Characterization of the integer-valued translation-invariant regular metrics on the discrete plane

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Abstract

We say that a metric space is regular if a straight-line (in the metric space sense) passing through the center of a sphere and any other point has at least two diametrically opposite points. Normed vector spaces have this property. Nevertheless, this property might not be satisfied in some metric spaces. In this work, we give a characterization of an integer-valued translation-invariant regular metric defined on the discrete plane, in terms of a symmetric subset $B$ that induces through, what we call, the Minkowski product, a chain of subsets that are morphologically closed with respect to $B$.

Keywords: Mathematical Morphology, symmetric subset, ball, regular metric space, integer-valued metric, translation-invariant metric, triangle inequality, Minkowski product, morphological closed subset, discrete plane, computational geometry, discrete geometry, digital geometry.

1. Introduction

The continuous plane or, more precisely, the two-dimensional Euclidean vector space, has good geometrical properties. For example, in such space, a closed ball is included in another one only if the radius of the latter is greater than or equal to the sum of the distance between their centers and the radius of the former. Furthermore, in such space, two closed balls intersect each other if the sum of their radii is greater than or equal to the distance between their centers. Nevertheless, not all metric spaces have these properties.

In the first part of this work, we introduce the concept of regular metric space in which the above two geometrical properties are satisfied.

We say that a metric space is regular if its metric satisfies three regularity axioms or equivalently if a straight-line (in the metric space sense) passing through the center of a sphere and any other point has at least two diametrically opposite points. Minkowski spaces (i.e., finite dimensional normed vector spaces) have this property.

This regularity is generally lost when a metric on the continuous plane is restricted to the discrete plane, as it is the case of the Euclidean metric.

In the second part of this work, we study the characterization of the integer-valued translation-invariant regular metrics on the discrete plane in terms of some appropriate symmetric subsets.

We show that every such metric can be characterized in terms of a symmetric subset $B$ that induces through, what we call, the Minkowski product, a chain of subsets that are morphologically closed with respect to $B$.

Our characterization shows the unique way to construct integer-valued translation-invariant regular metrics on the discrete plane.

This is an important issue in digital image analysis since the image domains are then discrete. In the sixties, Rosenfeld and Pfaltz [8] have already introduced a metric property and have used it to describe algorithms for computing some distance functions by performing repeated local operations. It appears that their property is precisely a necessary condition for a metric to be regular.

Actually, we came across the regularity property for a metric because we tried to find a class of metric whose metric dilations satisfy a semigroup relation (relation (9.19) of [3]) and whose balls are morphologically closed with respect to the unit ball. This class contains, for example, the chessboard distance.

In one dimension, we observed that the (discrete) convexity is not a necessary condition to have the morphological closure property, so it was useless to solve our problem.

For the sake of simplicity of the presentation, in this work, we limit ourselves to the class of integer-valued metrics. This is not a serious limitation because on the discrete plane the metrics assume only a countable number of values.

In Section 2, we give an axiomatic definition of regular metric space and we relate it to the Kiselman's properties of upper and lower regularity for the triangle inequality. In particular, we show that of the three axioms only two are sufficient to define the metric regularity. Independently of the definition of metric, in Section 3, we give a definition of ball based on the notions of set translation and set transposition. We introduce the Minkowski product in Section 4, and use it in Section 5 to define the notions of generated balls and radius of a ball. In Section 6, we study the properties of the balls of a regular metric space. Conversely, in Section 7, we study the properties of the metric spaces constructed from symmetric balls having a morphological closure property. Finally, in Section 8, we show the
existence of a bijection between the set of integer-valued translation-invariant regular metrics defined on the discrete plane and the set of symmetric balls that satisfy the morphological closure property.

2. Regular metric space

We first recall the definitions of distance, metric and metric space.

**Definition 1** (metric space) – Let $E$ be a nonempty set. A distance $d$ on $E$ is a mapping from $E \times E$ to $\mathbb{R}$ (the set of real numbers) satisfying, for any $x$ and $y$ in $E$:

(i) $d(x, y) \geq 0$, (positiveness)
(ii) $d(x, y) = 0 \iff x = y$,
(iii) $d(x, y) = d(y, x)$, (symmetry)

Furthermore, a distance $d$ on $E$ is a metric if in addition it satisfies, for any $x$, $y$ and $z$ in $E$:

(iv) $d(x, y) \leq d(x, z) + d(z, y)$. (triangle inequality)

A metric space $(E, d)$ is a set $E$ provided with a metric $d$ on $E$.

The mapping $d_0$ from $E \times E$ to $\mathbb{R}$ defined by, for any $x$ and $y$ in $E$:

$$d_0(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

is an example of metric, it is called the discrete metric.

In a metric space we can define the concepts of straight-line and sphere. Furthermore, based on these concepts, we can define what we call a regular metric and a regular metric space.

Let $(E, d)$ be a metric space. For any $x$ and $y$ in $E$, and any $i \in d(\{x\} \times E)$ (i.e., the image of $\{x\} \times E$ through $d$), let us define the following subsets of $E$:

$$L_i(x, y) = \{z \in E : d(z, x) = d(x, y) + d(y, z)\},$$
$$L_2(x, y) = \{z \in E : d(z, y) = d(z, x) + d(x, y)\},$$
$$L_3(x, y) = \{z \in E : d(z) = d(z, x) + d(x, y)\},$$

and

$$S(x, i) = \{z \in E : d(x, z) = i\}.$$

The subsets $L_i(x, y) = L_i(y, x) \cup L_2(x, y) \cup L_3(x, y)$ and $S(x, i)$ are, respectively, the straight-line passing through the points $x$ and $y$, and the sphere with center at $x$ and of radius $i$.

Figure 1 and Figure 2 in this section show examples of spheres and straight-lines. We must be aware that the above definition of straight-line is based on the concept of metric and the resulting object is generally different from the usual straight-line defined in the framework of linear vector space.

Because of the symmetry property of the distances, the straight-lines have some kind of symmetry as well.

**Proposition 2** (straight-line symmetry) – Let $(E, d)$ be a metric space. For any $x$ and $y$ in $E$,

(i) $L_0(x, y) = L_0(y, x)$
(ii) $L_2(x, y) = L_2(y, x)$
(iii) $L_3(x, y) = L(y, x)$.

**Proof**

(a) Let us prove (i). For any $x$, $y$ and $z$ in $E$:

$z \in L_0(x, y) \iff d(z, x) = d(x, y) + d(y, z)$ (definition of $L_0$)
$\iff d(z, x) = d(x, y) + d(z, y)$ (symmetry of $d$)
$\iff d(z, x) = d(z, y) + d(y, x)$ (commutativity of $+$)
$\iff z \in L_0(y, x)$ (definition of $L_0$)

that is, $L_0(x, y) = L_0(y, x)$.

(b) Let us prove (ii). For any $x$, $y$ and $z$ in $E$:

$z \in L_2(x, y) \iff d(y, x) = d(x, z) + d(z, y)$ (definition of $L_2$)
$\iff d(y, x) = d(z, x) + d(z, y)$ (symmetry of $d$)
$\iff d(y, x) = d(y, z) + d(z, x)$ (commutativity of $+$)
$\iff z \in L_2(y, x)$ (definition of $L_2$)

that is, $L_2(x, y) = L_2(y, x)$.

(c) Let us prove (iii). For any $x$ and $y$ in $E$:

$L_i(x, y) = L_i(y, x) \cup L_2(x, y) \cup L_3(x, y)$ (definition of $L_i$)
$= L_0(y, x) \cup L_2(y, x) \cup L_i(y, x)$ (from (i) and (ii))
$= L_i(y, x)$ (commutativity of $\cup$)
$= L_i(y, x)$ (definition of $L_i$)

that is, $L_i(x, y) = L_i(y, x)$.

We first define the regular metric space from three axioms.

**Definition 3** (regular metric space) – Let $(E, d)$ be a metric space. The metric $d$ on $E$ is

(i) **lower regular of type 1** if $S(x, i) \cap L_0(x, y) \neq \emptyset$, for any $x$ and $y$ in $E$, and any $i \in d(\{x\} \times E)$, such that $d(x, y) \leq i$;
(ii) **lower regular of type 2** if $S(x, i) \cap L_2(x, y) \neq \emptyset$, for any $x$ and $y$ in $E$, and any $i \in d(\{x\} \times E)$, such that $i \leq d(x, y)$;
(iii) **upper regular** if $S(x, i) \cap L_3(x, y) \neq \emptyset$, for any $x$ and $y$ in $E$, and any $i \in d(\{x\} \times E)$;
(iv) **regular** if it is lower regular (of type 1 and 2) and upper regular. A metric space $(E, d)$ is regular if its metric is regular.

Actually, the three regularity axioms are not independent each other as we show in the next proposition.

**Proposition 4** (axiom dependence) – Let $(E, d)$ be a metric space. The metric $d$ on $E$ is lower regular of type 1 if and only if (iff) it is upper regular.

**Proof**

(a) Let us prove that the lower regularity of type 1 implies the upper regularity. For any $x$ and $y$ in $E$, and any $i \in d(\{x\} \times E)$,

$d$ is lower regular of type 1 $\Rightarrow S(y, i + d(x, y)) \cap L_i(y, x) \neq \emptyset$

(definition of lower regularity of type 1)
$\iff S(y, i + d(x, y)) \cap L_3(y, x) \neq \emptyset$

(Property (i) of Proposition 2 - straight-line symmetry)
$\iff \exists z \in E : d(z, y) = i + d(z, x) + d(x, y)$ and $d(z, y) = d(z, x) + d(x, y)$

(definitions of $S$ and $L_3$)
$\iff \exists z \in E : d(z, y) = i + d(z, x) + d(x, y)$

(symmetry of $d$)
$\iff \exists z \in E : d(z, x) = i$ and $d(z, y) = d(z, x) + d(x, y)$

(transitivity of = and regularity of $+$)
$\iff \exists z \in E : d(z, x) = i$ and $d(z, y) = d(z, x) + d(x, y)$
⇔ \( S(x, i) \cap L_3(x, y) \neq \emptyset \), (definition of \( d \))

⇔ \( S(x, i) \cap L_3(x, y) \neq \emptyset \), (definitions of \( S \) and \( L_3 \))

that is, \( d \) is upper regular.

(b) Let us prove that the upper regularity implies the lower regularity of type 1. For any \( x \) and \( y \) in \( E \), and any \( i \in d(\{x\} \times E) \) such that \( d(x, y) \leq i \).

\( (d \) is upper regular) \( \Rightarrow S(y, i - d(x, y)) \cap L_1(x, y) \neq \emptyset \)

(definition of upper regularity)

\[ \exists \; z \in E : d(y, z) = i - d(x, y) \text{ and } d(z, x) = d(x, y) + d(y, z) \]

(definitions of \( S \) and \( L_1 \))

\[ \exists \; z \in E : d(y, z) = i \text{ and } d(z, x) = d(x, y) + d(y, z) \]

(\(+ \) versus -)

\[ \exists \; z \in E : d(z, x) = i \text{ and } d(z, x) = d(z, x) + d(y, z) \]

(transitivity of =)

\[ S(x, i) \cap L_1(x, y) \neq \emptyset, \]

(definitions of \( S \) and \( L_1 \))

that is, \( d \) is lower regular of type 1.

The axiom dependence allows us to make an equivalent definition of regular metric, but simpler with only two axioms, the lower regularity of type 2 being called simply lower regularity.

**Corollary 5** (first equivalent definition of regular metric) – Let \( (E, d) \) be a metric space. The metric \( d \) on \( E \) is:

(i) lower regular if \( S(x, i) \cap L_2(x, y) \neq \emptyset \), for any \( x \) and \( y \) in \( E \), and any \( i \in d(\{x\} \times E) \), such that \( i \leq d(x, y) \);

(ii) regular iff it is lower and upper regular.

**Proof**

(a) If \( d \) is regular in the sense of Definition 3, then \( d \) is regular in the sense of the corollary statement since lower regularity means lower regularity of type 2.

(b) Conversely, if \( d \) is regular in the sense of the corollary statement, then \( d \) is lower regular of type 2 and upper regular, therefore by Proposition 4 (axiom dependence) it is also lower regular of type 1, that is, it is regular in the sense of Definition 3.

In order to prove another equivalent definition of regular metric, we need the following lemma.

**Lemma 6** (straight-line and sphere intersection properties) – Let \( d \) be a metric on a set \( E \). For any \( x \) and \( y \) in \( E \), and any \( i \in d(\{x\} \times E) \),

(i) \( S(x, i) \cap L_1(x, y) \neq \emptyset \) \( \Rightarrow d(x, y) \leq i \);

(ii) \( S(x, i) \cap L_2(x, y) \neq \emptyset \) \( \Rightarrow i \leq d(x, y) \).

**Proof**

(a) Let us prove (i). For any \( x \) and \( y \) in \( E \), and any \( i \in d(\{x\} \times E) \),

\( S(x, i) \cap L_1(x, y) \neq \emptyset \)

\( \Rightarrow \exists \; u \in E : d(x, u) = d(x, y) + d(y, u) \text{ and } d(x, u) = i \)

(definitions of \( S \) and \( L_1 \))

\( \Rightarrow \exists \; u \in E : i = d(x, y) + d(y, u) \)

(substitution)

\( \Rightarrow d(x, y) \leq i \). (Definition 1 - positiveness of \( d \))

(b) Let us prove (ii). For any \( x \) and \( y \) in \( E \), and any \( i \in d(\{x\} \times E) \),

\( S(x, i) \cap L_2(x, y) \neq \emptyset \)

\( \Rightarrow \exists \; u \in E : d(x, y) = d(x, u) + d(u, y) \text{ and } d(x, u) = i \)

(definitions of \( S \) and \( L_2 \))

\( \Rightarrow \exists \; u \in E : d(x, y) = i + d(u, y) \)

(substitution)

\( \Rightarrow i \leq d(x, y) \). (Definition 1 - positiveness of \( d \))

The next proposition allows a geometrical interpretation for the regular metrics.

**Proposition 7** (second equivalent definition of regular metric) – A metric \( d \) on \( E \) is regular iff for any \( x \) and \( y \) in \( E \), and any \( i \in d(\{x\} \times E) \), the intersection between the straight-line \( L(x, y) \) and the sphere \( S(x, i) \) have at least two diametrically opposite points in the sense that it exists \( u \) and \( v \in S(x, i) \) such that \( u \in (L_1(x, y) \cup L_2(x, y)) \) and \( v \in L_3(x, y) \).

