will be denoted by lower case Greek letters $\alpha, \beta, \gamma, \ldots$. Finally, for any $\psi \in \Psi$, let $\psi(\mathcal{X})$ be the collection given by

$$\psi(\mathcal{X}) = \{Y \in \mathcal{P} : Y = \psi(X), X \in \mathcal{X}\}$$

Definition 1 *A mapping $\psi \in \Psi$ is an* **erosion** *in $(\mathcal{P}, \subset)$ if and only if*

$$\psi(\cap\mathcal{X}) = \cap\psi(\mathcal{X}), \text{ for } \mathcal{X} \subset \mathcal{P}$$

Definition 2 *A mapping $\psi \in \Psi$ is a* **dilation** *in $(\mathcal{P}, \subset)$ if and only if*

$$\psi(\cup\mathcal{X}) = \cup\psi(\mathcal{X}), \text{ for } \mathcal{X} \subset \mathcal{P}$$

These two definitions correspond to the particular case of erosion and dilation definitions for set mappings. The general definitions introduced by Serra (1988*a*) apply to any mapping between complete lattices and say that erosion and dilation commute, respectively, with infimum and supremum.

Let $\mathcal{E}$ and $\mathcal{D}$ be, respectively, the set of erosions and dilations. The mappings in $\mathcal{E}$ and $\mathcal{D}$ will be denoted, respectively, by ϵ and δ.

The set Ψ inherits the complete lattice structure of $(\mathcal{P}, \subset)$ by setting

$$\psi_1 < \psi_2 \quad \Leftrightarrow \quad \psi_1(X) \subset \psi_2(X), \ \textit{for } X \in \mathcal{P}$$

We know that $(\mathcal{E}, <)$ and $(\mathcal{D}, <)$ are complete lattices (Heijmans and Ronse, 1990*a*). The supremum for dilations is the supremum in $(\Psi, <)$, but the infimum is not. Similarly, the infimum for erosions is the infimum in $(\Psi, <)$, but the supremum is not.

A mapping ψ is *increasing (isotone)* in $(\mathcal{P}, \subset)$ if and only if one of the following three equivalent statements is satisfied (Heijman and Ronse, 1990*b*):

$$X \subset Y \quad \Rightarrow \quad \psi(X) \subset \psi(Y), \ \textit{for } X, Y \in \mathcal{P}$$

$$\psi(\cap \mathcal{X}) \subset \cap \psi(\mathcal{X}), \ \textit{for } \mathcal{X} \subset \mathcal{P}$$

$$\psi(\cup \mathcal{X}) \supset \cup \psi(\mathcal{X}), \ \textit{for } \mathcal{X} \subset \mathcal{P}$$

In particular, erosions and dilations are increasing mappings.

Definition 3 *An* **adjunction** *in $\mathcal{P}$ is a couple (ϵ, δ) of Ψ^2 such that ϵ and δ are increasing and*

$$\delta(\epsilon(X)) \subset X \subset \epsilon(\delta(X)), \ \textit{for } X \in \mathcal{P}$$

or, equivalently, a couple (ϵ, δ) of Ψ^2 such that

$$\delta(Y) \subset X \Leftrightarrow Y \subset \epsilon(X), \ \textit{for } X, Y \in \mathcal{P}$$

Proposition 1 *The set of adjunctions in $\mathcal{P}$ constitutes a dual isomorphism F between $(\mathcal{E}, <)$ and $(\mathcal{D}, <)$ which is given by*

$$F(\epsilon)(Y) = \cap\{X \in \mathcal{P} : Y \subset \epsilon(X)\}, \ \textit{for } Y \in \mathcal{P}, \ \epsilon \in \mathcal{E}$$

and has the inverse

$$G(\delta)(X) = \cup\{Y \in \mathcal{P} : \delta(Y) \subset X\}, \ \textit{for } X \in \mathcal{P}, \ \delta \in \mathcal{D}$$

For a proof of this property see (Heijmans and Ronse, 1990*c*).

Actually, an adjunction in $\mathcal{P}$ is exactly a Galois connection between $(\mathcal{P}, \supset)$ and $(\mathcal{P}, \subset)$ (Birkhoff, 1967*c*).

A mapping ψ is *idempotent* if and only if

$$\psi(X) = \psi(\psi(X)), \ \textit{for } X \in \mathcal{P}$$

Following Serra (1988*b*), an idempotent and increasing mapping is an *opening* if and only if, for any $X \subset E$, $\psi(X) \subset X$ (i.e., ψ is anti-extensive); and a *closing* (or, equivalently, a *closure operation* (Birkhoff, 1967*b*) or a *closure operator* (Everett, 1944)) if and only if, for any $X \subset E$, $X \subset \psi(X)$ (i.e., ψ is extensive).

If (ϵ, δ) is an adjunction then $\delta\epsilon$ is an opening and $\epsilon\delta$ is a closing. These two mappings are called morphological opening and closing by Serra (1988*a*).

The invariant subsets X of a morphological closing $\epsilon\delta$, that is, such that $X = \epsilon\delta(X)$, are said to be *closed* with respect to (ϵ, δ). In the same way, the invariant subsets of a morphological opening $\delta\epsilon$ are said to be *open* with respect to (ϵ, δ).

The subsets closed with respect to any adjunction form a complete lattice in which infimum means intersection (Birkhoff, 1967*d*). Similarly, by the duality principle, the subsets that are open with respect to any adjunction form a complete lattice in which supremum means union.

Proposition 2 *An adjunction* (ϵ, δ) *in* $\mathcal{P}$ *gives an (order) isomorphism between the complete lattices of the corresponding open and closed sets of* $\mathcal{P}$.

For the proof of Proposition 2 see Birkhoff (1967*e*).

Finally, we recall (Matheron, 1975*b*), that X is open (respectively, Y is closed) with respect to (ϵ, δ) if and only if there exists a $Y \in \mathcal{P}$ (respectively, $X \in \mathcal{P}$) such that $X = \delta(Y)$ (respectively, $Y = \epsilon(X)$. In other words, the collection of open (respectively, closed) subsets is the image of $\mathcal{P}$ by δ (respectively, ϵ).

The set of adjunctions in $\mathcal{P}$ is a complete lattice of couples (ϵ, δ) partially ordered by the rule that $(\epsilon_1, \delta_1) < (\epsilon_2, \delta_2)$ if and only if $\delta_1 < \delta_2$ (or, equivalently, $\epsilon_1 < \epsilon_2$ from Proposition 1).

Let $\mathcal{P}^E$ be the set of mappings from E to $\mathcal{P}$.

The mappings in $\mathcal{P}^E$ will be called *structuring functions* and will be denoted by lower case letters $a, b, c, \dots$.

Proposition 3 Representation Theorem for Erosions and Dilations. *The complete lattice* $\mathcal{P}^E$ *of structuring functions is isomorphic to the complete lattice of adjunctions in* $\mathcal{P}$, *by* $a \to (\epsilon, \delta)$ *with* $\epsilon(X) = \epsilon_a(X) = \{y \in E : X \supset a(y)\}$ *for any* $X \in \mathcal{P}$ *and* $\delta(Y) = \delta_a(Y) = \cup\{a(y) : y \in Y\}$ *for any* $Y \in \mathcal{P}$, *and by* $(\epsilon, \delta) \to a$ *with* $a(y) = a_{(\epsilon,\delta)}(y) = \delta(\{y\})$ *for any* $y \in E$.