**Proof**

(a) Let us assume that \( d \) is a regular metric on \( E \). Since, for any \( x \) and \( y \) in \( E \), and any \( i \in d(\{x\} \times E) \), at least one of the two conditions \( d(x, y) \leq i \) and \( i \leq d(x, y) \) is satisfied, by Definition 3 (regular metric space), at least one of the two properties \( L_1(x, y) \cap S(x, i) \neq \emptyset \) and \( L_2(x, y) \cap S(x, i) \neq \emptyset \) is satisfied, in other words, we have \( (L_1(x, y) \cup L_2(x, y)) \cap S(x, i) \neq \emptyset \). That is, \( \exists \; u \in S(x, i) : u \in (L_1(x, y) \cup L_2(x, y)) \). Furthermore, by Definition 3 again, for any \( x \) and \( y \) in \( E \), and any \( i \in d(\{x\} \times E) \), \( L_1(x, y) \cap S(x, i) \neq \emptyset \). That is, \( \exists \; v \in S(x, i) : v \in L_3(x, y) \).

(b) Conversely, let us assume that for any \( x \) and \( y \) in \( E \), and any \( i \in d(\{x\} \times E) \), \( \exists \; u \) and \( v \in S(x, i) : u \in (L_1(x, y) \cup L_2(x, y)) \) and \( v \in L_3(x, y) \).

(b1) If \( i < d(x, y) \),

true \( \iff S(x, i) \cap (L_1(x, y) \cup L_2(x, y)) \neq \emptyset \text{ and } i < d(x, y) \) (hypotheses)

\( \Rightarrow S(x, i) \cap (L_1(x, y) \cup L_2(x, y)) \neq \emptyset \) (Property (ii) of Lemma 6)

\( \Rightarrow S(x, i) \cap L_1(x, y) \neq \emptyset \text{ or } S(x, i) \cap L_2(x, y) \neq \emptyset \) (distributivity of \( \cup \))

\( \Rightarrow S(x, i) \cap L_2(x, y) \neq \emptyset \text{ (or } S(x, i) \cap L_1(x, y) \neq \emptyset) \) (\( \emptyset \) is unity for \( \cup \))

\( \Rightarrow d \text{ is lower regular.} \)

(Corollary 5 - first equivalent definition of regular metric)

If \( d(x, y) = i \),

true \( \iff \exists \; u \in E : d(x, y) = d(x, u) + d(u, y) \text{ and } d(x, u) = i \)

(namely \( u = y \) and Property (ii) of Definition 1)

\( \Rightarrow S(x, i) \cap L_2(x, y) \neq \emptyset \) (definitions of \( S \) and \( L_2 \))

\( \Rightarrow d \text{ is lower regular.} \)

(Corollary 5)

(b2)

true \( \iff S(x, i) \cap L_3(x, y) \neq \emptyset \) (hypothesis)

\( \Rightarrow d \text{ is upper regular.} \)

(Corollary 5)

That is, by Corollary 5, \( d \) is regular.
The Euclidean distance \( d \) on \( \mathbb{R}^2 \), given by, for any \( x = (x_1, x_2) \) and any \( y = (y_1, y_2) \in \mathbb{Z}^2 \),

\[
d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},
\]
is regular, nevertheless, when restricted to the discrete plane \( \mathbb{Z}^2 \) it is not regular. For example, the straight-line \( L(x, y) \) containing \( x = (0, 0) \) and \( y = (1, 1) \) consists of the points \((i, i)\) with \( i \in \mathbb{Z} \), and the sphere \( S(x, 1) \) of radius 1 with center at \( x \) consists of the points \((0, 1), (1, 0), (0, 1), (1, 0)\). We observe that this line and this sphere have no intersection. The left-hand side of Figure 1 illustrates this point. Another example of non-regular metric in the discrete plane can be built from the elliptic distance \( d \) given by, for any \( x = (x_1, x_2) \) and any \( y = (y_1, y_2) \in Z^2 \),

\[
d(x, y) = \sqrt{(x_1 - y_1)^2/a^2 + (x_2 - y_2)^2/b^2},
\]
with \( a = 5/3 \) and \( b = 5/4 \). For example, the straight-line \( L(x, y) \) containing \( x = (0, 0) \) and \( y = (2, 1) \) consists of the points \((2i, i)\) with \( i \in Z \), and the sphere \( S(x, 1) \) of radius 1 with center at \( x \) consists of the points \((1, 1), (1, -1), (-1, 1), (-1, 1)\). Again, both have no intersection. The right-hand side of Figure 1 illustrates this point.

![Figure 1 - Examples of non-regular metrics](image)

The discrete metric is not regular despite the fact that for any \( x \) and \( y \in E \), \( x \neq y \) and any \( i \in d \circ \{(x, x) \times E\} = \{0, 1\} \), \( S(x, i) \cap L(x, y) \neq \emptyset \) (since \( L(x, y) = \{x, y\} \), \( S(x, 0) = \{x\} \) and \( S(x, 1) = \{y\} \)). For the discrete metric, we cannot find two diametrically opposite points since \( L_0(x, y) = \{x\} \) and \( x \not\in S(x, 1) \).

From now on, we restrict ourself to translation-invariant metrics on an Abelian group. Such metric property suits most image analysis problems.

**Definition 8** (translation-invariant metric sapce) – A distance or metric \( d \) on an Abelian group \( (E, +) \) is translation-invariant (t.i.) if, for any \( u, x \) and \( y \in E \); \( d(x + u, y + u) = d(x, y) \). A translation-invariant metric space \((E, d)\) is a set \( E \) provided with a t.i. metric \( d \) on \( E \).

As it is well known, every translation-invariant metric on \( E \) can be characterized in terms of a mapping from \( E \) to \( \mathbb{R} \) as stated in the next proposition.

Let \((E, +, o)\) be an Abelian group where \( o \) is the unit element of +. The subtraction on \( E \), denoted \( -\), is the mapping \( E \times E \ni (x, y) \mapsto x + (-y) \in E \), where \( -y \) is the inverse of \( y \).

For any mapping \( d \) from \( E \times E \) to \( \mathbb{R} \) and any \( x \in E \), let \( f_d(x) = d(x, o) \).

For any mapping \( f \) from \( E \) to \( \mathbb{R} \) and any \( x \) and \( y \in E \) let \( d_f(x, y) = f(x - y) \).

**Proposition 9** (characterization of translation-invariant metric) – The mapping \( d \mapsto f_d \) from the set of translation-invariant distances on an Abelian group \((E, +, o)\) to the set of mappings \( f \) from \( E \) to \( \mathbb{R} \) satisfying the properties, for any \( x \in E \):

(i) \( f(x) \geq 0 \) (positiveness)
(ii) \( f(o) = 0 \) (hypothesis)
(iii) \( f(-x) = f(x) \) (symmetry)

is a bijection and its inverse is \( f \mapsto d_f \). Furthermore, \( d \mapsto f_d \) is, as well, a bijection from the set of translation-invariant metrics on the Abelian group \((E, +, o)\) to the set of mappings \( f \) from \( E \) to \( \mathbb{R} \) satisfying the properties (i) - (iii) and the property, for any \( x \) and \( y \in E \):

(iv) \( f(x + y) = f(x) + f(y) \) (subadditivity).

**Proof**

Let us divide the proof into four parts.

(a) Let \( d \) be a mapping from \( E \times E \) to \( \mathbb{R} \).

(a1) Let us assume that \( d \) satisfies Property (i) of Definition 1. For any \( x \in E \), \( f_d(x) = d(x, o) \) (definition of \( f_d \)) \geq 0, (hypothesis)

that is \( f_d \) satisfies Property (i).

(a2) Let us assume that \( d \) satisfies Property (ii) of Definition 1.

\( f_d(o) = d(o, o) \) (definition of \( f_d \)) = 0, (hypothesis)

that is \( f_d \) satisfies Property (ii).

(a3) Let us assume that \( d \) satisfies Property (iii) of Definition 1. For any \( x \in E \),

\[
\begin{align*}
\quad d_f(-x) &= d(-x, o) & (\text{definition of } f_d) \\
&= d(o, x) & (\text{d is t.i.}) \\
&= d(x, o) & (\text{hypothesis}) \\
&= f_d(x), & (\text{definition of } f_d)
\end{align*}
\]

that is \( f_d \) satisfies Property (iii).

(a4) Let us assume that \( d \) satisfies Property (iv) of Definition 1. For any \( x \) and \( y \in E \),

\[
\begin{align*}
\quad f_d(x + y) &= d(x + y, o) & (\text{definition of } f_d) \\
&\leq d(x + y, y) + d(y, o) & (\text{hypothesis}) \\
&= d(x, o) + d(y, o) & (\text{d is t.i.}) \\
&= f_d(x) + f_d(y), & (\text{definition of } f_d)
\end{align*}
\]

that is \( f_d \) satisfies Property (iv).

In other words, from (a1) - (a3), if \( d \) is a t.i. distance then \( f_d \) satisfies Properties (i) - (iii), and, from (a1) - (a4), if \( d \) is a t.i. metric then \( f_d \) satisfies Properties (i) - (iv).

(b) Let \( f \) be a mapping from \( E \) to \( \mathbb{R} \).

(b1) Let us assume that \( f \) satisfies Property (i). For any \( x \) and \( y \in E \),

\[
\begin{align*}
\quad d_f(x, y) &= f(x - y) & (\text{definition of } d_f) \\
&\geq 0, & (\text{hypothesis})
\end{align*}
\]

that is \( d_f \) satisfies Property (i) of Definition 1.

(b2) Let us assume that \( f \) satisfies Property (ii). For any \( x \in E \),

\[
\begin{align*}
\quad d_f(x, x) &= f(x - x) & (\text{definition of } d_f) \\
&= f(o) & (\text{group property}) \\
&= 0, & (\text{hypothesis})
\end{align*}
\]

that is \( d_f \) satisfies Property (ii) of Definition 1.
(b3) Let us assume that \( f \) satisfies Property (iii). For any \( x \) and \( y \in E \),
\[
   d_f(x, y) = f(x - y) = f(-(x - y)) = f(y - x) = d(y, x) \quad \text{(definition of } d)\
\]
that is, \( d_f \) satisfies Property (iii) of Definition 1.

(b4) Let us assume that \( f \) satisfies Property (iv). For any \( x, y \) and \( z \in E \),
\[
   d_f(x, y) = f(x - y) = f((x - z) + (z - y)) \leq f(x - z) + f(z - y) = d(x, z) + d(z, y) = d(x, y) \quad \text{(definition of } d_f)\
\]
that is, \( d_f \) satisfies Property (iv) of Definition 1.

In other words, from (b1) - (b3), if \( f \) satisfies Properties (i) - (iii) then \( d_f \) is a t.i. distance and, from (b1) - (b4), if \( f \) satisfies Properties (i) - (iv) then \( d_f \) is a t.i. metric.

(c) Let \( d_f \) be a t.i. distance on an Abelian group \((E, +, o)\). For any \( x \) and \( y \in E \),
\[
   d_f(x, y) = f(x - y) = f(x - z + (z - y)) = f(x - z) + f(z - y) = d(x, z) + d(z, y) = d(x, y) \quad \text{(definition of } d_f)\
\]
that is, \( f \mapsto d_f \) is the left inverse of \( d \mapsto d_f \).

(d) Let \( f \) be a mapping from \( E \) to \( \mathbb{R} \). For any \( x \in E \),
\[
   f_{d_f}(x) = d_f(x, o) = f(x - o) = f(x) \quad \text{(definition of } f_{d_f})\
\]
that is, \( f \mapsto f_{d_f} \) is the right inverse of \( d_f \mapsto f \).

In other words, from (a) - (d), both mappings \( d \mapsto f \) from the set of distances and the set of metrics are bijections.

In the case of t.i. metrics, we can make an explicit relationship between the above concept of regular metric and the concepts of upper and lower regularity proposed by Kiselman [5] in order to compare balls with different centers. Let us recall the Kiselman’s definitions.

\textbf{Definition 10} (first definition of Kiselman’s regularity for a metric) – Let \((E, +, o)\) be an Abelian Group and let \( f \) a mapping from \( E \) to \( \mathbb{R} \) satisfying Properties (i) - (iv) of Proposition 9 (characterization of t.i. metric), then the translation-invariant metric \( d_f \) on \((E, +, o)\) is

(i) \textit{lower regular (of type 1) for the triangle inequality} if, for any \( x \) and \( y \in E \) such that \( f(y) \leq f(x) \), there exists a point \( \bar{x} \in E \) such that \( f(\bar{x}) = f(x) \) and
\[
   f(\bar{x}) = f(\bar{x} - y) + f(y);\]

(ii) \textit{lower regular (of type 2) for the triangle inequality} if, for any \( x \) and \( y \in E \) such that \( f(x) \leq f(y) \), there exists a point \( \bar{x} \in E \) such that \( f(\bar{x}) = f(x) \) and
\[
   f(\bar{x}) = f(\bar{x} + y) + f(y - \bar{x});\]

(iii) \textit{upper regular} for the triangle inequality if, for any \( x \) and \( y \in E \), there exists a point \( \bar{y} \in E \) such that \( f(\bar{y}) = f(y) \) and
\[
   f(x + \bar{y}) = f(x) + f(\bar{y}).\]

\textbf{Definition 11} (second definition of Kiselman’s regularity for a metric) – A translation-invariant metric \( d \) on an Abelian group \((E, +, o)\) is

(i) \textit{lower regular (of type 1) for the triangle inequality} if, for any \( x \) and \( y \in E \) such that \( f(y) \leq f(x) \), there exists \( \bar{x} \in E \) such that \( f(\bar{x}) = f(x) \) and \( f(\bar{x}) = f(\bar{x} - y) + f(y); \)

(ii) \textit{lower regular (of type 2) for the triangle inequality} if, for any \( x \) and \( y \in E \) such that \( f(x) \leq f(y) \), there exists \( \bar{x} \in E \) such that \( f(\bar{x}) = f(x) \) and \( f(\bar{x}) = f(\bar{x} + y) + f(y - \bar{x}); \)

(iii) \textit{upper regular} for the triangle inequality if, for any \( x \) and \( y \in E \), there exists \( \bar{y} \in E \) such that \( f(\bar{y}) = f(y) \) and \( f(x + \bar{y}) = f(x) + f(\bar{y}). \)

In [5], Kiselman has defined Axioms (i) and (iii). For the sake of completeness, we have added here Axiom (ii) and the expressions "type 1" and "type 2".

Actually, we can rewrite Definition 10 as shown below.

\textbf{Proposition 12} (definition equivalence) – Let \( A \) be the subset of mappings \( f \) from \( E \) to \( \mathbb{R} \) satisfying Properties (i) - (iv) of Proposition 9 (characterization of t.i. metric) and let \( B \) be the subset of mappings \( f \) in \( A \) satisfying Conditions (i) - (iii) of Definition 10, then the set \((d \mapsto d_f)(B)\) (the image of \( B \) through the inverse of \( d \mapsto f \)) is equal to the set \((d \mapsto f)(A)\) (the inverse image of \( B \) through \( d \mapsto f \)).

\textbf{Proof} By Proposition 9 (characterization of t.i. metric), \( f \mapsto d_f \) is a left and right inverse for the mapping \( d \mapsto f \) from the set of translation-invariant metrics to the set \( A \), therefore, for any subset \( X \) of \( A \), \( (d \mapsto d_f)(X) = (d \mapsto f)(X) \). So, the equality is also true for \( B \).

By using the second definition of Kiselman’s regularity for a metric, we show now that the t.i. regular metrics satisfy the Kiselman’s regularity axioms and conversely.

\textbf{Proposition 13} (equivalent definition of translation-invariant regular metric) – Let \( d \) be a translation-invariant metric on an Abelian group \((E, +, o)\), then,

(i) \( d \) is lower regular of type 1 in the sense of Definition 3 (regular metric space) iff \( d \) is lower regular of type 1 for the triangle inequality;

(ii) \( d \) is lower regular of type 2 in the sense of Definition 3 iff \( d \) is lower regular of type 2 for the triangle inequality;

(iii) \( d \) is upper regular in the sense of Definition 3 iff \( d \) is upper regular for the triangle inequality;

(iv) \( d \) is regular iff \( d \) is lower regular of type 2 and upper regular for the triangle inequality.

\textbf{Proof} Let us divide the proof into three parts.

(a) Let us prove (i).
(a1) Let $x$ and $y \in E$, then $f_d(x)$ and $f_d(y) \in d(\{o\} \times E)$. Let us assume that $f_d(y) \leq f_d(x)$.
(d is lower regular of type 1 in the sense of Definition 3)
\[ S(o, f_d(x)) \cap L_2(o, y) \neq \emptyset \text { and } d(o, y) \leq f_d(x) \]
(definition of $S$ and $L_2$)
\[ \exists \bar{x} \in E: d(o, \bar{x}) = d(o, y) + d(y, \bar{x}) \text { and } d(o, \bar{x}) = f_d(x) \]
(definitions of $S$ and $L_1$)
\[ \exists \bar{x} \in E: d(\bar{x}, o) = d(y, o) + d(\bar{x}, y) \text { and } d(\bar{x}, o) = f_d(x) \]
(Definition 1 - symmetry of $d$)
\[ \exists \bar{x} \in E: d(\bar{x}, o) = d(y, o) + d(\bar{x} - x, o) \text { and } d(\bar{x}, o) = i \]
(definition of $f_d$)
\[ \exists \bar{x} \in E: d(\bar{x}, o) = d(x, o) + d(\bar{x} + x, o) \text { and } d(\bar{x}, o) = i \]
(group properties)
\[ \exists \bar{x} \in E: d(\bar{x} + x, o) \neq \emptyset \text { and } d(\bar{x}, o) = f_d(x) \]
(definitions of $S$ and $L_1$)
\[ \exists \bar{x} \in E: d(\bar{x}, o) = d(y, o) + d(\bar{x} - x, o) \text { and } d(\bar{x}, o) = f_d(x) \]
(Definition 1 - symmetry of $d$)
\[ \exists \bar{x} \in E: f_d(\bar{x}) = f_d(x) \text { and } f_d(\bar{x} - y) \text { and } f_d(\bar{x}) = f_d(x) \]
(definition of $f_d$)
\[ \exists \bar{x} \in E: d(\bar{x}, o) = d(y, o) + d(\bar{x} - y, o) \text { and } d(\bar{x}, o) = i \]
(definition of $f_d$)
\[ \exists \bar{x} \in E: d(\bar{x}, o) = d(\bar{x} + x, o) \text { and } d(\bar{x}, o) = i \]
(group properties)
\[ \exists \bar{x} \in E: d(\bar{x} + x, o) = d(y, x) + d(\bar{x} + x, x) \text { and } d(\bar{x} + x, x) = i \]
(definition of $f_d$)
\[ \exists \bar{x} \in E: d(\bar{x} + x, o) \neq \emptyset \text { and } d(\bar{x}, o) = f_d(x) \]
(definitions of $S$ and $L_1$)
\[ \exists \bar{x} \in E: f_d(\bar{x}) = f_d(x) \text { and } f_d(\bar{x} - y) \text { and } f_d(\bar{x}) = f_d(x) \]
(definition of $f_d$)
\[ \exists \bar{x} \in E: d(y, o) = d(\bar{x}, o) + d(\bar{x} - x, o) \text { and } d(\bar{x}, o) = i \]
(definition of $f_d$)
\[ \exists \bar{x} \in E: d(y, o) = d(\bar{x}, o) + d(y - x, o) \text { and } d(\bar{x}, o) = i \]
(definition of $f_d$)
\[ \exists \bar{x} \in E: d(y, o) = d(\bar{x}, o) + d(y - x, \bar{x}) + d(\bar{x}, o) = i \]
(definition of $f_d$)
\[ \exists \bar{x} \in E: d(y, o) = d(\bar{x}, o) + d(y - (x + \bar{x}), x) \text { and } d(\bar{x}, o) = i \]
(group properties)
\[ \exists \bar{x} \in E: d(y, o) = d(\bar{x}, o) + d(y, \bar{x}) + d(\bar{x}, x) \text { and } d(\bar{x}, x) = i \]
(group properties)
\[ \exists \bar{x} \in E: d(y, o) = d(\bar{x}, o) + d(y, \bar{x}) + d(\bar{x}, x) \text { and } d(\bar{x}, x) = i \]
(group properties)

(c2) Conversely, let $x$ and $y \in E$, and $i \in d(\{x\} \times E)$ such that $i \leq d(x, y)$, then $i \leq f_d(x)$ (since $d$ is t.i. and symmetric); furthermore, $d$ is upper regular for the triangle inequality.

(Definition 11)
Actually, we have reused in this work the word "regular" of the Kiselman's definitions for the metric satisfying Definition 3 because of the previous proposition.

As a consequence of Property (iv), we observe that the important Kiselman's axioms are the lower regularity of type 2 and the upper regularity for the triangle inequality and not the lower regularity of type 1 for the triangle inequality.

Proposition 13 may be a short cut for the proof that some t.i. metrics are regular as we show now for the chessboard distance which is of particular interest in image processing.

The set $\mathbb{Z}^2$ equipped with the addition $((x_1, x_2), (y_1, y_2)) \mapsto (x_1 + y_1, x_2 + y_2)$, denoted $+$, forms an Abelian group $(\mathbb{Z}^2, +, o)$ with $o = (0, 0)$ as unitary element.

**Definition 14** (chessboard distance) – The chessboard distance $d_8(x, y)$ between $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{Z}^2$ is the natural number defined by

$$d_8(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$ 

The mapping $d_8: (x, y) \mapsto d_8(x, y)$ from $\mathbb{Z}^2 \times \mathbb{Z}^2$ to $\mathbb{R}$ is called the chessboard distance.

Actually, the chessboard distance $d_8$ is a metric. Furthermore, with respect to the Abelian group $(\mathbb{Z}^2, +, o)$, $d_8$ is by construction translation-invariant.

Let $f_8 = f_{d_8}$, and let $z = (z_1, z_2)$ be a point in $\mathbb{Z}^2$ then,

$$f_8(z) = d_{8}(z, o) = \max\{|z_1 - 0|, |z_2 - 0|\} = \max\{|z_1|, |z_2|\}.$$ 

In other words, $f_8$ is given by, for any $z = (z_1, z_2) \in \mathbb{Z}^2$,

$$f_8(z) = \max\{|z_1|, |z_2|\}.$$ 

Let us show that $d_8$ is regular.

**Proposition 15** (example of regular metric space) – The metric space $(\mathbb{Z}^2, d_8)$ is regular.

**Proof**

(a) Let us show that $d_8$ is lower regular (of type 2) for the triangle inequality. Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{Z}^2$ such that $f_8(x) \leq f_8(y)$, let $i = f_8(x)$ and let $\tilde{x} = (\text{sign}(y_1), \min\{|y_1|, i\}, \text{sign}(y_2), \min\{|y_2|, i\})$, then

$$f_8(\tilde{x}) = \max\{|\text{sign}(y_1)\min\{|y_1|, i\}, |\text{sign}(y_2)\min\{|y_2|, i\}|\}$$

(expressions of $f_8$ and $\tilde{x}$)

$$= \max\{|\text{sign}(y_1)|, \min\{|y_1|, i\}, |\text{sign}(y_2)|, \min\{|y_2|, i\}|\}$$

(by hypothesis, $i \leq \max\{|y_1|, |y_2|\}$)

$$= i$$

(see above, $k = 2$)

If $|y_1| \leq |y_2|$, then

$$|y_1 - \tilde{x}_1| = |y_1 - \text{sign}(y_1)\min\{|y_1|, i\}|$$

(see above, $k = 1$)

$$= \begin{cases} 
0 & \text{if } |y_1| \leq i \\
|y_1| - i & \text{otherwise}
\end{cases}$$

(definition of min)

and

$$|y_2 - \tilde{x}_2| = |y_2 - \min\{|y_2|, i\}|$$

(see above, $k = 2$)

$$= |y_2 - i|$$

(by hypothesis, $i \leq |y_2|$)

therefore,

$$|y_1 - \tilde{x}_1| \leq |y_2 - \tilde{x}_2|.$$ 

In other words, if $|y_1| \leq |y_2|$, then

$$\max\{|y_1 - \tilde{x}_1|, |y_2 - \tilde{x}_2|\} = |y_2 - \tilde{x}_2|$$

(see above)

$$= |y_2 - i|$$

(see above)

$$= \max\{|y_1|, |y_2|\} - i.$$ (hypothesis)

If $|y_2| \leq |y_1|$, then, under the same arguments,

$$\max\{|y_1 - \tilde{x}_1|, |y_2 - \tilde{x}_2|\} = \max\{|y_1|, |y_2|\} - i.$$ Hence, in any case, $\max\{|y_1 - \tilde{x}_1|, |y_2 - \tilde{x}_2|\} = \max\{|y_1|, |y_2|\} - i$.

Finally,

$$f_8(\tilde{x}) = f_8(y - \tilde{x}) = i + f_8(y - \tilde{x})$$

(see above)

$$= i + \max\{|y_1 - \tilde{x}_1|, |y_2 - \tilde{x}_2|\}$$

(expression of $f_8$)

$$= i + \max\{|y_1|, |y_2|\} - i$$

(see above)

$$= \max\{|y_1|, |y_2|\}$$

(cancellation)

$$= f_8(y)$$

(expression of $f_8$)

That is, for any $x$ and $y \in \mathbb{Z}^2$ such that $f_8(x) \leq f_8(y)$, there exists a point $\tilde{x} \in \mathbb{Z}^2$ such that $f_8(\tilde{x}) = f_8(x)$ and $f_8(y) = f_8(x) + f_8(y - \tilde{x})$. In other words, $d_8$ is lower regular (of type 2) for the triangle inequality.

(b) Let us show that $d_8$ is upper regular for the triangle inequality. Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{Z}^2$, let $i = f_8(y)$ and let $\tilde{y} = (\text{sign}(x_1)i, \text{sign}(x_2)i)$, then

$$f_8(\tilde{y}) = \max\{|\text{sign}(x_1)i|, |\text{sign}(x_2)i|\}$$

(see above)

$$= \max\{|i|, i\}$$

(i $\geq 0$)

$$= i$$

(max is idempotent)

$$= f_8(y).$$

(hypothesis)

Furthermore,

$$f_8(x + \tilde{y}) = \max\{|x_1 + \tilde{y}_1|, |x_2 + \tilde{y}_2|\}$$

(see above)

$$= \max\{|x_1 + \text{sign}(x_1)i, |x_2 + \text{sign}(x_2)i|\}$$

(see above)

$$= \max\{|x_1 + i, |x_2 + i|\}$$

(definition of $\|\|$)

Therefore, by applying Property (iv) of Proposition 13 (equivalent definition of translation-invariant regular metric), we may conclude that $d_8$ is regular.
diametrically opposite points, for instance, the points \( u \) and \( v \). In this figure, \( u \in L_2(x, y) \) and \( v \in L_2(x, y) \).

Rosenfeld and Pfaltz [8] have shown in their Proposition 6 that \( d_S \) satisfies the following property: for any \( y \in \mathbb{Z}^2 \) such that \( 1 \leq f_g(y) \), there exists a point \( \tilde{x} \in \mathbb{Z}^2 \) such that \( f_g(\tilde{x}) = 1 \) and \( f_g(y) = f_g(\tilde{x}) + f_g(y - \tilde{x}) \). This result is a consequence of the lower regularity (of type 2) of \( d_g \). More interesting for us is their counter example showing that the octagonal distance \( d_{oct} = \sup\{d_g, g\} \) where \( g \) is the distance given by, for any \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) in \( \mathbb{Z}^2 \), 
\[
g(x, y) = 2(\lfloor |x_1 - y_1| + |x_2 - y_2| \rfloor + 1)/3,
\]
doesn’t satisfy this property. In other words, such octagonal distance is an example of non-regular metric.

![Figure 2 - Intersection of a diameter and a sphere.](image)

Before ending this section, we show that the good geometrical properties of the Euclidean vector space mentioned at the beginning of the introduction are satisfied in a regular metric space.

First, we show that the lower regularity property for a distance is a necessary and sufficient condition to have the usual ball intersection property of the Euclidean vector space.