For the proof of Proposition 3 see (Achache, 1982). In Achache the result is relative to Galois connections between $(\mathcal{P}, \subset)$ and $(\mathcal{P}, \supset)$. Related results are given in Serra (1988*c*).

In fuzzy set theory, the structuring functions from E to $\mathcal{P}$ are called *fuzzy subsets* of E (or *L-fuzzy subsets* of E, since the valuation set $\mathcal{P}$ is a lattice) and up to a converse order relation the above representation theorem is equivalent to the Negoita and Ralescu representation theorem for fuzzy sets (1975).

By applying Proposition 1 the structuring function $a_{(\epsilon,\delta)}$ of the representation theorem can be defined equivalently from ϵ by

$$a_{(\epsilon,\delta)}(y) = F(\epsilon)(\{y\}) = \cap\{X \in \mathcal{P} : y \in \epsilon(X)\}, \ \textit{for } y \in E$$

Let a^s, the *transpose* of a structuring function a, be the structuring function defined by

$$a^s(x) = \{y \in E / x \in a(y)\}, \ \textit{for } x \in E$$

For any structuring function a, let a^c be the structuring function defined by

$$a^c(x) = (a(x))^c, \ \textit{for } x \in E$$

For any a in $\mathcal{P}^E$ we have $(a^s)^s = a$, $(a^c)^c = a$, and $(a^s)^c = (a^c)^s$.

The erosion ϵ_a and the dilation δ_a of the above representation theorem are called, respectively, the erosion and the dilation *by* the structuring function a, and can be defined equivalently by, for any $X \in \mathcal{P}$,

$$
\begin{aligned}
\epsilon_a(X) &= \{y \in E : \forall x \in X^c, x \in a^c(y)\} \\
&= \{y \in E : \forall x \in X^c, y \in (a^c)^s(x)\} \\
&= \cap\{(a^s)^c(x) : x \notin X\}
\end{aligned}
$$

and, for any $Y \in \mathcal{P}$,

$$
\begin{aligned}
\delta_d(Y) &= \{y \in E : \exists y \in Y, x \in a(y)\} \\
&= \{x \in E : \exists y \in Y, y \in a^s(x)\} \\
&= \{x \in E : a^s(x) \cap Y \neq \emptyset\}
\end{aligned}
$$

Let $\psi \in \Psi$ and ψ^c be the mapping defined by

$$\psi^c(X) = (\psi(X))^c, \ \textit{for } X \in \mathcal{P}$$

Let ψ^* be the *dual mapping* of ψ defined by

$$\psi^*(X) = \psi^c(X^c) = (\psi(X^c))^c, \ \textit{for } X \in \mathcal{P}$$

From the above, we observe that, for any $a \in \mathcal{P}^E$,

$$(\epsilon_a)^*(Y) = \cup\{a^s(y) : y \in Y\} = \delta_{a^s}(Y), \ \textit{for } Y \in \mathcal{P}$$

and

$$(\delta_a)^*(X) = \{y \in E : a^s(y) \cap Y^c = \emptyset\} = \epsilon_{a^s}(X), \ \textit{for } X \in \mathcal{P}$$

In other words, for any $a \in \mathcal{P}^E$, $(\epsilon_a)^* = \delta_{a^s}$ and $(\delta_a)^* = \epsilon_{a^s}$.

Following Serra (1987), if ϵ and δ are, respectively, erosion and dilation, the mapping ϵ^c and δ^c are called, respectively, *anti-erosion* and *anti-dilation* and we have, for any $a \in \mathcal{P}^E$, $((\epsilon_a)^c)^* = (\delta_{a^s})^c$ and $((\delta_a)^c)^* = (\epsilon_{a^s})^c$.

3 Decomposition Theorems

3.1 Decomposition by a Set of Sup-Generating Mappings

We will now be slightly more specific and consider, instead of the set Ψ of mappings ψ from $\mathcal{P}(E)$ to $\mathcal{P}(E)$, the set of restrictions $\psi/\mathcal{A}$ of ψ to the subcollection $\mathcal{A}$ of $\mathcal{P}(E)$ (i.e., $\mathcal{A} \subset \mathcal{P}(E)$), with at least two elements. In other words, we are considering the set $\mathcal{P}(E)^{\mathcal{A}}$ of the set mappings from $\mathcal{A}$ to $\mathcal{P}(E)$. Furthemore, by changing the role of $\mathcal{A}$ and E we consider also the set $\mathcal{P}(\mathcal{A})^E$ of the mappings from E to $\mathcal{P}(\mathcal{A})$. As Ψ, $\mathcal{P}(E)^{\mathcal{A}}$ and $\mathcal{P}(\mathcal{A})^E$ inherit the complete lattice structure of a power set; the order relation will be denoted in both cases by $<$. We will write $\vee$ and $\wedge$ for the supremum and infimum in $\mathcal{P}(E)^{\mathcal{A}}$ and $\mathcal{P}(\mathcal{A})^E$. The generic mappings in $\mathcal{P}(\mathcal{A})^E$ will be denoted by the lower case Greek letter ρ.

Let $\mathcal{K}$ and $\mathcal{L}$ be two mappings, respectively, from $\mathcal{P}(E)^{\mathcal{A}}$ to $\mathcal{P}(\mathcal{A})^E$ and from $\mathcal{P}(\mathcal{A})^E$ to $\mathcal{P}(E)^{\mathcal{A}}$ defined by

$$\mathcal{K}(\psi)(y) = \{X \in \mathcal{A} : y \in \psi(X)\}, \ \textit{for } y \in E, \psi \in \mathcal{P}(E)^{\mathcal{A}}$$

$$\mathcal{L}(\rho)(X) = \{y \in E : X \in \rho(Y)\}, \text{ for } X \in \mathcal{A}, \rho \in \mathcal{P}(\mathcal{A})^E$$

The mapping $\mathcal{K}(\psi)$ is called the *kernel* of ψ. This generalizes the notion of a kernel given by Matheron (1975*a*) for translation invariant set mappings (see Section 5). In the case that $\mathcal{A} = \mathcal{P}(E)$, we observe that the structuring function $a_{(\epsilon,\delta)}$ of the representation theorem of Section 2 can be obtained from the $\simeq \Lambda \mathcal{K}(\epsilon)$ kernel of the erosion ϵ by $a_{(\epsilon,\delta)} = \underset{\Lambda}{\,}\mathcal{K}(\epsilon)$.

Proposition 4 *The mapping $\mathcal{K}$ is a lattice-isomorphism between $(\mathcal{P}(E)^{\mathcal{A}}, <)$ and $(\mathcal{P}(\mathcal{A})^E, <)$; its inverse is $\mathcal{L}$.*

Proof. We show that $\mathcal{K}$ is a bijection. For any $X \in \mathcal{A}$ and $\psi \in \mathcal{P}(E)^{\mathcal{A}}$,

$$\begin{aligned}\mathcal{L}(\mathcal{K}(\psi))(X) &= \{y \in E : X \in \mathcal{K}(\psi)(y)\} \\ &= \{y \in E : y \in \psi(X)\} \\ &= \psi(X)\end{aligned}$$

that is,

$$\mathcal{L}(\mathcal{K}(\psi)) = \psi, \text{ for } \psi \in \mathcal{P}(E)^{\mathcal{A}}$$

Under the same arguments, changing the role of E and $\mathcal{A}$,

$$\mathcal{K}(\mathcal{L}(\rho)) = \rho, \text{ for } \rho \in \mathcal{P}(\mathcal{A})^E$$

Therefore, $\mathcal{K}$ is a bijection and $\mathcal{L} = \mathcal{K}^{-1}$.

Now we show that $\mathcal{K}$ is increasing two-sided. For any ~~Suppose~~ ψ_1 and $\psi_2 \in \mathcal{P}(E)^{\mathcal{A}}$ we have ~~satisfy $\mathcal{K}(\psi_1) < \mathcal{K}(\psi_2)$. Then~~

$$\begin{aligned}\mathcal{K}(\psi_1) < \mathcal{K}(\psi_2) &\Leftrightarrow \mathcal{K}(\psi_1)(y) \subset \mathcal{K}(\psi_2)(y), \text{ for } y \in E \\ &\Leftrightarrow y \in \psi_1(X) \Rightarrow y \in \psi_2(X), \text{ for } X \in \mathcal{A},\ y \in E \\ &\Leftrightarrow \psi_1(X) \subset \psi_2(X), \text{ for } X \in \mathcal{A} \\ &\Leftrightarrow \psi_1 < \psi_2\end{aligned}$$

that is, $\mathcal{K}$ is increasing two-sided. □

Given $A \subset B$ in $\mathcal{A}$ the subcollection $\{X \in \mathcal{A} : A \subset X \subset B\}$ is called a *closed interval* of $(\mathcal{A}, \subset)$ and is denoted by $[A, B]$ (Birkhoff, 1967*f*).

Given two structuring functions a and b from E to $\mathcal{A} \subset \mathcal{P}(E)$ ~~such that $a < b$,~~ the mapping $[a,b]_{\mathcal{A}}$ or, simply, $[a,b]$ from E to $\mathcal{P}(\mathcal{A})$, determined by

$$[a,b](y) = \begin{cases} [a(y), b(y)] & \text{if } a(y) \subset b(y) \\ \emptyset & \text{otherwise} \end{cases} \qquad [a,b](y) = \begin{cases} [a(y), b(y)] & \text{if } a(y) \subset b(y) \\ \cancel{= \{X \in \mathcal{A} : a(y) \subset X \subset b(y)\}} & \\ \emptyset & \text{otherwise} \end{cases}$$

function is called a point to interval ~~mapping~~ function or, simply, an *interval-~~mapping,~~* function because the image of a point y is a closed interval of $(\mathcal{A}, \subset)$.

Lemma 1 *Let ρ be a mapping form E to $\mathcal{P}(\mathcal{A})$. ~~If $\emptyset \notin \rho(E)$, that is, the values pointed by ρ are never the empty collection of subsets of E,~~ then* Then

$$\rho = \vee\{[a,b] : [a,b] < \rho\}$$

Proof. By supremum definition,

$$\rho > \vee\{[a,b] : [a,b] < \rho\}$$

since ρ is an upper bound for $\{[a,b] : [a,b] < \rho\}$.

For any $y \in E$, let $X \subset \rho(y)$ (~~this is always possible since~~ assuming that $\rho(y) \neq \emptyset$ ~~for any $y \in E$~~) then there always exists an a and a b in $\mathcal{A}^E$ such that $[a,b] < \rho$ and $a(y) = b(y) = X$, therefore

$$X \in \cup\{[a,b](y) : [a,b] < \rho\}$$

that is, for any $y \in E$,

$$\begin{aligned} \rho(y) &\subset \cup\{[a,b](y) : [a,b] < \rho\} \\ &= \rho(\vee\{[a,b] : [a,b] < \rho\})(y) \end{aligned}$$

The above inclusion still holds even for $\rho(y) = \phi$.

In other words,

$$\rho < \vee\{[a,b] : [a,b] < \rho\}$$

□

Let a and b be two structuring functions from E to $\mathcal{P}(E)$. We now introduce the set mapping $\alpha_{(a,b)}$ in Ψ defined by

$$\alpha_{(a,b)}(X) = \{y \in E : a(y) \subset X \subset b(y)\}, \; \textit{for } X \in \mathcal{P}$$

The mappings of the type $\alpha_{(a,b)}$ are called *sup-generating* mappings because, as shown in the next theorem, any mapping in Ψ can be represented as a supremum of a family of these mappings.

We observe that if a and b are elements of $\mathcal{A}^E$, the kernel of the restriction of $\alpha_{(a,b)}$ to $\mathcal{A}$ is the mapping $[a,b]_{\mathcal{A}}$:

$$\mathcal{K}(\alpha_{(a,b)}/\mathcal{A})(y) = \{X \in \mathcal{A} : a(y) \subset X \subset b(y)\} = [a,b]_{\mathcal{A}}(y), \; \textit{for } y \in E$$

Theorem 1 Decomposition by a set of sup-generating mappings. *Let ψ be any set mapping from $\mathcal{A}$ to $\mathcal{P}(E)$ and let $\mathcal{K}(\psi)$ be its kernel. ~~If $\emptyset \notin \mathcal{K}(\psi)(E)$, t~~Then ψ can be decomposed by a set of sup-generating mappings $\alpha_{(a,b)}$ restricted to $\mathcal{A}$ and the decomposition expression can be written*

$$\psi = \vee\{\alpha_{(a,b)}/\mathcal{A} : [a,b]_{\mathcal{A}} < \mathcal{K}(\psi)\}$$

Proof. By using Lemma 1, the mapping $\mathcal{K}(\psi)$ can be written

$$\mathcal{K}(\psi) = \vee\{[a,b]_{\mathcal{A}} : [a,b]_{\mathcal{A}} < \mathcal{K}(\psi)\}$$

or again, from the above observation,

$$\mathcal{K}(\psi) = \vee\{\mathcal{K}(\alpha_{(a,b)}/\mathcal{A} : [a,b]_{\mathcal{A}} < \mathcal{K}(\psi)\}$$

Since, by Proposition 4, $\mathcal{K}$ is a lattice-isomorphism, by applying the inverse mapping $\mathcal{K}$ to both sides we get the above mentioned result. □

The case of the mappings for which $\emptyset \in \mathcal{K}(\psi)(E)$ can be considered pathological, since it means that there exists at least one $y \in E$ which never belongs to the transformation of any set X in $\mathcal{A}$.

It is interesting to note that the sup-generating mapping $\alpha_{(a,b)}$ is actually the infimum of an erosion and an anti-dilation (Serra, 1987):

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$$\begin{aligned} \alpha_{(a,b)}(X) &= \{y \in E : a(y) \subset X\} \cap \{y \in E : b^c(y) \cap X \neq \emptyset\} \\ &= \epsilon_a(X) \cap (\delta_{(b^c)^*}(X))^c, \; \textit{for } X \in \mathcal{P} \end{aligned}$$

that is,

$$\alpha_{(a,b)} = \epsilon_a \wedge (\delta_{(b^c)^*})^c$$

3.2 Decomposition by a Set of Inf-Generating Mappings

Let a and b be two structuring functions from E to $\mathcal{P}(E)$. We now introduce the set mapping $\beta_{(a,b)}$ in Ψ which is defined by

$$\beta_{(a,b)} = (\alpha_{a^*,b^*})^*$$

Mappings of this type are called inf-generating mappings.