**Proposition 16** (ball intersection in a lower regular metric space) – Let \((E, d)\) be a metric space then,  
(i) for any \( x \) and \( y \) in \( E \), any \( i \in d(\{x\} \times E) \) and any \( j \in d(E \times \{y\}) \),  
\[
\exists z \in E: d(x, z) \leq i \text{ and } d(z, y) \leq j \implies d(x, y) \leq i + j,
\]
(ii) \((E, d)\) is lower regular iff, for any \( x \) and \( y \) in \( E \), any \( i \in d(\{x\} \times E) \) and any \( j \in d(E \times \{y\}) \),
\[
d(x, y) \leq i + j \implies \exists z \in E: d(x, z) \leq i \text{ and } d(z, y) \leq j.
\]

**Proof**
(a) Let us prove Property (i).
For any \( x \) and \( y \) in \( E \), any \( i \in d(\{x\} \times E) \) and any \( j \in d(E \times \{y\}) \),
\[
\exists z \in E: d(x, z) \leq i \text{ and } d(z, y) \leq j
\]
\[
\implies \exists z \in E: d(x, z) + d(z, y) \leq i + j \text{ and } d(z, y) \leq i + j
\]
(addition is increasing)
\[
\implies \exists z \in E: d(x, z) + d(z, y) \leq i + j \quad \text{(transitivity of \( \leq \))}
\]
\[
\implies d(x, y) \leq i + j.
\]

(b) Let us prove the if part of Property (ii).
For any \( x \) and \( y \) in \( E \), any \( i \in d(\{x\} \times E) \) and any \( j \in d(E \times \{y\}) \),
\[
d(x, y) \leq i + j \implies \exists z \in E: d(x, z) \leq i \text{ and } d(z, y) \leq j
\]
\[
\implies d(x, y) = i + j \implies \exists z \in E: d(x, z) \leq i \text{ and } d(z, y) \leq j
\]
(particular case)
\[
\implies d(x, y) = i + j \implies \exists z \in E: d(x, z) \leq i \text{ and } d(z, y) \leq j
\]
\[
d(x, y) + d(z, y) \leq i + j
\]
\[
\implies d(x, y) = i + j \implies \exists z \in E: d(x, z) \leq i \text{ and } d(z, y) \leq j
\]
\[
d(x, y) \leq d(x, z) + d(z, y) \leq i + j
\]
\[
\implies d(x, y) = i + j \implies \exists z \in E: d(x, z) \leq i \text{ and } d(z, y) \leq j
\]
\[
d(x, y) = i + j \implies \exists z \in E: d(x, z) \leq i \text{ and } d(z, y) \leq j
\]
\[
d(x, y) \leq d(x, z) + d(z, y) \leq i + j
\]
\[
\text{(logical derivation)}
\]
\[
\implies d(x, y) = i + j \implies \exists z \in E: d(x, z) \leq i \text{ and } d(z, y) \leq j
\]
\[
d(x, y) + d(z, y) \leq i + j
\]
\[
\implies d(x, y) = i + j \implies \exists z \in E: d(x, z) = i \text{ and } d(z, y) = j
\]
\[
d(x, y) + d(z, y) = d(x, y)
\]
\[
\text{(logical derivation)}
\]
\[
\implies d(x, y) = i + j \implies \exists z \in E: d(x, z) = i \text{ and } d(z, y) = j
\]
\[
d(x, y) + d(z, y) = d(x, y)
\]
\[
\text{(logical derivation)}
\]
That is, under the hypothesis, for any \( x \) and \( y \) in \( E \), any \( i \in d(\{x\} \times E) \) and any \( j \in d(E \times \{y\}) \) such that \( d(x, y) = i + j \), we have \( \exists z \in E: d(x, z) = i \) and \( d(z, y) = j \), but this implies that, for any \( x \) and \( y \) in \( E \), any \( i \in d(\{x\} \times E) \), \( i \leq d(x, y) \), and \( j \in d(E \times \{y\}) \), we have \( \exists z \in E: d(x, z) = i \) and \( d(z, y) = j \), or equivalently by definitions of \( S \) and \( L_2 \), for any \( x \) and \( y \) in \( E \), and any \( i \in d(\{x\} \times E) \), \( i \leq d(x, y) \), we have \( S(x, i) \cap L_2(x, y) \neq \emptyset \) which proves, by Corollary 5 (first equivalent definition of regular metric), that \( d \) is lower regular.
(c) Let us prove the only-if part Property (ii).
For any \( x \) and \( y \) in \( E \), any \( i \in d(\{x\} \times E) \) and any \( j \in d(E \times \{y\}) \), such that \( d(x, y) \leq i + j \),
\[(b1) \text{ if } i = j = 0, \text{ then } y = x, \text{ therefore } \exists z \in E: d(x, z) \leq i \text{ and } d(z, y) \leq j, \text{ namely } z = y = x; \]
\[(b2) \text{ if } i = 0 \text{ and } j \neq 0, \text{ then } d(x, y) \leq j, \text{ therefore } \exists z \in E: d(x, z) \leq i \text{ and } d(z, y) \leq j, \text{ namely } z = x; \]
\[(b3) \text{ if } i \neq 0 \text{ and } j = 0, \text{ then } d(x, y) \leq i, \text{ therefore } \exists z \in E: d(x, z) \leq i \text{ and } d(z, y) \leq j, \text{ namely } z = y; \]
\[(b4) \text{ if } i = 0 \text{ and } j \neq 0, \text{ and if } d(x, y) < i \text{ or } d(x, y) < j \text{ then } \exists z \in E: d(x, z) \leq i \text{ and } d(z, y) < j, \text{ or } d(x, z) < i \text{ and } d(z, y) \leq j, \text{ namely } z = y \text{ or } z = x; \]
\[(b5) \text{ if } i \leq d(x, y) \text{ and } j \leq d(x, y), \text{ d is lower regular} \]
\[
\exists S(x, i) \cap L_2(x, y) \neq \emptyset \quad \text{(Corollary 5)}
\]
\[
\exists z \in E: d(x, z) + d(z, y) = d(x, y) \text{ and } d(x, z) = i
\]
\[
\text{(definitions of } S \text{ and } L_2)\]
\[
\exists z \in E: i + d(z, y) = d(x, y) \text{ and } d(x, z) = i;
\]
\[
\text{(substitution)}
\]

Furthermore,
\[d(x, y) \leq i + j \Rightarrow \exists z \in E: d(x, z) = i \text{ and } d(z, y) \leq j\]

(lower regularity of \(d\) (see above) and + is double-side increasing)

\[\Rightarrow \exists z \in E: d(x, z) \leq i \text{ and } d(z, y) \leq j\]

(reflexivity of \(\leq\))

\[\text{Definition of ball.}\]

In order to illustrate better the previous proposition, we recall the definition of ball derived from a metric and we give a corollary. We call ball of center \(x \in E\) and radius \(i \in d(\{x\} \times E)\), derived from a metric \(d\), the subset \(B_d(x, i) = \{z \in E: d(x, z) \leq i\}\); in the next section we will give another definition of ball.

**Corollary 17** (ball intersection in a lower regular metric space) – Let \((E, d)\) be a metric space then,

(i) for any \(x, y \in E\), any \(i \in d(\{x\} \times E)\) and any \(j \in d(E \times \{y\})\),

\[B_d(x, i) \cap B_d(y, j) \neq \emptyset \Rightarrow d(x, y) \leq i + j,\]

(ii) \((E, d)\) is lower regular iff, for any \(x, y \in E\), any \(i \in d(\{x\} \times E)\) and any \(j \in d(E \times \{y\})\),

\[d(x, y) \leq i + j \Rightarrow B_d(x, i) \cap B_d(y, j) \neq \emptyset.\]

**Proof**

Properties (i) and (ii) are, respectively, equivalent to Properties (i) and (ii) of Proposition 16 (ball intersection in a lower regular metric space) since \((\exists z \in E: d(x, z) \leq i \text{ and } d(z, y) \leq j)\) is equivalent to \((B_d(x, i) \cap B_d(y, j) \neq \emptyset)\) by definition of \(B_d(x, i)\).

---

Figure 3 illustrates a counter example showing that the elliptic metric restricted to the discrete plane depicted in Figure 1 doesn’t satisfy Property (ii) of Corollary 17. This is another way to conclude that this elliptic metric is not regular.

In Figure 3 the ellipses are the locus of points at distance 1 to their centers of symmetry \(x\) and \(y\) and the elliptic distance between these centers is \(d(x, y) = 9/5\) which is less than 2. In other words, the condition \(d(x, y) \leq i + j\) (with \(i = j = 1\)) is satisfied. Nevertheless, as we can see on this figure, the ball intersection \(B_d(x, 1) \cap B_d(y, 1)\) is empty.

---

Figure 4 shows two examples of lack of intersection in the octagonal metric space \((\mathbb{Z}, d_{oct})\). On the left hand-side the two balls are of radius \(i = j = 1\) and their centers \(x\) and \(y\), 2 units apart. On the right-hand-side the two balls are of radius \(i = 1\) and \(j = 3\), and their centers \(x\) and \(y\), 4 units apart.

On both cases the two balls have no intersections despite the fact that \(d_{oct}(x, y) = i + j\). The right-hand side example is due to Rosenfeld and Pfaltz [8].

Proposition 16 will be a key result to prove Proposition 53 in Section 6 and consequently Proposition 62 of Section 8 showing that the lower regular metrics can be reconstructed from their unit balls using the Minkowski product of Section 4.

---

Figure 4 - Lack of intersection (octagonal distance).

Finally, we show that the upper regularity property for a distance is a sufficient condition to have the usual ball inclusion property of the Euclidean vector space, see also [5, Theorem 5.19].

**Proposition 18** (ball inclusion in an upper regular metric space) – Let \((E, d)\) be a metric space then, for any \(x, y \in E\), any \(i \in d(\{x\} \times E)\) and any \(j \in d(E \times \{y\})\),

(i) \(i + d(x, y) \leq j \Rightarrow B_d(x, i) \subseteq B_d(y, j),\)

(ii) if \((E, d)\) is upper regular then

\[B_d(x, i) \subseteq B_d(y, j) \Rightarrow i + d(x, y) \leq j.\]

**Proof**

(a) Let us prove Property (i).

For any \(x, y \in E\), any \(i \in d(\{x\} \times E)\) and any \(j \in d(E \times \{y\})\),

\[z \in B_d(x, i) \Leftrightarrow d(x, z) \leq i\]

\[(\text{def. of } B_d)\]

\[\Leftrightarrow d(x, y) + d(x, z) \leq d(x, y) + i\]

\[(\text{positiveness of } d \text{ and } + \text{ is increasing})\]

\[\Leftrightarrow d(y, x) + d(z, y) \leq d(y, x) + i\]

\[(\text{symmetry of } d)\]

\[\Leftrightarrow d(y, z) \leq d(y, x) + d(z, y) \leq d(y, x) + i\]

\[(\text{triangle inequality})\]

\[\Rightarrow d(y, z) \leq d(y, x) + i\]

\[(\text{transitivity of } \leq)\]

\[\Rightarrow d(y, z) \leq d(y, x) + i \leq j\]

\[(\text{commutativity of } +)\]

\[\Rightarrow d(y, z) \leq j\]

\[(\text{hypothesis})\]

\[\Leftrightarrow z \in B_d(y, j),\]

\[(\text{def. of } B_d)\]

that is, by inclusion definition, under the hypothesis, \(B_d(x, i) \subseteq B_d(y, j)\).

(b) Let us prove Property (ii).

Let \(d\) be an upper regular metric, let \(x\) and \(y \in E\), let \(i \in d(\{x\} \times E)\) and let \(j \in d(E \times \{y\})\). We divide the proof in five parts.

(b1) Let us prove that \(B_d(x, i) \subseteq B_d(y, j) \Rightarrow d(y, x) \leq j:\)

\[B_d(x, i) \subseteq B_d(y, j) \Rightarrow x \in B_d(y, j)\]

\[\Leftrightarrow d(y, x) \leq j.\]

\[(\text{def. of } B_d)\]
(b2) By Proposition 4 (axiom dependence), if \( d \) is upper regular, then \( d \) is lower regular of type 1. Furthermore, true \( \iff S(y, j) \cap L_1(y, x) \neq \emptyset \) and \( S(x, i) \cap L_3(x, y) \neq \emptyset \) 
(d is lower regular of type 1 and by (b1) \( d(y, x) \leq j \), and \( d \) is upper regular)
\( \iff \exists u \in E: d(y, u) = j \) and \( d(y, u) = d(y, x) + d(x, u) \) and \( \exists v \in E: d(x, v) = i \) and \( d(v, y) = d(x, v) + d(x, y) \) \( (\text{definitions of } S, L_1 \text{ and } L_3) \)
(b3) For any \( v \in E \),
\[ d(x, v) = i \implies v \in B_d(x, i) \] \( (\text{def. of } B_d) \)
\( \iff v \in B_d(y, j) \) \( (B_d(x, i) \subset B_d(y, j)) \)
\( \iff d(y, v) \leq j \) \( (\text{def. of } B_d) \)
(b4) For some \( u \) and \( v \in E \),
\[ d(x, v) + d(x, y) = d(y, x) + d(x, y) \] \( (\text{symmetry of } d) \)
\( = d(y, v) \) \( ((b2)) \)
\( = d(y, v) \) \( (\text{symmetry of } d) \)
\( \leq j \) \( ((b3)) \)
\[ = d(y, u) \] \( ((b2)) \)
\[ = d(y, x) + d(x, u) \] \( ((b2)) \)
\[ = d(y, x) + d(x, y) \] \( (\text{symmetry of } d) \)
\[ = d(y, u) \] \( ((b2)) \)
that is, for some \( u \) and \( v \in E \), by regularity of \( + \), \( d(x, v) \leq d(x, u) \).
(b5) For some \( u \) and \( v \in E \),
\[ i + d(x, y) = d(x, v) + d(x, y) \] \( ((b2)) \)
\[ \leq d(x, u) + d(x, y) \] \( ((b4)) \)
\[ = d(x, y) + d(x, u) \] \( (\text{commutativity of } +) \)
\[ = d(y, x) + d(x, u) \] \( (\text{symmetry of } d) \)
\[ = d(y, u) \] \( ((b2)) \)
\[ = j \] \( ((b2)) \)
that is, if \( d \) is upper regular, then \( B_d(x, i) \subset B_d(y, j) \implies i + d(x, y) \leq d(x, y) \).

3. Balls

The ball definition in this section is independent of the concept of distance seen in the previous section. Instead, it is based on the concepts of set translation and transposition. Because of our interest in digital image processing, the balls will be considered as subsets of the discrete plane \( \mathbb{Z}^2 \) (the Cartesian product of the set of integers by itself).

We denote by \( P(\mathbb{Z}^2) \) the collection of all the subsets of \( \mathbb{Z}^2 \).

Let \( N \) be the set of natural numbers: 0, 1, 2, ... A subset \( X \) of \( \mathbb{Z}^2 \) is finite if there exists a natural number \( n \) and a bijection between \( X \) and the subset of natural numbers \( \{1, 2, \ldots, n\} \). We denote by \#X the natural number \( n \), and by \#X \( \ll \) \( X \) the finiteness of \( X \).

The Abelian group \( \mathbb{Z}^2 \) equipped with the scalar multiplication \( \langle j, (x_1, x_2) \rangle \mapsto j(x_1, x_2) = (jx_1, jx_2) \) with operand \( j \) in \( \mathbb{Z} \) forms a module over the ring \( \mathbb{Z} \) [6, p. 166].

We begin recalling the definition of translated version of a subset.

**Definition 19** (translated version) – The translated version of a subset \( X \) of \( \mathbb{Z}^2 \) by a point \( u \) in \( \mathbb{Z}^2 \) is the subset \( X_u = \{ y \in \mathbb{Z}^2 : y - u \in X \} \).

The next proposition shows a regularity property of the translated version: if \( X_u = X \), then \( u = v \). This property will be important to prove the uniqueness of the concept of radius of a ball in Section 5. The proof of this property relies on a basic poset theorem.

Let \( X \) and \( Y \) be two subsets of \( \mathbb{Z}^2 \). The complement of \( X \) in \( \mathbb{Z}^2 \) is denoted \( X^c \), that is, for any \( x \in \mathbb{Z}^2 \), \( x \in X \iff x \notin X \), and the difference between \( X \) and \( Y \) denoted \( X - Y \), that is, \( X - Y = X \cap Y^c \).

**Proposition 20** (translated version property) – If the subset \( X \) of \( \mathbb{Z}^2 \) is finite, then the mapping \( u \mapsto X_u \) from \( \mathbb{Z}^2 \) to \( P(\mathbb{Z}^2) \) is injective. Furthermore, for any \( u \) and \( v \) in \( \mathbb{Z}^2 \) such that \( u \neq v \), \( X_u - X_v \neq \emptyset \) and \( X_u - X_v \neq \emptyset \) or, equivalently, \( X_u \cap X_v \neq X_u \cap X_v \).