Let $\mathcal{A}^*$ be the image of $\mathcal{A}$ through complementation. In other words,

$$\mathcal{A}^* = \{X \in \mathcal{P} : X^c \in \mathcal{A}\}$$

For any $\mathcal{A} \subset \mathcal{P}(E)$, we have $(\mathcal{A}^*)^* = \mathcal{A}$. If ψ is from $\mathcal{P}(E)$ to $\mathcal{P}(E)$ and $\mathcal{A} \subset \mathcal{P}(E)$, we have $(\psi/\mathcal{A})^* = \psi^*/\mathcal{A}^*$.

Theorem 2 Decomposition by a set of inf-generating mappings. *Let ψ be any set mapping from $\mathcal{A}$ to $\mathcal{P}(E)$ and $\mathcal{K}(\psi^*)$ be the kernel of its dual.* ~~*If $\emptyset \notin \mathcal{K}(\psi^*)(E)$,*~~ *then ψ can be decomposed by a set of inf-generating mappings $\beta_{(a,b)}$ restricted to $\mathcal{A}$ and the decomposition expression can be written*

$$\psi = \wedge\{\beta_{(a^*,b^*)}/\mathcal{A} : [a,b]_{\mathcal{A}^*} < \mathcal{K}(\psi^*)\}$$

Proof. By Theorem 1,

$$\psi^* = \vee\{\alpha_{(a,b)}/\mathcal{A}^* : [a,b]_{\mathcal{A}^*} < \mathcal{K}(\psi^*)\}$$

By applying the inverse of the dual lattice isomorphism $\psi \to \psi^*$ between the complete lattices $\mathcal{P}^{\mathcal{A}}$ and $\mathcal{P}^{\mathcal{A}^*}$ to both sides there results

$$\psi^* = \wedge\{(\alpha_{(a,b)}/\mathcal{A}^*)^* : [a,b]_{\mathcal{A}^*} < \mathcal{K}(\psi^*)\}$$

We finally get the mentioned result from the above observations. □

It is interesting to note that the inf-generating mapping $\beta_{(a,b)}$ is actually the supremum of a dilation and an anti-erosion (Serra, 1987):

$$\begin{aligned}\beta_{(a,b)} &= (\epsilon_{a^*} \wedge (\delta_{b^c})^c)^* \\ &= \delta_a \vee (\epsilon_{(b^c)^*})^c\end{aligned}$$

4 Minimal Decomposition Theorems

4.1 Algebraic Aspects

The decomposition theorems of the previous section may lead to redundant decomposition for most set mappings in the sense that the decomposition may be carried out by a smaller set of generating mappings.

In the case of the decomposition by a set of sup-generating mappings (for example, if $[a,b] < [a',b']$), $\alpha_{(a,b)} < \alpha_{(a',b')}$. Hence, if $[a,b]$ and $[a',b']$ are both less than or equal to $\mathcal{K}(\psi)$, in the decomposition of ψ by a set of sup-generating mappings, $\alpha_{(a,b)}$ appears to be redundant.

In order to derive minimal decompositions for set mappings, following Banon and Barrera (1991), we introduce some new definitions. ~~Throughout this section we assume that the mappings ψ under study~~

satisfy the condition $\emptyset \notin \mathcal{K}(\psi)(E)$ (or $\emptyset \notin \mathcal{K}(\psi^*)(E)$ for dual decomposition), otherwise there would be no interval mapping less than or equal to $\mathcal{K}(\psi)$ (or less than or equal to $\mathcal{K}(\psi^*)$).

The set $\mathbf{B}(\psi)$ of all the maximal interval functions less than or equal to $\mathcal{K}(\psi)$ is called the *basis* of ψ. An interval function less than or equal to $\mathcal{K}(\psi)$ is *maximal* if no other interval function less than or equal to $\mathcal{K}(\psi)$ is greater than it.

The set $\mathbf{B}$ of interval functions less than or equal to $\mathcal{K}(\psi)$ is said to satisfy the *decomposition condition* for ψ if and only if for any interval function less than or equal to $\mathcal{K}(\psi)$ there exists an interval function in $\mathbf{B}$ which is greater than it.

Theorem 3 Decomposition by a least set of sup-generating mappings. *Let ψ be any set mapping from $\mathcal{A}$ to $\mathcal{P}(E)$, let $\mathcal{K}(\psi)$ be its kernel, and let $\mathbf{B}$ be a set of interval functions less than or equal to $\mathcal{K}(\psi)$ satisfying the decomposition condition for ψ. Then*

$$\psi = \vee\{\alpha_{(a,b)}/\mathcal{A} : [a,b]_{\mathcal{A}} \in \mathbf{B}\}$$

Furthermore, if $\mathbf{B}(\psi)$ is its basis and satisfies the decomposition condition for ψ, then

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$$\begin{array}{rcl} \mathbf{B}(\psi) & \subset & \mathbf{B}, \quad \longrightarrow \\ \psi & = & \vee\{\alpha_{(a,b)}/\mathcal{A} : [a,b]_{\mathcal{A}} \in \mathbf{B}(\psi)\} \end{array}$$

and ψ is said to have a minimal decomposition by a set of sup-generating mappings restricted to $\mathcal{A}$.

For the proof of a similar result see (Banon and Barrera, 1991).

The dual form of the minimal decomposition is now presented.

Theorem 4 Decomposition by a least set of inf-generating mappings. *Let ψ be any mapping from $\mathcal{A}$ to $\mathcal{P}(E)$, let $\mathcal{K}(\psi^*)$ be the kernel of its dual, and let $\mathbf{B}$ be a set of interval functions less than or equal to $\mathcal{K}(\psi^*)$ satisfying the decomposition condition for ψ^*. Then*

$$\psi = \wedge\{\beta_{(a^*,b^*)}/\mathcal{A} : [a,b]_{\mathcal{A}^*} \in \mathbf{B}\}$$

is and

Furthermore, if $\mathbf{B}(\psi^*)$, is the basis of its dual and satisfies the decomposition condition for ψ^*, then

centrelized

$$\begin{array}{rcl} \mathbf{B}(\psi^*) & \subset & \mathbf{B}, \quad \longrightarrow \\ \psi & = & \wedge\{\beta_{(a^*,b^*)}/\mathcal{A} : [a,b]_{\mathcal{A}^*} \in \mathbf{B}(\psi^*)\} \end{array}$$

and ψ is said to have a minimal decomposition by a set of inf-generating mappings restricted to $\mathcal{A}$.

For the proof of a similar result see (Banon and Barrera, 1991).

Before ending this subsection, we would like to make the following observation. The condition $[a,b] \in \mathbf{B}(\psi)$ is equivalent to $[a(y), b(y)]$ being maximal in $\mathcal{K}(\psi)(y)$ for any $y \in E$. Therefore, the computation of $\psi(X)$ through the minimal decomposition (if any) can be simplified in the sense that, for any $X \in \mathcal{A}$,

$$\begin{array}{rcl} \psi(X) & = & \cup\{\alpha_{(a,b)}(X) : [a,b] \in \mathbf{B}(\psi)\} \\ & = & \{y \in E : \exists [A,B] \ \text{ maximal in } \ \mathcal{K}(\psi)(y) : A \subset X \subset B\} \end{array}$$

In the case of finite set E, the number of maximal elements (closed intervals) involved in the latter expression may be much smaller than the number of maximal elements (interval functions) involved in the former one. Similar observation holds for the minimal decomposition by a set of inf-generating mappings.

4.2 Topological Aspects

We now show that under a condition of upper semicontinuity on a mapping from $\mathcal{F}(E)$, or simply $\mathcal{F}$, the collection of closed subsets of E) to $\mathcal{P}(E)$, its basis satisfies the decomposition condition. Actually, this condition appears to be the same as for the translation invariant case analyzed in (Banon and Barrera, 1991).

In order to describe such sufficient condition we use the Hit-Miss topology on the collection $\mathcal{F}$ of closed subsets of E. Throughout this subsection E will be a locally compact (i.e., each point in E admits a compact neighborhood), Hausdorff and separable (i.e., the topology of E admits a countable base) topological space.

The Hit-Miss topology on $\mathcal{F}$ is generated by the set of collections of the type

$$\mathcal{F}^K = \{X \in \mathcal{F} : X \cap K = \emptyset\}$$

where K is a compact subset of E, and

$$\mathcal{F}_G = \{X \in \mathcal{F} : X \cap G \neq \emptyset\}$$

where G is an open subset of E (Matheron, 1975*a*; Serra, 1982).

A mapping ψ from $\mathcal{F}$ to $\mathcal{F}$ is *upper semicontinuous* (u.s.c.) if and only if for any compact subset K of E, the set $\psi^{-1}(\mathcal{F}^K)$ is open in $\mathcal{F}$ (Matheron, 1975*c*).

The set of closed intervals of $\mathcal{F}$ (provided with $\subset$) and the set of interval functions (provided with $<$) are complete joint semilattices.

Theorem 5 Property of the basis of an u.s.c. mapping. *Let ψ be an u.s.c. mapping from $\mathcal{F}$ to $\mathcal{F}$.* ~~*If $\emptyset \notin \mathcal{K}(\psi)(E)$*~~ *Then its basis $\mathbf{B}(\psi)$ satisfies the decomposition condition for ψ.*

Proof. Let $[a,b]$ be an interval ~~mapping~~ function less than or equal to $\mathcal{K}(\psi)$. It is always possible to construct a linearly ordered set $\mathbf{L}$ of interval functions less than or equal to $\mathcal{K}(\psi)$ such that $[a,b] \in \mathbf{L}$. By Lemma 2.1 in Maragos (1985), there exists a maximal linearly ordered set $\mathbf{M}$ of interval functions less than or equal to $\mathcal{K}(\psi)$ such that $\mathbf{L} \subset \mathbf{M}$. Therefore, there exists an interval function $[a',b']$, namely $[a',b'] = \sup \mathbf{M}$, which is greater than or equal to $[a,b]$:

$$[a,b] < \sup \mathbf{L} < \sup \mathbf{M} = [a',b']$$

If ψ is u.s.c. from $\mathcal{F}$ to $\mathcal{F}$, then for any compact subset K of E,

$$\begin{aligned}
(\psi^{-1}(\mathcal{F}^K))^c &= \{X \in \mathcal{F} : \psi(X) \in \mathcal{F}^K\}^c \\
&= \{X \in \mathcal{F} : \psi(X) \cap K = \emptyset\}^c \\
&= \{X \in \mathcal{F} : \psi(X) \cap K \neq \emptyset\}
\end{aligned}$$

is closed in $\mathcal{F}$. In particular, for any $y \in E$,

$$\begin{aligned}
\{X \in \mathcal{F} : \psi(X) \cap \{y\} \neq \emptyset\} &= \{X \in \mathcal{F} : y \in \psi(X)\} \\
&= \mathcal{K}(\psi)(y)
\end{aligned}$$

is closed in $\mathcal{F}$, that is, $\overline{\mathcal{K}(\psi)(y)} = \mathcal{K}(\psi)(y)$ in $\mathcal{F}$.

On the other hand, for any $y \in E$, $[a',b'](y) = (\sup \mathrm{M})(y) = \sup\{\rho(y) : \rho \in \mathrm{M}\}$. By construction of M, the set $\{\rho(y) : \rho \in \mathrm{M}\}$ is a linearly ordered set of closed intervals of $\mathcal{F}$ contained in $\overline{\mathcal{K}(\psi)(y)}$. Therefore, by Lemma 4.3 in (Banon and Barrera, 1990, 1991), its supremum is contained in $\overline{\mathcal{K}(\psi)(y)}$ in $\mathcal{F}$; that is, in $\mathcal{F}$,

$$[a',b'](y) \subset \overline{\mathcal{K}(\psi)(y)}, \; \textit{for } y \in E$$

(assuming that $K(\psi)(y) \neq \emptyset$)

or again, under the u.s.c. assumption on ψ,

$$[a',b'](y) \subset \overset{K}{\cancel{\mathcal{K}}}(\psi)(y), \; \textit{for } y \in E$$

The above inclusion still holds even for $K(\psi)(y) = \emptyset$.

In other words, $[a',b'] < \mathcal{K}(\psi)$.

Furthermore, M being a maximal set of interval functions less than or equal to $\mathcal{K}(\psi)$, we have $[a',b'] \in \mathrm{B}(\psi)$, because otherwise there would exist another interval function less than or equal to $\mathcal{K}(\psi)$ which is greater than $[a',b']$. In other words, there would exist a linearly ordered set of interval functions less than or equal to $\mathcal{K}(\psi)$ that properly contains M and M would not be maximal. □

In terms of minimal decomposition the above theorem leads to the following result. (Let $\mathcal{G}$ be the collection of open subsets of E.)

Corollary 1

1. *If ψ is a u.s.c. mapping from $\mathcal{F}$ to $\mathcal{F}$ ~~(such that $\emptyset \notin \mathcal{K}(\psi)(E)$)~~, then ψ has a minimal decomposition by a set of sup-generating mappings restricted to $\mathcal{F}$.*
2. *If ψ is a mapping from $\mathcal{G}$ to $\mathcal{G}$ which has a u.s.c. dual ψ^*, ~~(such that $\emptyset \notin \mathcal{K}(\psi^*)(E)$),~~ then ψ has a minimal decomposition by a set of inf-generating mappings restricted to $\mathcal{G}$.*

5 Translation Invariant Mappings

To introduce the case of translation invariant mappings we now assume that E is an Abelian group, with respect to a binary operation denoted $+$. The null element of $(E,+)$ is denoted by 0 and the inverse of any y in $(E,+)$ is denoted by $-y$.

For any $h \in E$ and $X \subset E$, the set

$$X_h = \{y \in E : y = x + h, \; x \in X\}$$

is called the *translate* of X by h. In particular, $X_0 = X$.

For any $h \in E$ and $\mathcal{A} \subset \mathcal{P}$, we denote by $\mathcal{A}_h$ the set of translates of X_h with X running over $\mathcal{A}$, i.e.,

$$\mathcal{A} = \{X \in \mathcal{P} : X_{\underset{-h}{\cancel{h}}} \in \mathcal{A}\}$$

We now consider the mappings whose domain is a collection $\mathcal{A}$ *closed under translation*, that is $\mathcal{A}_h = \mathcal{A}$ for any $h \in E$, which are translation invariant in the sense that

$$\psi(X_h) = (\psi(X))_h, \; \textit{for } X \in \mathcal{A}, \; h \in E$$

The translation invariant mappings from $\mathcal{A}$ to $\mathcal{P}$ form a complete sublattice of $(\Psi, <)$.