**Proof**

Let \( u \) be a point in \( \mathbb{Z}^2 \) and let \( X \) be a subset of \( \mathbb{Z}^2 \), by the regularity of the addition on \( \mathbb{Z}^2 \), the mapping \( x \mapsto x + u \) from \( X \) to \( X_u \) is a bijection, therefore, if \( X \) is finite then \( X_u \) is finite as well.

Let \( u \in \mathbb{Z}^2 \) and let \( R_u \) be the binary relation on \( \mathbb{Z}^2 \) given by, for any \( x \) and \( y \in \mathbb{Z}^2 \), \( R_u \iff \exists j \in \mathbb{N} : y = x + ju \). Let us first prove that \( R_u \) is a partial order. For any \( x \in \mathbb{Z}^2 \), \( x, x \) is true since for \( j = 0, x = x + ju \) by group property, that is, \( R_u \) is reflexive. If \( u \neq o \), for any \( x, y \in \mathbb{Z}^2 \), \( R_u y \) and \( R_u x \) imply \( x = y \) (and \( y \) is \( x \)). If \( u \neq o \), for any \( x \) and \( y \in \mathbb{Z}^2 \), \( x R_u y \) and \( y R_u x \) \( \iff \exists j, k \in \mathbb{N} : y = x + ju \) and \( x = y + ku \) (by substitution and module property)
\( \iff \exists j, k \in \mathbb{N} : y = x + ju \) and \( x = y + ku \) and \( j + k = 0 \) (natural number property)
\( \iff y = x + ju \) and \( x = y + ku \) and \( j + k = 0 \) (natural number property)
\( \iff y = x \) (natural number property)
that is, \( R_u \) is antisymmetric. For any \( x, y, z \in \mathbb{Z}^2 \), \( x R_u y \) and \( y R_u z \) \( \iff \exists j, k \in \mathbb{N} : x = y + ju \) and \( z = y + ku \) (definition of \( R_u \))
\[ \iff \exists j, k \in \mathbb{N} : x = y + (j + k)u \] (by substitution)
\[ \iff \exists i \in \mathbb{N} : z = x + iu \] (namely, \( i = j + k \))
\[ \iff x R_u z \] (definition of \( R_u \))
that is, \( R_u \) is transitive.

Let \( u \) and \( v \) be two points in \( \mathbb{Z}^2 \), \( u \neq v \) and let \( X \) be a finite subset of \( \mathbb{Z}^2 \), then \( X_u \) is finite as well. Therefore, by Theorem 3 of Birkhoff [4] or Theorem 3 of Szász [7], \( \exists x \in X_u \) such that \( x \) is maximal with respect to \( R_{v-u} \).

Let \( y \) be the point in \( \mathbb{Z}^2 \) such that \( y = x + v - u \), then \( x R_{v-u} y \) (doing \( j = 1 \) in the definition of \( R_{v-u} \)), \( y \in X_u \) (by translated version definition) and \( y \neq x \) (by regularity of the addition on \( \mathbb{Z}^2 \)). Since \( u \neq v \) and \( x \) is maximal, \( x R_{v-u} z \) for no \( z \) in \( X_u \). Consequently \( y \) doesn't belong to \( X_u \). In other words, there exists a point in \( X_u \) (namely \( y \)) which doesn't belong to \( X_u \), that is, \( X_u - X_u \neq \emptyset \) and \( X_u \neq X \). Furthermore, for any \( u, v \in \mathbb{Z}^2 \),
\[ X \setminus X_u \neq \emptyset \iff X_u \subset X_u \quad (B - A) = \emptyset \iff B \subset A \]
\[ \emptyset = X_u \cap X \neq X_u. \quad (B \subset A) \iff (A \cap B) = B \]
Finally, by changing the role of \( u \) and \( v \), for any \( u \) and \( v \in \mathbb{Z}^2 \), \( u \neq v \), \( X_u \setminus X_v \neq \emptyset \) and \( X_u \cap X_v \neq X_u \).

We go on recalling the definition of transpose of a subset.

**Definition 21** (transpose) – The transpose of a subset \( X \) of \( \mathbb{Z}^2 \) is the subset \( X^t = \{ x \in \mathbb{Z}^2 : -x \in X \} \).

A subset \( X \) of \( \mathbb{Z}^2 \) is symmetric (with respect to the origin \( o \)) iff it is equal to its transpose, that is \( X = X^t \) or equivalently, iff \( x \in X \iff -x \in X \). \( \mathbb{Z}^2 \) is an example of symmetric subset. We denote by \( S \) the sub collection of all the finite symmetric subsets of \( \mathbb{Z}^2 \), i.e., \( S = \{ X \in \mathcal{P}(\mathbb{Z}^2) \setminus \{ \emptyset \} \mid X \neq \infty \} \) and by \( S^* \) the sub collection \( S + (\mathbb{Z}^2) \).

From the operations of translation and transposition, we can build the sub collection of symmetric finite subsets and their translated versions, that we call balls. The symmetry assumption was made in order to establish, in Sections 6 and 7, the relationship with distance (which is a symmetric mapping).

**Definition 22** (ball) – A subset \( X \) of \( \mathbb{Z}^2 \) is a ball iff \( \exists u \in \mathbb{Z}^2 \) such that \( X_u \in S^* \).

By definition of \( S^* \), the set \( \mathbb{Z}^2 \) is a ball. Symmetric subsets are balls (doing \( u = o \)).

The 3 by 3 discrete square, denoted by \( B_3 \), consisting of the points \((0, 0), (0, 1), (1, 1) (1, 0), (1, -1), (0, -1), (-1, -1), (-1, 0), (1, -1) \) is ball since it is a symmetric subset.

Another important examples of balls (important from the theoretical point of view) are the singletons (the subsets containing just one point).

**Proposition 23** (ball example) – Any singleton of \( \mathbb{Z}^2 \) is a ball.

**Proof**

For any \( x \in \mathbb{Z}^2 \),
\[
\{(x), x\} = \{x - x\}^t \quad \text{(Definition 19 - translated version)}
\]
\[
= \{0\} \quad \text{(group property)}
\]
\[
= \{-o\} \quad \text{(Definition 21 - transpose)}
\]
\[
= \{0\} \quad \text{(group property)}
\]
\[
= \{x - x\} \quad \text{(group property)}
\]
\[
= \{x\} \quad \text{(Definition 19)}
\]

in other words, for any \( x \in \mathbb{Z}^2 \), \( \exists u \in \mathbb{Z}^2 \) (namely \( u = -x \)) such \( \{(x), x\} = \{x\}_u \), that is, by Definition 22, \( \{x\} \) is a ball of \( \mathbb{Z}^2 \).

We denote by \( B^s \) the sub collection of all the balls and by \( B \) the sub collection of finite balls, i.e., \( B = B^s - \{\mathbb{Z}^2\} \). As a consequence of these definitions, \( S \subset B \subset B^s \subset \mathcal{P}(\mathbb{Z}^2) \).

The opposite – \( u \) of the point \( u \) appearing in the ball definition is a center of symmetry for the ball, it is called its center.

**Definition 24** (ball center) – Let \( X \) be a ball \( (X \in B^s) \), if \( X \) is finite, \( u \in \mathbb{Z}^2 \) is the center of \( X \) iff \( X_u \in S \), and if \( X \) is \( \mathbb{Z}^2 \), its center is the origin \( o \).

Hence, by Definition 22 (ball), a ball of \( \mathbb{Z}^2 \) has always a center, furthermore, if the ball is finite, this center is unique as shown in the next proposition. If the ball is \( \mathbb{Z}^2 \), by convention, we assume that its center is the origin \( o \).

**Proposition 25** (ball center construction) – Let \( X \) be a finite ball (i.e., \( X \in B \)), \( u \) is the center of \( X \) iff \( u = \sum_{x \in X} x / \# X \).

**Proof**

Let \( R \) be a relation on \( \mathbb{Z}^2 \) given by \( x R y \iff y = x \) or \( y = -x \). By group properties, \( R \) is an equivalent relation.

Let \( u_1 \) be a center of \( X \) and let \( u_2 = \sum_{x \in X} x / \# X \).

Let us prove that \( u_1 = u_2 \).

\[
u_2 - u_1 = \left( \sum_{x \in X} x / \# X \right) - u_1 \quad \text{(def. of } u_2)\]
\[
= \left( \sum_{x \in X} x - u_1 \right) / \# X \quad \text{(group property)}
\]
\[
= \sum_{x \in X - u_1} v / \# X \quad \text{ (} v = x - u_1 \text{)}
\]
\[
= \sum_{v \in X - u_1} (v + (-v)) / \# X \quad \text{(center of } X \Rightarrow X - u_1 \in S)\]
\[
= o, \quad \text{(group properties)}
\]

that is, \( u_1 = u_2 \).

We denote by \( \text{center}(X) \) the center of \( X \in B^s \). If the center of \( X \) is \( u \), then we say that \( X \) is with center at \( u \).

The center of a ball is not sufficient to characterize a ball, we need one more parameter that we call here the ball matrix.

**Definition 26** (ball matrix) – Let \( X \) be a ball \( (X \in B^s) \), the matrix of \( X \) denoted by \( \text{matrix}(X) \), is the ball \( X \text{-center}(X) \), that is, \( \text{matrix}(X) = X \text{-center}(X) \).

The ball matrix is a symmetric ball.

**Proposition 27** (ball matrix property) – Let \( X \) be a ball (i.e., \( X \in B^s \)), then \( \text{matrix}(X) \in S^* \).

**Proof**

For any \( X \in B^s \),
\[
\text{matrix}(X)^t = (X \text{-center}(X))^t \quad \text{(ball matrix definition)}
\]
\[
= X \text{-center}(X) \quad \text{(center definition)}
\]
\[
= \text{matrix}(X), \quad \text{(ball matrix definition)}
\]

that is, \( \text{matrix}(X) \in S^* \).
The balls with center at origin $o$ are symmetric and every finite symmetric subset is a ball with center at origin. This, and other properties of the balls with center at origin are shown in the next proposition.

**Proposition 28** (properties of a ball with center at origin) – Let $B$ be a ball (i.e., $B \in B^3$), then the following statements are equivalent:
(i) for any $x, y \in \mathbb{Z}^2$, $x \in B \Leftrightarrow y \in B$,
(ii) $B \in S^d$,
(iii) $\text{center}(B) = o$, 
(iv) $\text{matrix}(B) = B$.

**Proof**
(a) Let us prove (i) $\Rightarrow$ (ii). For any finite ball $B$ (i.e., $B \in B$) and any $x$ and $y \in \mathbb{Z}^2$,
\[ x \in B \Leftrightarrow y \in B, \] (hypothesis (i))
\[ \Leftrightarrow x \in (\mathbb{B}^3), \] (Property (2) of Prop. 4.10 of [2])
that is, for $y = o, B \in B$, in other words, $B \in S$. Furthermore, for $B = \mathbb{Z}^2, B \in S^d$ as a consequence of the definition of $S^d$.
(b) Let us prove (i) $\Leftarrow$ (ii). For any symmetric ball $B$ and any $x$ and $y \in \mathbb{Z}^2$,
\[ x \in B \Leftrightarrow x - y \in B \quad \text{(Definition 19 - translated version)} \]
\[ \Leftrightarrow y - x \in B \] (hypothesis (ii))
\[ \Leftrightarrow y \in B. \] (Definition 19)
(c) Let us prove (ii) $\Rightarrow$ (iii). For any finite ball $B$,
\[ B \in S \Leftrightarrow B = B^* \quad \text{(definition of $S^d$)} \]
\[ \Leftrightarrow B = (B^*)^t \] (Property (1) of Prop. 4.6 of [2])
\[ \Leftrightarrow \text{center}(B) = o. \] (Definition 24 - ball center)
For $B = \mathbb{Z}^2, B \in S^d$ and $\text{center}(B) = o$ are both true as a consequence of the definitions of $S^d$ and center of a ball.
(d) Let us prove (iii) $\Rightarrow$ (iv). For any ball $B$,
\[ \text{matrix}(B) = B_{\text{center}(B)} \quad \text{(Definition 26 - ball matrix)} \]
\[ = B - o \quad \text{(hypothesis (iii))} \]
\[ = B_o \quad \text{(group property)} \]
\[ = B \quad \text{(Property (1) of Prop. 4.6 of [2])} \]
(e) Let us prove (iii) $\Leftarrow$ (iv). For any ball $B$,
\[ B_{\text{center}(B)} = \text{matrix}(B) \quad \text{(Definition 26)} \]
\[ = B \quad \text{(hypothesis (iv))} \]
\[ = B_o \quad \text{(Property (1) of Prop. 4.6 of [2])} \]
\[ = B_{-o} \quad \text{(group property)} \]
that is, if $B$ is finite, by Proposition 20 (translated version property), $\text{center}(B) = o$, if $B = \mathbb{Z}^2$, by convention, $\text{center}(B) = o$ as well.

The subcollection of balls is closed under translation as we show in the next proposition.

**Proposition 29** (ball translation) – Let $X$ be a finite ball (i.e., $X \in B$) and let $v \in \mathbb{Z}^2$, then $X_v$ is a finite ball (i.e., $X_v \in B$). Furthermore, $\text{center}(X_v) = \text{center}(X) + v$ and $\text{matrix}(X_v) = \text{matrix}(X)$.

**Proof**
Let $X$ be a finite ball and let $u = \text{center}(X)$. For any $v \in \mathbb{Z}^2$,
\[ ((X_u)_v)^t = (X_u)^t \quad \text{(Prop. 4.6 of [2])} \]
\[ = X_{-u} \quad \text{(X is a ball and } u \text{ is its center)} \]
\[ = (X_v)_{-u} \quad \text{(Prop. 4.6 of [2])} \]
in other words, for any $X \in B$ and any $v \in \mathbb{Z}^2$, $\exists u \in \mathbb{Z}^2$ such that $(X_u)_v = (X_v)_u$, namely $u' = -u + v$, that is, $X_v \in B$ and by Definition 24 (ball center), $\text{center}(X_v) = \text{center}(X) + v$.

Let us prove the last statement. For any $X \in B$ and any $v \in \mathbb{Z}^2$,
\[ \text{matrix}(X_v) = (X_v)_{-\text{center}(X_v)} \quad \text{(ball matrix definition)} \]
\[ = X_{v - \text{center}(X_v)} \quad \text{(Property (2) of Prop. 4.6 of [2])} \]
\[ = X_{v - \text{center}(X)} - v \quad \text{(first statement)} \]
\[ = X_{-\text{center}(X)} \quad \text{(group property)} \]
\[ = \text{matrix}(X). \quad \text{(ball matrix definition)} \]

Their center and matrix characterize the finite balls.