The kernel of a translation invariant mapping ψ satisfies, for any $y \in E$,

$$\begin{aligned} \mathcal{K}(\psi)(y) &= \{X \in \mathcal{A} : 0 \in (\psi(X))_{-y}\} \\ &= \{X \in \mathcal{A} : 0 \in \psi(X_{-y})\} \\ &= \{X \in \mathcal{A} : 0 \in \psi(X)\}_y \\ &= (\mathcal{K}(\psi)(0))_y \end{aligned}$$

The collection $\mathcal{K}(\psi)(0)$ is called by Matheron (1975*a*) the kernel of the translation invariant mapping ψ.

For each translation invariant dilation δ, the corresponding structuring function a satisfies, from the representation theorem for erosions and dilations of Section 2,

$$\begin{aligned} a(y) &= \delta(\{y\}) \\ &= \delta(\{0\}_y) \\ &= (\delta(\{0\}))_y \\ &= a(0)_y \end{aligned}$$

Consequently, the complete sublattice of translation invariant adjunctions (ϵ, δ) is isomorphic to the complete lattice $\mathcal{P}$ of subsets A of E, by $(\epsilon, \delta) \to A = \delta(\{0\})$ and by $A \to (\epsilon, \delta)$ with

$$\epsilon(X) = \epsilon_A(X) = \{y \in E : A_y \subset X\}, \textit{ for any } X \in \mathcal{P}$$

and

$$\delta(Y) = \delta_A(Y) = \cup\{A_y : y \in Y\}, \textit{ for any } Y \in \mathcal{P}$$

The set A is called a *structuring element.*

For any X in $\mathcal{P}$, $\delta_A(X)$, the *dilation* of X by A (following Sternberg's definition (1982)[1]), can be expressed as a Minkowski addition (Hadwiger, 1950):

$$\delta_A(X) = X \oplus A$$

and $\epsilon_A(X)$, the *erosion* of X by A, can be expressed as a Minkowski subtraction (following Hadwiger's definition (1950)[2]):

$$\epsilon_A(X) = X \ominus A$$

The minimal decompositions for translation invariant mapping introduced in (Banon and Barrera, 1990, 1991) can be derived from our general setting. If ψ is translation invariant and its basis satisfies the decomposition condition for ψ of the previous section, then for any X in $\mathcal{A}$,

$$\begin{aligned} \psi(X) &= \{y \in E : \exists [A, B] \text{ maximal in } \mathcal{K}(\psi)(0) \overset{:}{\not\ni} A_y \subset X \subset B_y\} \\ &= \cup\{(X \ominus A) \cap (X^c \ominus B^c) : [A, B] \text{ maximal in } \mathcal{K}(\psi)(0)\} \end{aligned}$$

Actually, the subset $(X \ominus A) \cap (X^c \ominus B^c)$ in the above expression is the image of X by ~~a restriction to A of~~ a translation invariant sup-generating mapping, since $(X \ominus A) \cap (X^c \ominus B^c) = \{y \in E : A_y \subset X \subset B_y\}$ for

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[1] **Matheron and Serra's definition for dilation by A is slightly different.**

[2] **Matheron and Serra's definition for Minkowski subtraction is slightly different.**

any $X \in \mathcal{A}$. By using Banon and Barrera's notation (1990, 1991) for translation invariant sup-generating mappings,

$$X \textcircled{A} (A, B) = (X \ominus A) \cap (X^c \ominus B^c), \ \textit{for } X \in \mathcal{P}$$

t The above minimal decomposition expression becomes, for any $X \in \mathcal{A}$,

$$\psi(X) = \cup\{X \textcircled{A} (A, B)/\mathcal{A} : [A, B] \text{ maximal in } \mathcal{K}(\psi)(0)\}$$

In other words, ψ has a minimal decomposition by a set of translation invariant sup-generating mappings.

In the same way, if the basis of ψ^* satisfies the decomposition condition for ψ^* then ψ has a minimal decomposition by a set of translation invariant inf-generating mappings. By using Banon and Berrera's notation (1990, 1991) for translation invariant inf-generating mappings,

$$X \textcircled{V} (A, B) = (X \oplus A) \cap (X^c \oplus B^c), \ \textit{for } \overset{X}{Y} \in \mathcal{P}$$

t The corresponding minimal decomposition expression is, for $\underset{y}{\text{and}}$ $X \in \mathcal{A}$,

$$\psi(X) = \cap\{X \textcircled{V} (A^s, B^s)/\mathcal{A} : [A, B] \ \text{ maximal in } \mathcal{K}(\psi^*)(0)\}$$

6 Examples

6.1 Morphological Openings

Let a be a structuring function from E to $\mathcal{P}$ $(= \mathcal{P}(E))$ and γ_a the opening $\delta_a \epsilon_a$ (see Section 2). For any $y \in E$, the kernel of γ_a is given by

$$\begin{aligned}
\mathcal{K}(\gamma_a)(y) &= \{X \in \mathcal{P} : y \in \gamma_a(X)\} \\
&= \{X \in \mathcal{P} : y \in \{x \in E : a^s(x) \cap \epsilon_a(X) \neq \emptyset\}\} \\
&= \{X \in \mathcal{P} : a^s(y) \cap \epsilon_a(X) \neq \emptyset\} \\
&= \{X \in \mathcal{P} : \exists \ x \in a^s(y) \not\ni x \in \epsilon_a(X)\} \\
&= \cup\{\mathcal{K}(\epsilon_a)(x) : x \in a^s(y)\}
\end{aligned}$$

Observing that the kernel of ϵ_a is given by

$$\mathcal{K}(\epsilon_a)(x) = \{X \in \mathcal{P} : a(x) \subset X\}, \ \textit{for any } x \in E$$

we obtain that the following set $\mathbf{B}$ of interval ~~mappings~~ functions less than or equal to $\mathcal{K}(\gamma_a)$ satisfies the decomposition condition for γ_a:

$$\mathbf{B} = \{[u, v] : \forall y \in E, y \in u(y) \in a(E) \text{ and } \ v(y) = E\}$$

If for any $y \in E$ all the subsets in $a(E)$ which contain y are not comparable (under inclusion), then $\mathbf{B}$ is the basis of γ_a. This is the case for translation invariant openings. On the contrary, the basis of γ_a may not satisfy the decomposition condition for γ_a, but if it satisfies this condition, then it is properly included in $\mathbf{B}$. For example, for a u.s.c. opening we have

$$\mathbf{B}(\gamma_a) = \{[u, v] : \forall y \in E, \ u(y) \text{ is a minimal element of } \{a(x) : x \in a^s(y)\} \text{ and } v(y) = E\} \subset \mathbf{B}$$

If E is the set of vertices of a graph defined by a set Γ of pairs of vertices satisfying

1. $\forall x \in E$, $(x,x) \in \Gamma$ (there is a loop attached to each vertex)
2. $\forall (x,y) \in \Gamma$, $(y,x) \in \Gamma$ (the graph is nonoriented)