**Proposition 30** (ball characterization) – The mapping from $B$ to $\mathbb{Z}^2 \times S, X \mapsto (\text{center}(X), \text{matrix}(X))$ is a bijection. Its inverse is $(u, B) \mapsto B_u$.

**Proof**
Let us divide the proof in four parts.
(a) By definition of center, for any $X \in B, \text{center}(X) \in \mathbb{Z}^2$.
Furthermore, by Proposition 27 (ball matrix property), for any $X \in B$, $\text{matrix}(X) \in S$.
(b) Let us prove that for any $u \in \mathbb{Z}^2$ and any $B, B_u \in B$. For any $u \in \mathbb{Z}^2$ and any $B \in S$, let $v = u - t$,
\[ ((B_u)_v)^t = (B_u + v)^t \quad \text{(Property (2) of Prop. 4.6 of [2])} \]
\[ = (B_v)^t \quad \text{(using the same arguments as above)} \]
\[ = B \quad \text{(Property (1) of Prop. 4.6 of [2])} \]
\[ = (B_u)^t, \] (using the same arguments as above) that is, there exists a point $v$ in $\mathbb{Z}^2$ such that $(B_u)_v = (B_u)_v$.
(c) Let us prove that $(u, B) \mapsto B_u$ is a left inverse for $X \mapsto (\text{center}(X), \text{matrix}(X))$. For any $X \in B$,
\[ \text{matrix}(X)_{\text{center}(X)} = (X_{-\text{center}(X)})_{\text{center}(X)} \quad \text{(Definition 26 - ball matrix)} \]
\[ = X_{-\text{center}(X)} \quad \text{(Property (2) of Prop. 4.6 of [2])} \]
\[ = X_v \quad \text{(inverse definition)} \]
\[ = X \quad \text{(Property (1) of Prop. 4.6 of [2])} \]
(d) Let us prove that $(u, B) \mapsto B_u$ is a right inverse for $X \mapsto (\text{center}(X), \text{matrix}(X))$. For any $u \in \mathbb{Z}^2$ and any $B \in S$,
\[ (\text{center}(B_u), \text{matrix}(B_u)) = (\text{center}(B) + u, \text{matrix}(B)) \quad \text{(Proposition 29 - ball translation)} \]
\[ = (o + u, B) \quad \text{(B in S implies (iii) and (iv) of Proposition 28 - properties of a ball with center at origin)} \]
\[ = (u, B). \quad \text{(group property)} \]
Therefore, from (a) - (d), $X \mapsto (\text{center}(X), \text{matrix}(X))$ has an inverse (which is $(u, B) \mapsto B_u$), consequently it is a bijection.

In order to combine geometrically the balls, we recall the Minkowski addition and subtraction definitions [2], [3], [6].
Definition 31 (Minkowski addition and subtraction) – Let A and B be two subsets of $\mathbb{Z}^d$, their Minkowski sum is the subset $A \oplus B = \{a + b: a \in A, b \in B\}$ and their Minkowski difference is the subset $A \ominus B = \{x \in \mathbb{Z}^d: \exists b \in B, \exists a \in A: x = a - b\}$. (A, B) \mapsto A \oplus B is the Minkowski addition and (A, B) \mapsto A \ominus B is the Minkowski subtraction from $P(\mathbb{Z}^d) \times P(\mathbb{Z}^d)$ to $P(\mathbb{Z}^d)$.

The sub collection of balls with center at origin is closed under Minkowski’s addition as we show in the next proposition.

Proposition 32 (symmetric ball addition) – Let X and Y be two symmetric balls (i.e., $X, Y \in S^*$), then $X \oplus Y$ is a symmetric ball (i.e., $X \oplus Y \in S^*$).

Proof
For any $X$ and $Y \in S^*$, $(X \oplus Y)^t = X^t \oplus Y^t$ (Minkowski’s addition property) 
= $X \oplus Y$, ($X$ and $Y \in S$)
that is, $X \oplus Y \in S^*$.

As a consequence of the above symmetric ball addition result, in the next proposition we show that the sub collection of balls is closed under Minkowski’s addition as well.

Proposition 33 (ball addition) – Let X and Y be two finite balls (i.e., $X$ and $Y \in B$), then $X \oplus Y$ is a ball (i.e., $X \oplus Y \in B$), furthermore, $\text{center}(X \oplus Y) = \text{center}(X) + \text{center}(Y)$ and $\text{matrix}(X \oplus Y) = \text{matrix}(X) \oplus \text{matrix}(Y)$.

Proof
Let divide the proof into four parts.
(a) For any $X$ and $Y \in B$, 
$X \oplus Y = \text{matrix}(X)_{\text{center}(X)} \oplus \text{matrix}(Y)_{\text{center}(Y)}$
(Proposition 30 - ball characterization)
= $(\text{matrix}(X)_{\text{center}(X)} \oplus \text{matrix}(Y)_{\text{center}(Y)})$
(Property (5) of Prop. 4.12 of [2])
= $(\text{matrix}(X) \oplus \text{matrix}(Y))_{\text{center}(X) + \text{center}(Y)}$
(Property (5) of Prop. 4.12 of [2])
= $(\text{matrix}(X) \oplus \text{matrix}(Y))_{\text{center}(X) + \text{center}(Y)}$.
(Property (2) of Prop. 4.6 of [2])
(b) For any $X, Y \in B$, by Proposition 27 (ball matrix property) and by Proposition 32 (symmetric ball addition) $\text{matrix}(X) \oplus \text{matrix}(Y) \in S$, therefore, by Part (a) of the proof, it exists $u \in \mathbb{Z}^d$ (namely, $u = -\text{center}(X) + \text{center}(Y)$) such that $(X \oplus Y)_u \in S$. That is, by Definition 22 (ball definition), $X \oplus Y \in B$.
(c) For any $X, Y \in B$, $\text{center}(X \oplus Y) = \text{center}((\text{matrix}(X) \oplus \text{matrix}(Y))_{\text{center}(X) + \text{center}(Y)})$
(Part (a) of the proof)
= $\text{center}(\text{matrix}(X) \oplus \text{matrix}(Y)) + \text{center}(X) + \text{center}(Y)$
(Proposition 29 - ball translation)
= $o + \text{center}(X) + \text{center}(Y)$
(Proposition 29 - ball translation)

Before continuing our study about the ball collection, we need to introduce the Minkowski product.

4. Minkowski product

Within the collection of balls we can identify sub collections in which the balls are mutually related through an external binary operation between a natural number and a subset, that we call the Minkowski product.

We denote by $\mathbb{N}^*$ the set of extended natural numbers (i.e., the natural numbers plus an element denoted $\infty$) with the usual addition extended in such a way that, for any $j \in \mathbb{N}^*$, $j + \infty = \infty = j$ and with the usual order extended in such a way that, for any $j \in \mathbb{N}^*$, $j \leq \infty$.

Definition 34 (Minkowski product) – Let $B \in P(\mathbb{Z}^d)$ such that $B \neq \emptyset$, and let $j \in \mathbb{N}^*$, the Minkowski product of $B$ by $j$ is the subset $jB$ of $\mathbb{Z}^d$ given by

$jB = \begin{cases} \{o\} & \text{if } j = 0 \\ B & \text{if } j = 1 \\ ((j-1)B) \oplus B & \text{if } 1 < j < \infty \\ \mathbb{Z}^2 & \text{if } j = \infty \end{cases}$

The Minkowski product $jB$ should be distinguished from the result of the usual scaling operation of $B$ by $j$ used in the continuous case.

The scaling transform of a subset $B$ by a real number $j$ is the subset denoted scaling$(j, B)$ and given by scaling$(j, B) = \{jx: x \in B\}$.

In the discrete case, both results, $jB$ and scaling$(j, B)$, may be different as it can be verified when $B$ is the 3 by 3 discrete square $B_3$. The difference goes on even if we try to preserve the “interior” of the square using a slightly modified version of the scaling transform given by scaling$2(j, B) = \{ix: x \in B\}$ and $i \in [0, j]$. Figure 5 shows (from left to right and top to bottom) the $B_3$ square, its transformation when applying the above two scaling transformations and at last the Minkowski product, with $j = 2$. We observe that these expansion definitions lead to different results.
In the discrete domain $\mathbb{Z}^+$, the Minkowski product is more appropriate for our purpose than the scaling operation (see Section 7).

We now verify the mix distributivity of this product and then some other elementary properties. Based on the mix distributivity, we could establish in Section 5 Proposition 44 and Proposition 48 about the intersection and inclusion of generated balls.

**Proposition 35** (mix distributivity of the Minkowski product) – Let $B \in P(\mathbb{Z}^2)$ such that $B \neq \emptyset$, for any $i$ and $j \in \mathbb{N}^*$, $(i + j)B = (iB) \oplus (jB)$. (Part (a) of the proof)

```
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image.png}
\caption{Minkowski product versus scaling}
\end{figure}
```

**(Proof)** Let us divide the proof in three parts.

(a) Let us prove that for any $i \in \mathbb{N}$, $(i + 1)B = (iB) \oplus B$. If $i = 0$,

\[
(i + 1)B = 1B = B \quad \text{(hypothesis)}
\]

\[
= \{o\} \oplus B \quad \text{(Definition 34 - Minkowski's)}
\]

\[
= (0B) \oplus B \quad \text{(Property (4) of Prop. 4.12 of [2])}
\]

\[
= (iB) \oplus B \quad \text{(hypothesis)}
\]

If $i > 0$,

\[(i + 1)B = ((i + 1) - 1)B \oplus B \quad \text{(product definition)}
\]

\[
= (iB) \oplus B \quad \text{(natural number property)}
\]

(b) Let us prove the proposition statement for any $i$ and $j \in \mathbb{N}$. If $j = 0$,

\[
(i + j)B = IB \quad \text{(hypothesis)}
\]

\[
= (iB) \oplus \{o\} \quad \text{(Property (4) of Prop. 4.12 of [2])}
\]

\[
= (iB) \oplus (jB) \quad \text{(hypothesis)}
\]

if $j = 1$,

\[
(i + j)B = (i + 1)B \quad \text{(hypothesis)}
\]

\[
= (iB) \oplus (iB) \quad \text{\textbf{(Part (a) of the proof)}}
\]

\[
= (iB) \oplus (iB) \quad \text{(Definition 34)}
\]

\[
= (iB) \oplus (jB) \quad \text{(hypothesis)}
\]

if $j > 1$,

\[
(i + j)B = ((i + j - 1) + 1)B \quad \text{(natural number property)}
\]

\[
= ((i + j - 1)B) \oplus B \quad \text{\textbf{(Part (a) of the proof)}}
\]

\[
= (iB) \oplus ((j - 1)B) \quad \text{\textbf{(is true for $j - 1$)}}
\]

\[
= (iB) \oplus ((j - 1)B) \quad \text{(associativity of $\oplus$)}
\]

\[
= (iB) \oplus (jB) \quad \text{\textbf{(Part (a) of the proof)}}
\]

(c) Let us prove the proposition statement for any $i \in \mathbb{N}^*$ and $j = \infty$:

\[
(i + j)B = (i + \infty)B \quad \text{(hypothesis)}
\]

\[
= \infty B \quad \text{(assumption on the extended addition)}
\]

\[
= \mathbb{Z}^2 \quad \text{(Definition 34)}
\]

\[
= (iB) \oplus (\infty B) \quad \text{(Definition 34)}
\]

\[
= (iB) \oplus (jB) \quad \text{(hypothesis)}
\]

The same result can be obtained by changing the hypotheses on $i$ and $j$.

That is, from (b) and (c), the product is mix distributive. (Part (a) of the proof)

For convenience, let us assume that $\#\mathbb{Z}^2 = \infty$.

**Proposition 36** (elementary properties of the Minkowski product) – Let $B \in P(\mathbb{Z}^2)$ such that $B \neq \emptyset$, for any $i \in \mathbb{N}$ and any $j \in \mathbb{N}^*$,

(i) if $o \in B$ then $o \in jB$;

(ii) if $o \in B$ then the mapping $j \mapsto jB$ from $\mathbb{N}^*$ to $P(\mathbb{Z}^2)$ is increasing;

(iii) if $j \neq 0$ and $\#B > 1$ then $\#(jB) > 1$;

(iv) $\#(jB) \leq (\#B)$;

(v) if $B \subseteq S$ then $jB \subseteq S^2$.

**Proof**

(a) Let us prove Property (i).

By Definition 34 (Minkowski product), $o \in jB$ is true for $j = 0, 1$ and $\infty$. Let us assume that it was true for $j - 1 (1 < j < \infty)$, $o \in (j - 1)B$, $o \in B$ and Property (6) of Prop. 4.12 of [2]

\[
\left. \begin{array}{l}
(j - 1)B \subseteq (j - 1)B \oplus B \\
\end{array} \right\} \text{(definition 34)}
\]

that is, by inclusion definition, $o \in jB$.

(b) Let us prove Property (ii).

Let $i$ and $j \in \mathbb{N}$, such that $i \leq j$.

\[
iB \subseteq iB \oplus (j - i)B \quad \text{(Property (i))}
\]

\[
= o \in (j - i)B \quad \text{(Property (6) of Prop. 4.12 of [2])}
\]

\[
\left. \begin{array}{l}
(j = i + (j - i) \text{ and Proposition 35)}
\end{array} \right\}
\]

that is the mapping $j \mapsto jB$ from $\mathbb{N}^*$ to $P(\mathbb{Z}^2)$ is increasing.

(c) Let us prove Property (iii).

By hypothesis, $\#(jB) > 1$ is true for $j = 1$ and it is always true for $j = \infty$. Let us assume that it was true for $j - 1 (1 < j < \infty)$, and let $x \in (j - 1)B$.

\[
\#(jB) = \#(x) \oplus B \quad \text{(Definition 34)}
\]

\[
\geq \#(x) \quad \text{\textbf{(Minkowski’s addition is increasing (Property (6) of Exerc. 4.9 of [2]) and cardinality property)}}
\]

\[
= \#B \quad \text{(Property (1) of Prop. 4.12 of [2])}
\]

\[
\left. \begin{array}{l}
\text{($X \mapsto X$ is a bijection)}
\end{array} \right\}
\]

\[
> 1. \quad \text{(hypothesis)}
\]

(d) Let us prove Property (iv).
By Property (iii) of Proposition 36, if \((\#B) \leq (\#B')\) is true for \(i = 1\). Let us assume that it was true for \(i - 1\) \((1 < i)\),

\[
\#(iB) = \#(iB_1) + (iB_2 - (iB_1)) = \#(iB_1) + \#((iB_2 - (iB_1))_1)
\]

(Definition 34)

(e) Let us prove Property (v).

Let \(B = B', \) by Minkowski product definition, \(jB = (jB)'\) is true for \(j = 0\) (since \(\{0\} \) is symmetric), 1 and \(\infty\) (since \(Z^2\) is symmetric as well). Let assume that it was true for \(j - 1\) \((1 < j < \infty)\),

\[
(jB)' = ((j - 1)B) \oplus B = (j - 1)B \oplus B
\]

(hypotheses and Proposition 32 (symmetric ball addition))

\[
= jB.
\]

(Definition 34)

Property (v) says that if a subset is symmetric then its product by any natural number is still symmetric.

Next proposition about the regularity of the Minkowski product will be useful for the definition of radius of a ball in the next section.

Proposition 37 (regularity of the Minkowski product) – Let \(B \in P(Z^2)\), such that \(B\) is finite and \(\#B > 1\), then the mapping \(j \mapsto \#(jB)\) from \(N^*\) to \(N^*\), is injective and increasing. Furthermore, under the above assumption on \(B\), the mapping \(j \mapsto jB\) from \(N^*\) to \(P(Z^2)\) is injective (i.e. \(iB = jB \Rightarrow i = j\)).

Proof

Let \(i, j \in N\), such that \(i < j\),

\[
jB = iB \oplus (j - i)B = \bigcup_{u \in (j - i)B}(iB)u
\]

(Definition 34)

(see above inclusion)

Let \(i \in N\) and let \(j = \infty\), then \(i < j\), furthermore,

\[
\#(iB) \leq (\#B)
\]

(by Property (iv) of Proposition 36)

\[
< \infty
\]

(\(B\) is finite)

\[
= \#Z^2
\]

(by convention)

\[
= \#(\infty B)
\]

(Definition 34 - Minkowski product)

\[
= \#(jB).
\]

(hypothesis)

Furthermore, for any \(i, j \in N^*\)

\[
i \neq j \iff i < j \text{ or } j < i
\]

\[
\Rightarrow \#(iB) \neq \#(jB) \text{ or } \#(iB) < \#(iB)
\]

(see above inclusion)

That is, the mapping \(j \mapsto \#(jB)\) from \(N^*\) to \(N^*\), is injective and increasing, and the mapping \(j \mapsto jB\) from \(N^*\) to \(P(Z^2)\), is injective.

Based on the Minkowski product, we define, in the next section, the notions of generated balls and of ball radius. In Section 7, the ball radius will be used to derive a metric from a symmetric ball.

5. Generated balls

Let \(A\) and \(B\) be two subsets of \(Z^2\). If \#B > 1 and \(B\) is finite, then we say that \(A\) is a multiple of \(B\) iff there exists an element \(j \in N^*\) such that \(A = jB\). In this case, we say that the set \(B\) divide \(A\) and we call the natural number \(j\) (which is unique under the restrictions on \(B\)) the quotient of \(A\) by \(B\) and we denote it \(\frac{A}{B}\). The uniqueness of quotient is a consequence of Proposition 37 (regularity of the Minkowski product): let \(i, j\) be two quotients of \(A\) by \(B\), by definition of the quotient, \(iB = A\) and \(jB = A\), that is, \(iB = jB\), therefore, since \(j \mapsto jB\) is injective (Proposition 37), \(i = j\).

With a finite symmetric ball \(B\) we can associate, through the Minkowski product, a sub collection of balls. We say that \(B\) induces a sub collection or that the sub collection is generated by \(B\).

Definition 38 (sub collection generated by a ball) – Let \(B\) be a finite symmetric ball (i.e., \(B \in S\)), such that \(B \neq \{0\}\). The sub collection generated by \(B\), denoted by \(B_n\), is the set of all the balls whose matrices are multiple of \(B\), that is, \(B_n = \{X \in B' : \exists j \in N^*, \text{matrix}(X) = jB\}\).

For any \(j \in N\) and any \(B \in S\), by Property (v) of Proposition 36 (elementary properties of the Minkowski product) and by Proposition 28 (property of a ball with center at origin), matrix\((jB) = jB\), that is \(jB \in B_{jB}\), in other words, \(B_{jB}\) is never empty, moreover, it always contains \(Z^2\). We call the elements of \(B_{jB}\) generated balls and we say that they are generated from the prototype ball \(B\).

The assumption that \(Z^2\) was a ball is convenient in the sense that, in this way, every point in \(Z^2\) is contained in at least one generated ball.
**Proposition 39** (example of generated balls) – Let $B$ be a finite symmetric ball (i.e., $B \in S$), such that $B \neq \{o\}$. Any singleton of $\mathbb{Z}^2$ belongs to $B_b$ (the sub collection generated by $B$).

**Proof**
For any $x \in \mathbb{Z}^2$, by Proposition 23 (ball example) $\{x\} \in B^*$. Furthermore, for any $x \in \mathbb{Z}^2$, $\text{matrix}(\{x\}) = \text{matrix}(\{o\})$.

(Definition 19 - translated version)

(Proposition 29 - ball translation)

(Proposition 28 (property of a ball with center at origin), since $\{o\} \subset S$)

$= 0B$, (Definition 34 - Minkowski product) that is, $\exists j \in \mathbb{N}^*$, $\text{matrix}(\{x\}) = jB$ (namely, $j = 0$).

Consequently, by Definition 38 (sub collection generated by a ball), for any $x \in \mathbb{Z}^2$, $\{x\} \in B_b$. ♦

As a direct consequence of the definition of a generated ball, we can parameterize its matrix by a natural number that we call its radius.

**Definition 40** (radius of a generated ball) – Let $B$ be a finite symmetric ball (i.e., $B \in S$), such that $B \neq \{o\}$. The radius of $X \in B_b$, denoted $\text{radius}_B(X)$ is the quotient of $\text{matrix}(X)$ by $B$, that is, $\text{radius}_B(X) = \frac{\text{matrix}(X)}{B}$. ♦

The uniqueness of the quotient guarantees the uniqueness of the radius.

We observe that the same ball generated from different ball prototypes may have different radius. A ball prototype plays the role of a unit ball.

We are now ready to show that the generated balls can be completely characterized in terms of their centers and their radius.

**Proposition 41** (characterization of the generated balls) – Let $B$ be a finite symmetric ball (i.e., $B \in S$), such that $B \neq \{o\}$. The mapping from $B_b - \{\mathbb{Z}^2\}$ to $\mathbb{Z}^2 \times \mathbb{N}^* X \mapsto (\text{center}(X), \text{radius}_B(X))$ is a bijection and its inverse is $(x, j) \mapsto (jB)_j$. Furthermore, for any $X \in B_b$, $X = \text{matrix}(X)B_{\text{center}(X)}$, and for any $(x, j) \in \mathbb{Z}^2 \times \mathbb{N}$, $(x, j) = (\text{center}((jB)_j), \text{radius}_B((jB)_j))$.

**Proof**
Let us divide the proof in four parts.

(a) For any $X \in B_b - \{\mathbb{Z}^2\}$, by Definition 24 (ball center), $\text{center}(X) \in \mathbb{Z}^2$ and by Definition 40 (radius of generated ball), $\text{radius}_B(X) \in \mathbb{N}$.

(b) For any $x \in \mathbb{Z}^2$ and any $j \in \mathbb{N}$, $\text{matrix}((jB)_j) = \text{matrix}(jB)$ (Proposition 29 - ball translation).

(Property (v) of Proposition 36 (elementary properties of the Minkowski product) and Proposition 28 (property of a ball with center at origin))

that is, by Proposition 37 (regularity of the Minkowski product) $\exists i \in \mathbb{N}$, $\text{matrix}((jB)_j) = iB$, namely $i = j$, and by Definition 38 (sub collection generated by a ball) $(jB)_j \in B_b - \{\mathbb{Z}^2\}$.

(c) Let us prove that $(x, j) \mapsto (jB)_j$ is a left inverse for $X \mapsto (\text{center}(X), \text{radius}_B(X))$. For any $X \in B_b$ and any $x \in \mathbb{Z}^2$, $x \in (\text{radius}_B(X)B)_{\text{center}(X)}$

$\iff x \in \left(\frac{\text{matrix}(X)}{B}\right)_{\text{center}(X)}$ (Definition 40)

$\iff x \in \left(\frac{X - \text{center}(X)}{B}\right)_{\text{center}(X)}$ (Definition 26 - ball matrix)

$\iff x \in (X_{\text{center}(X)})_{\text{center}(X)}$ (quotient property)

$\iff x \in X_{(-\text{center}(X)) + \text{center}(X)}$ (translated version property)

$\iff x \in X$, (group property)

that is, $(x, j) \mapsto (jB)_j$ is a left inverse.

(d) Let us prove that the $(x, j) \mapsto (jB)_j$ is a right inverse for $X \mapsto (\text{center}(X), \text{radius}_B(X))$. For any $x \in \mathbb{Z}^2$ and any $j \in \mathbb{N}$, $\text{center}((jB)_j) = \text{center}(jB) + x$

(Proposition 29 - ball translation)

(Property (v) of Proposition 36 and Proposition 28 (properties of a ball with center at origin))

For any $x \in \mathbb{Z}^2$ and any $j \in \mathbb{N}$,

$\text{radius}_B((jB)_j) = \left((jB)_j\right)_{\text{center}((jB)_j)}$ (Definition 40)

$= \left((jB)_j\right)_{\text{center}((jB)_j) - x}$ (above result)

$= \left((jB)_j\right)_{\text{center}((jB)_j) - x}$ (translated version property)

$= \frac{jB}{B}$, (group property)

in other words, by quotient definition, $\text{radius}_B((jB)_j)B = jB$, therefore, by uniqueness of the quotient (consequence of Proposition 37), $\text{radius}_B((jB)_j) = j$. That is, $(x, j) \mapsto (jB)_j$ is a right inverse.

Therefore, from (a) - (d), the mapping $X \mapsto (\text{center}(X), \text{radius}_B(X))$ has an inverse (which is $(x, j) \mapsto (jB)_j$) and consequently is a bijection. Furthermore, from (c), for any $X \in B_b$, $X = \text{matrix}(X)B_{\text{center}(X)}$, and from (d), for any $(x, j) \in \mathbb{Z}^2 \times \mathbb{N}$, $(x, j) = (\text{center}((jB)_j), \text{radius}_B((jB)_j))$. ♦

The radius inherits some properties of the Minkowski’s addition.

**Proposition 42** (radius properties) – Let $B$ be a finite symmetric ball (i.e., $B \in S$), such that $B \neq \{o\}$, then

(i) $\text{radius}_B(\{o\}) = 0$

(ii) $\text{radius}_B(B) = 1$

(iii) for any $X, Y \in B_b - \{\mathbb{Z}^2\}$, $\text{radius}_B(X \oplus Y) = \text{radius}_B(X) + \text{radius}_B(Y)$. ♦
Proof
(a) Let us prove (i).
\[
\text{radius}_B(o) = \text{radius}_B(0B) = 0.
\]
(Definition 34 - Minkowski product)

(Proposition 41 - characterization of the generated balls)

(b) Let us prove (ii).
\[
\text{radius}_B(B) = \text{radius}_B(1B) = 1.
\]
(Definition 34)

(c) Let us prove (iii). For any \(X, Y \in B_B - \{Z\}\),
\[
\text{radius}_B(X \oplus Y) = \frac{(X \oplus Y) - \text{center}(X \oplus Y)}{B}.
\]
(Definition 40)

(Proposition 41 - mix distributivity of the Minkowski product)

in other words, by quotient definition, \(\text{radius}_B(X \oplus Y)B = (\text{radius}_B(X) + \text{radius}_B(Y))B\), therefore, by uniqueness of the quotient (consequence of Proposition 37), \(\text{radius}_B(X \oplus Y) = \text{radius}_B(X) + \text{radius}_Y(Y)\).

Within the poset \((B_B, \sqsubset)\) generated by a ball \(B\), we can identify some useful chains: the chains of generated balls with center at given points.

Let \(B\) be a finite symmetric ball (i.e., \(B \in S\)), such that \(B \neq \{o\}\), and let \(B_B(x)\) be the sub collection of \(B_B\) consisting of all the balls with center at a given point \(x\) of \(Z^2\), that is, \(B_B(x) = \{X \in B_B : \text{center}(X) = x\}\).

In particular, \(B_B(o) = \{B : j \in N\}\) (by Property (iv) of Proposition 28 - properties of the generated balls with center at origin).

Proposition 43 (chain of generated balls with center at a given point) – Let \(B\) be a finite symmetric ball (i.e., \(B \in S\)), such that \(o \in B\) and \(B \neq \{o\}\). Then the mapping \(j \mapsto (jB)_x\) from \((N^*, \leq)\) to \((B_B(o), \sqsubset)\) when \(x = o\), and from \((N, \leq)\) to \((B_B(x), \sqsubset)\) when \(x \neq o\), is a poset isomorphism and the sub collection \((B_B(x), \sqsubset)\) is a chain. Its inverse is \(X \mapsto \text{radius}_B(X)\). The sub collection \(B_B(x)\) has a smaller element which is \(\{x\}\) and a greater one which is \(Z^2\) when \(x = o\).

Proof
Let \(B\) be a finite symmetric ball (i.e., \(B \in S\)), such that \(o \in B\) and \(B \neq \{o\}\), and let \(x \in Z^2\). We observe that \(o \in B\) and \(B \neq \{o\}\) \(\Rightarrow \#B > 1\). For any \(j_1\) and \(j_2 \in N^*\), \(j_1 \neq j_2 \Rightarrow j_1B \neq j_2B\)

(Proposition 37 - regularity of the Minkowski product)
\[
\Leftrightarrow (j_1B)_x \neq (j_2B)_x.
\]
(translation is injective)

that is, the mapping \(j \mapsto (jB)_x\) from \(N^*\) to \(B_B(o)\) when \(x = o\), and from \(N\) to \(B_B(x)\) when \(x \neq o\), is injective. By definition of \(B_B(x)\), it is surjective, that is it a bijection. From Proposition 41 (characterization of the generated balls), its inverse is \(X \mapsto \text{radius}_B(X)\).

Furthermore, for any \(j_1, j_2 \in N^*\),
\[
j_1 \leq j_2 \Rightarrow j_1B \sqsubseteq j_2B
\]

\((j \mapsto jB\) is increasing (Property (ii) of Proposition 36))
\[
\Rightarrow (j_1B)_x \sqsubseteq (j_2B)_x,
\]
(translation is increasing)

that is, the mapping \(j \mapsto (jB)_x\) from \(N^*\) to \(B_B(o)\) when \(x = o\), and from \(N\) to \(B_B(x)\) when \(x \neq o\), is increasing.