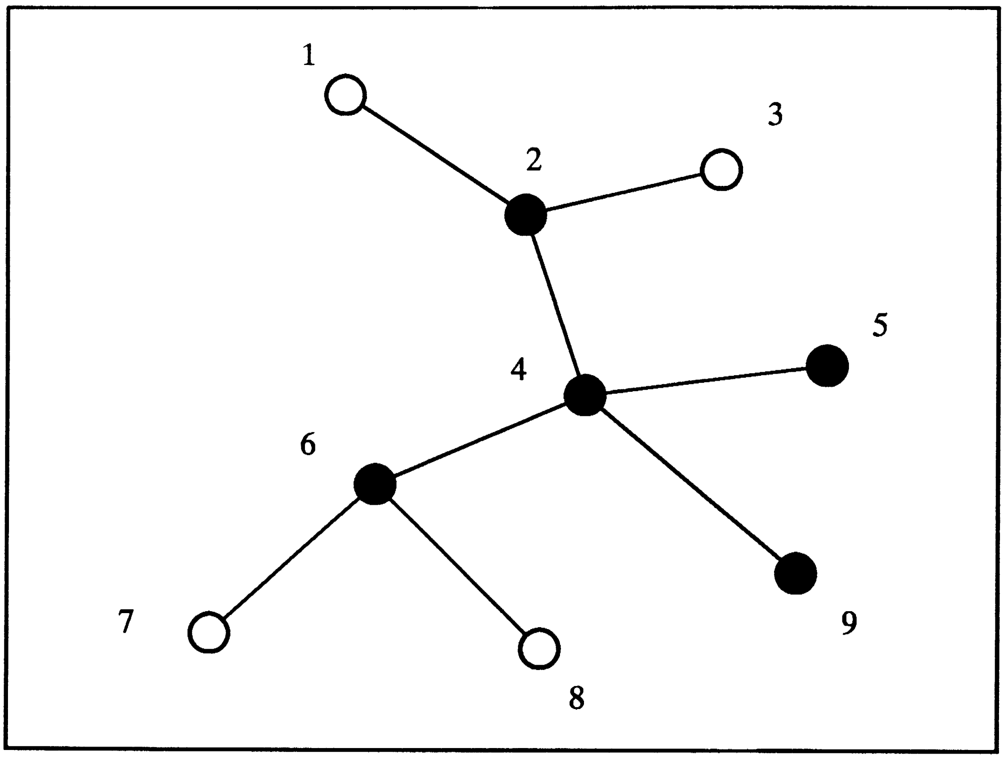
then the order-1 neighborhood in Γ of any vertex $x \in E$, $a(x) = \{y \in E : (x,y) \in \Gamma\}$, is an interesting example of an extensive ($\forall x \in E, x \in a(x)$) and symmetric ($a = a^s$) structuring function (Vincent, 1990).

For example, the values at vertex 4 of all possible interval ~~mappings~~ functions $[u,v]$ in $\mathbf{B}(\gamma_a)$ for the graph of Figure 1 are (noting that $a^s(4) = a(4)$)

$$[\{1,2,3,4\}, E]$$
$$[\{4,5\}, E]$$
$$[\{4,9\}, E]$$
$$[\{4,6,7,8\}, E]$$

Figure 1: Example of a graph with 9 vertices. For the sake of simplicity the loop attached to each vertex is not represented. The set $\{2,4,5,6,9\}$ (the black vertices) is the value $a(4)$ of the structuring function a associated with the graph at vertex 4.

It can be observed, in the case of the graph of Figure 1, that $\mathbf{B}(\gamma_a)$ is properly included in $\mathbf{B}$


1
2
3
4
5
6
7
8
9

since, for example, there is no interval ~~mapping~~ function in $B(\gamma_a)$ with value $[\{2,4,5,6,9\},E]$ at vertex 4, for $[\{2,4,5,6,9\},E]$ is included in $[\{4,5\},E]$ or $[\{4,9\},E]$.

6.2 Shape Recognition

Crimmins and Brown (1985) have introduced the so-called window transformation to solve automatic shape recognition. Let $(\mathcal{Z}^2,+)$ be the Abelian group of pairs of integers. A mapping ψ from $\mathcal{A} \subset \mathcal{P}(\mathcal{Z}^2)$ to $\mathcal{P}(\mathcal{Z}^2)$ is called a *window transformation* with respect to the window W if and only if there exists a subcollection $\mathcal{D} \subset \mathcal{P}(W)$ such that

$$\psi(X) = \{y \in \mathcal{Z}^2 : W \cap X_y \in \mathcal{D}\}, \textit{ for } X \in \mathcal{A}$$

The mapping ψ recognizes in particular all the shapes in $\mathcal{A}$ which are in $\mathcal{D}$ (up to a translation) by producing a point marker. If $\mathcal{A}$ is closed under translation, then ψ is translation invariant.

From now on, for pratical reasons, we assume that $0 \in W$, E is a finite *field of view* defined in $\mathcal{Z}^2$ ($E \subset \mathcal{Z}^2$), and $\mathcal{A}$ is a subcollection of $\mathcal{P} = \mathcal{P}(E)$. In this case $\mathcal{A}$ is no longer closed under translation and it does not make sense to consider the usual translation invariant property for ψ. So we must consider ψ in the general framework of this study. Its kernel $\mathcal{K}(\psi)$ is given by

$$\mathcal{K}(\psi)(y) = \{X \subset E : \; W \cap X_{-y} \in \mathcal{D}\}, \textit{ for } y \in E$$

Usually, the meaningful relationship between E and W is that there exists $y \in E$ such that $W_y \subset E$. For such y, $\mathcal{K}(\psi)(y)$ is never empty.

~~In order to guarantee that $\emptyset \notin \mathcal{K}(\psi)(E)$, that is, for any $y \in E$, $\mathcal{K}(\psi)(y)$ is nonempty, we must add in $\mathcal{D}$ additional subsets of E.~~

To be coherent with our shape recognition objective we should add in $\mathcal{D}$ all the nonempty subsets of the type $X \cap E_y$ for which X belongs to the original $\mathcal{D}$ when y runs over E. If $0 \in X$ for any $X \in \mathcal{D}$, then the previous subsets $X \cap E_{-y}$ are never empty and they correspond to the so-called ***partially observed*** shapes (see Figure 2).

Let $\mathcal{D}'$ be the new subcollection of interest:

$$\mathcal{D}' = \{Y \in \mathcal{P}(W) : \; Y = U \cap E_{-y}, \; U \in \mathcal{D}, \; y \in E\}$$

With such $\mathcal{D}'$, ~~$\emptyset \notin \mathcal{K}(\psi)(E)$ and~~ ψ is able to recognize (and mark) even partially observed shapes. The price to pay is that we are now unduly recognizing all the subsets of the form $(U \cap E_{-y})_x$ for which U belongs to $\mathcal{D}$ and $x \neq y$, when x and y run over E. To fix this problem we introduce the mapping $\mathcal{M}$ from E to $\mathcal{P}(\mathcal{P})$ given by

$$\mathcal{M}(y) = \{Y \in \mathcal{P} : \; Y = U_y \cap E, \; U \in \mathcal{D}\}, \textit{ for } Y \in E$$

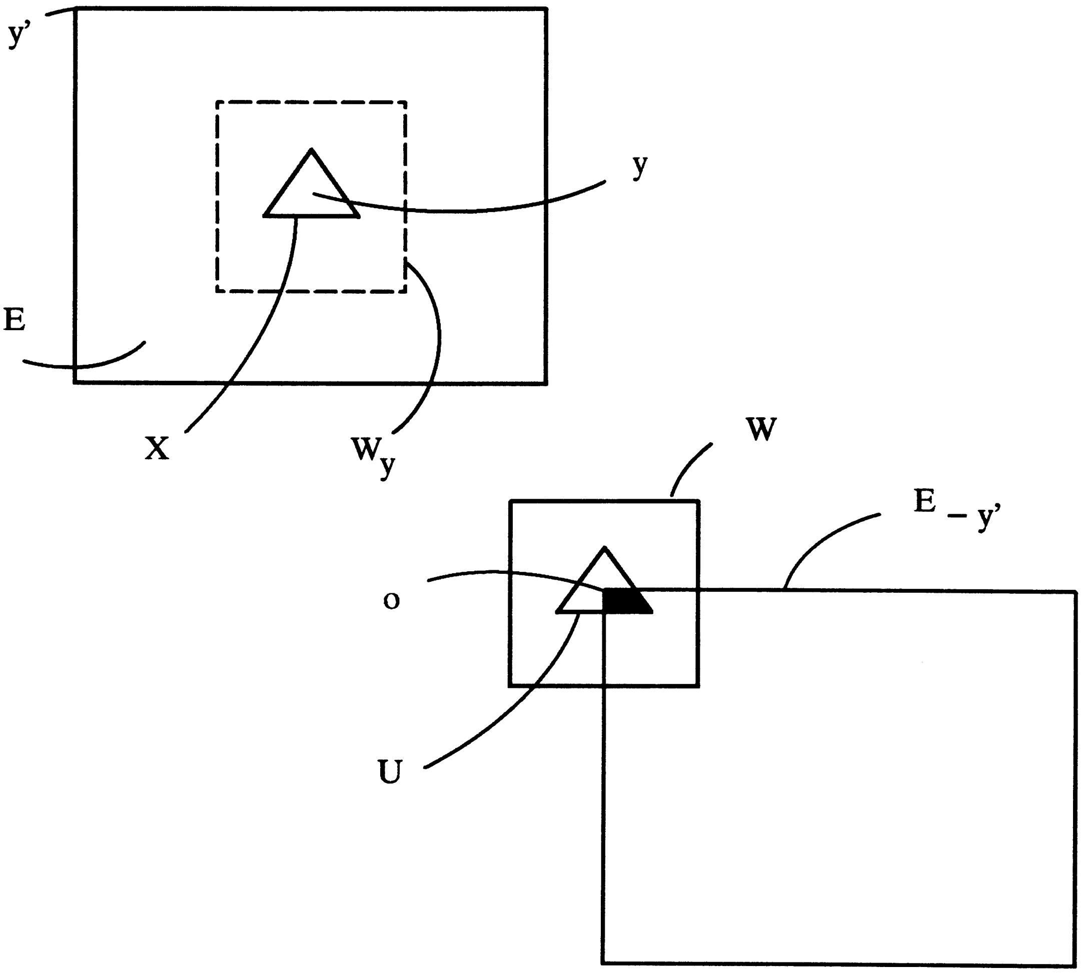
and we define the following new window transformation $\psi_{\mathcal{M}}$ that we will call the *adaptive window transformation*:

$$\psi_{\mathcal{M}}(X) = \{y \in E : \; W_y \cap X \in \mathcal{M}(y)\}, \textit{ for } X \subset E$$

Its kernel is given by

$$\mathcal{K}(\psi_{\mathcal{M}})(y) = \{X \subset E : \; W_y \cap X \in \mathcal{M}(y)\}, \textit{ for } y \in E$$

Figure 2: Example of the shape recognition of a triangle X. For the sake of simplicity the set $\mathbb{Z}^2$ is not represented. The rectangular shape E is the field of view, W is the window, and U is a trianglular shape in $\mathcal{D}$. The triangle X is recognized because $W \cap X_{-y} = U$. If y' is the upper left corner of E, the subset $U \cap E_{-y'}$ (in black) corresponds to a partially observed triangular shape.


y'
y
E
X
W_y
W
$E_{-y'}$
o
U

The following set $\mathbf{B}$ of interval functions less than or equal to $\mathcal{K}(\psi)$ satisfies the decomposition condition for ψ:

$$\mathbf{B} = \{[a,b] : \forall y \in E,\ a(y) \in \mathcal{M}(y) \text{ and } b(y) = E - (W_y - a(y))\}$$

or, in another way,

$$\mathbf{B} = \left\{[a,b] : \forall y \in E,\ a(y) \in \mathcal{M}(y) \text{ and } b(y) = a(y) + (W_y^c \cap E)\right\}$$

where $+$ stands for the union of disjoint sets. If, for any $y \in E$, all the elements in $\mathcal{M}(y)$ are not comparable (under inclusion), then $\mathbf{B}$ is the basis of $\psi_{\mathcal{M}}$. On the contrary, the basis of ψ is properly included in $\mathbf{B}$ and still satisfies the decomposition condition since E is finite.

$$\mathbf{B}(\psi_{\mathcal{M}}) = \left\{[a,b] : \forall y \in E,\ a(y) \text{ is a maximal element of } \mathcal{M}(y) \text{ and } b(y) = a(y) + (W_y^c \cap E)\right\}$$

Before ending this subsection we point out that the adaptative window transformation may recognize shapes not in $\mathcal{D}$. For example, a subset X in $\mathcal{Z}^2$ (with no translates in $\mathcal{D}$) is seen through the field of view E as the subset $X \cap E$ and its intersection with W_y may belong to $\mathcal{M}(y)$ for a given y in E. This drawback is the price to pay to work with a finite field of view. If the recognition problem can be described as a random experiment with its probability law, then it is possible to associate to each y in E a probability of misclassification. In this case, for a uniform distribution law, we can observe that the probability of misclassification increases as y becomes closer to the edges of E. For high probability of misclassification, the current classification could then be disregarded.

Finally, we observe that within $E \ominus W$, $\psi_{\mathcal{M}}$ and the translation invariant mapping ψ assume the same values. This follows, since from the definition of $\mathcal{M}$ we have

$$\mathcal{M}(y) = \mathcal{D}_y,\ for\ Y \in E \ominus W$$

In this sense, $\psi_{\mathcal{M}}$ may be said to be *almost translation invariant.* This sort of mapping actually plays a very important role in image processing.

7 Conclusion

prototypes

The main contribution of this paper was to prove that the elementary mappings of mathematical morphology (i.e., erosions, dilations, anti-erosions, and anti-dilations) are the ~~ptotypes~~ *****prototypes?***** to decompose any set mapping. Therefore, any set mapping can be performed, at least theoretically, by the existing specialized machines.

A generalization of the concepts of kernel and basis was given in order to prove that any set mapping (not necessarily translation invariant) can be decomposed by a set of (non) translation invariant) sup-generating mappings. A sup-generating mapping is actually the infimum of an erosion and an anti-dilation, hence is uniquely characterized by a couple of structuring functions. The sup-generating mappings involved in the decomposition of a set mapping are those characterized by couples of structuring functions that form interval functions less than or equal to the kernel.

A dual decomposition result, which involves the so called inf-generating mappings, was also given.

This paper opens some perspectives for future work, such as the extension of the present results to the much more abstract domain of complete lattices and the study of adaptive filters from the morphological decomposition point of view.

The decompositions presented here require an enormous degree of parallelism that makes them almost unfeasible in practice. However, there should exist other equivalent decompositions in terms of the elementary mappings that may lead to more feasible implementations.

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