Conversely, let \(B_1\) and \(B_2\) be two balls in \(B_B\) with center at \(x\), and let \(j_1\) and \(j_2\) their respective radius
\[
j_2 \leq j_1\text{ and }j_1 \neq j_2 \Rightarrow B_2 \sqsubseteq B_1\text{ and }B_1 \neq B_2.
\]

\((j \mapsto (jB)_x\) is injective and increasing)
\[
\Leftrightarrow B_2 \sqsubseteq B_1\text{ and }B_1 \neq B_2\text{ or }B_2 \neq B_1.
\]
(anti-symmetry of inclusion)

\(\Leftrightarrow B_2 \sqsubseteq B_1\text{ and }B_1 \neq B_2\text{ (logical derivation)}
\]

\(\Rightarrow B_1 \neq B_2\),

(translation is injective)

in other words,
\[
B_1 \sqsubseteq B_2 \Rightarrow j_1 < j_2 \text{ or }j_1 = j_2
\]

(above result)
\[
\Leftrightarrow j_1 \leq j_2,
\]

\((N^*, \leq)\) is a chain)

this proves that the mapping \(j \mapsto (jB)_x\) is two-sided increasing.

Consequently, the mapping \(j \mapsto (jB)_x\) is a poset isomorphism, and the sub collection \((B_B(x), \sqsubset)\) is isomorphic to the chain \((N^*, \leq)\). That is, this sub collection is a chain as well. The smaller element \((0)\) of \(N^*\) maps to \(\{x\}\) which is therefore the smaller element of \(B_B(x)\), and the greater element \((\infty)\) of \(N^*\) maps to \(Z^2\) which is therefore the greater element of \(B_B(x)\).

Based on the mix distributivity of the Minkowski product, we can establish the following geometrical property. This proposition shows that the intersection between generated balls occurs under the same condition as for the balls derived from an Euclidean distance on the continuous plane for example.

Proposition 44 (generated balls versus intersection) – Let \(B\) be a finite symmetric ball (i.e., \(B \in S\)), for any points \(x\) and \(y\) in \(Z^2\) and any numbers \(i\) and \(j\) in \(N^*\), \(y \in ((i + j)B)_x \Leftrightarrow (iB)_x \cap (jB)_x \neq \emptyset\).

Proof
Let \(x\) and \(y\) \(\in Z^2\) and let \(i \) and \(j \in N^*\),
\[
y \in ((i + j)B)_x \Leftrightarrow y \in ((iB)_x \oplus (jB)_x)
\]
(Proposition 35 - mix distributivity of the product)
\[
\Leftrightarrow y \in (iB)_x \oplus (jB)_x
\]
(translation invariance of \(\oplus\))

(Proposition 5 of Prop. 4.12 of [2])
\[
\Leftrightarrow y \in \bigcup_{z \in (iB)_x} (jB)_z
\]
(equivalent definition of \(\oplus\))

(Proposition 1 of Prop. 4.12 of [2])
\[
\Leftrightarrow \exists z \in (iB)_x : y \in (jB)_z
\]
We now recall the definitions of erosion by a structuring element and of B-border. We will use the latter definition in the study of the generated ball border.

**Definition 45** (erosion by a structuring element) – Let B be a subset of $\mathbf{Z}^2$. The erosion by B is the mappings from $\mathbf{P}(\mathbf{Z}^2)$ to $\mathbf{P}(\mathbf{Z}^2)$ $\varepsilon_B : X \mapsto X \ominus B$. The subset $B$ is the structuring element of $\varepsilon_B$.

**Definition 46** (B-border) – Let B be a subset of $\mathbf{Z}^2$. The B-border of a subset $X$ of $\mathbf{Z}^2$ is the subset $\partial_B(X) = X - \varepsilon_B(X)$.

We observe that the B-border is an inner border: $\partial_B(X) \subset X$.

**Proposition 47** (B-border properties) – Let $B$ and $X$ be two subsets of $\mathbf{Z}^2$, and let $x$ be a point in $\mathbf{Z}^2$, then

(i) if $\#B > 1$ then $\partial_B\{\{o\}\} = \{o\}$ ($\{o\}$ is an invariant)

(ii) $\partial_B(B(x)) = \partial_B(X)$ (translation-invariance)

(iii) $X$ and $B \in S \Rightarrow \partial_B(X) \in S$. (symmetry preservation)

**Proof**

(a) Let us prove (i). If $\#B > 1$ then $\partial_B\{\{o\}\} = \{o\}$.

(b) Let us prove (ii). For any $x \in \mathbf{Z}^2$,

$\partial_B(X) = \{x - \varepsilon_B(X)\}$ (Definition 46)

$= \{x - \varepsilon_B(X)\}$ (translation-invariance of $-\$)

$= \{x - \varepsilon_B(X)\}$ (Definition 45)

(c) Let us prove (iii). For any $u \in \mathbf{Z}^2$,

$u \in \partial_B(X) \Rightarrow u \in X - \varepsilon_B(X)$ (Definition 46)

$\Rightarrow u \in X - (X \ominus B)$ (Definition 45)

$\Rightarrow u \in X$ and $u \in (X \ominus B)$ (def. of $\ominus$)

$\Rightarrow u \in X \text{ and } \forall b \in B, \exists x \in X : u = x - b$ (writing $x' = x - b$ and $b' = -b$, and hypothesis $X \text{ and } B \in S$)

$\Rightarrow u \in \partial_B(X)$ (Definition 45)

In order to be able to characterize the integer-valued t.i. regular metrics later on, we now introduce the concept of "closed" sub collection of balls. A sub collection of balls $B_B$ is $B$-closed if any members of $B_B$ morphologically $B$-closed, that is, for any $X \in B_B$, $X$ satisfies the equation $X = (X \ominus B) \ominus B$. If $B_B$ is $B$-closed then we say for convenience that $B$ has the closure property.

The balls multiple of the 3 by 3 discrete square $B_3$ of Section 3, are examples of morphologically closed balls with respect to $B_3$ (see Section 6).

But, for example, the ball $B = (2B_{a} - 1B_{a}) + 0B_{a}$ is not morphologically closed with respect to itself since, in this case, $(B \ominus B) \ominus B = 2B_{a}$ and consequently $(B \ominus B) \ominus B \neq B$.

Under the closure property the sufficient condition to have nested balls becomes necessary.

**Proposition 48** (generated balls versus inclusion) – Let $B$ be a finite symmetric ball (i.e., $B \in S$). For any points $x$ and $y$ in $\mathbf{Z}^2$, and any numbers $i$ in $\mathbf{N}$ and $j$ in $\mathbf{N}^*$,

(i) $x \in (iB)_{i} \Rightarrow (iB)_{i} \subset ((i + j)B)_{i}$

(ii) if $B$ has the closure property, then

$(iB)_{i} \subset ((i + j)B) \Rightarrow x \in (iB)_{i}$.

**Proof**

Let $B$ be a finite symmetric ball (i.e., $B \in S$).

(a) Let us prove Property (i). For any $x$ and $y$ in $\mathbf{Z}^2$, any $i$ in $\mathbf{N}$ and $j$ in $\mathbf{N}^*$,

$(iB)_{i} \subset \bigcup_{u \in (jB)_{j}} \{u \in (iB)_{i}\}$ (x \in (jB)_{j})

$= (jB)_{j} \oplus iB$ (equivalent definition of $\oplus$ (Property 1 of Prop. 4.12 of [2]))

$= (jB \oplus iB)$ (translation invariance of $\oplus$

(Property 5 of Prop. 4.12 of [2]))

(b) Let us prove Property (ii). If $x \in (iB)_{i}$, let $i \in \mathbf{N}$ and $j \in \mathbf{N}^*$. If $i = 0$,

$x \in \{x\}$ (singleton definition)

$= \{x\}$ (Definition 19 - translated version)

$= \emptyset B_{i}$ (Definition 34 - Minkowski product) (hypothesis)

$\subset (i(0 + j)B)$ (natural number property that is, $(0B)_{0} \subset ((0 + j)B) \Rightarrow x \in (jB)_{j}$).

If $i \neq 0$,

$x \in \{x\}$ (singleton definition)

$= (iB)_{i} \ominus iB$ (property)

$\subset ((i + j)B)_{j} \ominus iB$ (hypothesis and $\ominus$ is increasing)

$= (((i - 1) + j)B)_{j} \ominus iB$ (properties of $+$ on $\mathbf{N}^*$)

$= (((i - 1) + j)B)_{j} \ominus iB$ (Proposition 35)
We observe that the last implication in Property (ii) of Proposition 48 may be false when $B$ has not the closure property. We can construct the following counter example: let $B = (2B_8 - 1B_8) + 0B_8$, let $x \in 1B_8 - 0B_8$ and let $y = o$, then $(1B_8) \subset (2B_8)$, but $x \notin (1B_8)$.

Figure 6 illustrates this counter example. On the left-hand side we see that the dark gray ball $(1B_8)$, is included in the light gray ball $(2B_8)$, the 9 by 9 square), nevertheless we see on the right-hand side that $x$ doesn’t belong to $(1B_8)$.

The closure property for a prototype ball allows us to "go backward" along a chain of generated balls.

**Proposition 49** (erosion of a generated ball) – Let $B$ be a subset of $Z^2$. For any $x \in Z^2$ and $j \in N - \{0\}$,

(i) $\varepsilon_B((jB)_x) = ((j - 1)B)_x$

(ii) if $B$ has the closure property, then $\varepsilon_B((jB)_x) = ((j - 1)B)_x$.

**Proof**

For any subset $B$ of $Z^2$ having the closure property, any $x \in Z^2$ and any $j \in N - \{0\}$,

$\varepsilon_B((jB)_x) = (jB)_x \ominus B$ (Definition 45 - erosion by a subset)

$= (jB \ominus B)_x$

(translation invariance of \( \ominus \) (Property (4) of Prop. 4.13 of [2]))

$= (((j - 1)B) \ominus B)_x$ (Property (5) of Prop. 4.12 of [2])

$\subseteq (j - 1)B_x$

(\( \subseteq \)) since $B$ has the closure property and Proposition 41 (characterization of the generated balls); \( \supseteq \) since closing is extensive.

Next proposition will be useful in Section 7 to show that a metric induced by a ball having the closure property is regular (Proposition 61).

**Proposition 50** (generated balls versus $B$-border) – Let $B$ be a finite symmetric ball (i.e., $B \in S$) 1 having the closure property and such that $\#B > 1$. For any points $x$ and $y$ in $Z^2$ and any numbers $i$ in $N$ and $j$ in $N^*$,

(i) $x \in \partial_B((i + j)B)_x \Rightarrow \partial_B((i + j)B)_x \cap \partial_B((iB)_y) \neq \emptyset$;

(ii) $x \in \partial_B((i + j)B)_x \Rightarrow \partial_B((iB)_y) \cap \partial_B((jB)_y) \neq \emptyset$.

**Proof**

(a) Let us prove Property (i). Let $x$ and $y \in Z^2$, let $i \in N$ and let $j \in N^*$. If $j \neq 0$,

$x \in \partial_B((0B)_y) \Leftrightarrow x \in \partial_B((0B)_y)$. (Definition 44 - Minkowski product)

$x \in \partial_B((iB)_y)$ (by hypothesis, as shown above $x = y$)

(\#B > 1 and Property (i) of Proposition 47)

$x \in \partial_B((iB)_y) \Leftrightarrow x \in \partial_B((0B)_y)$. (Definition 19 - translated version)

$x \in \partial_B((iB)_y)$. (singleton definition)

Furthermore, if $x \in \partial_B((0B)_y)$

$\partial_B((i + 1)B)_x \cap \partial_B((i1B)_x) = \partial_B((iB)_x) \cap \partial_B((iB)_x)$

(natural number property)

$= \partial_B((iB)_x)$ (Proposition 37 - regularity of the Minkowski product)

(by hypothesis, as shown above $x = y$)

$= \partial_B((iB)_x)$ (\( \cap \) is idempotent)

$= \partial_B((iB)_x)$ (Definition 46 - 9-Border)

$= (iB)_x - ((i - 1)B)_x$

(B has the closure property and Property (ii) of Proposition 49 - erosion of a generated ball)

$\neq \emptyset$.

(Proposition 37 - regularity of the Minkowski product)

that is, $x \in \partial_B((0B)_y) \Rightarrow \partial_B((i + 1)B)_x \cap \partial_B((iB)_x) \neq \emptyset$.

If $j \neq 0$,

$x \in \partial_B((0B)_y) \Leftrightarrow x \in \partial_B((iB)_y) - \varepsilon_B((jB)_y)$ (Definition 46)

$\Rightarrow x \notin \varepsilon_B((jB)_y)$ (def. of –)

$\Rightarrow x \notin ((j - 1)B)_y$ (Property (i) of Proposition 49)

$\Rightarrow (iB)_x \subset ((i + j - 1)B)_y$

(B has the closure property and Property (ii) of Proposition 48 - generated balls versus inclusion)

$\Rightarrow (iB)_x \subset \varepsilon_B((i + j)B)_y$ (Property (ii) of Proposition 49)

$\Rightarrow (iB)_x \cap \varepsilon_B((i + j)B)_y$ (inclusion property)

that is, $x \in \partial_B((jB)_y) \Rightarrow X \cap \varepsilon_B((Y)_x) \neq \emptyset$, where $X = (iB)_x$ and $Y = ((i + j)B)_y$. Furthermore, for any $j \in N^*$, $x \in \partial_B((jB)_y) \Rightarrow x \in \partial_B((jB)_y) - \varepsilon_B((jB)_y)$ (Definition 46)

$\Rightarrow x \in (jB)_y$ (def. of –)

$\Rightarrow (iB)_x \subset ((i + jB)_y$ (Property (i) of Proposition 48)

$\Rightarrow X \subset Y$ (definitions of $X$ and $Y$)

$\Rightarrow \varepsilon_B(X) \subset \varepsilon_B(Y)$ (erosion is increasing (Proposition. 3.3 of [2]))

$\Rightarrow (\varepsilon_B(Y))^+ \subset (\varepsilon_B(X))$. (complementation is decreasing)

Hence,
\( x \in \partial_B((i + j)B_j) \Rightarrow \begin{cases} \quad X \cap (\mathcal{E}_B(Y))^c \neq \emptyset \\ \quad X \subset Y \\ \quad (\text{see above}) \\ \quad (\mathcal{E}_B(Y))^c \subset (\mathcal{E}_B(X))^c \end{cases} \)

\( X \cap (\mathcal{E}_B(Y))^c \neq \emptyset \) (intersection property)

\( (X \cap Y) \cap ((\mathcal{E}_B(Y))^c \cap (\mathcal{E}_B(X))^c) \neq \emptyset \) (substitution)

\( \Rightarrow (X \cap (\mathcal{E}_B(Y))^c) \cap (Y \cap (\mathcal{E}_B(Y))^c) \neq \emptyset \)

\( \Rightarrow \partial_B(X) \cap \partial_B(Y) \neq \emptyset. \) (Definition 46)

That is, for any \( j \in \mathbb{N}^*, \ x \in \partial_B((i + j)B_j) \Rightarrow \partial_B(((i + j)B_j) \cap \partial_B((i + 1)B_j) \neq \emptyset.

(b) Let us prove Property (ii). For any \( x \) and \( y \) in \( \mathbb{Z}^2 \), any \( i \in \mathbb{N} \) and \( j \in \mathbb{N}^* \). If \( j = 0 \),

\( x \in \partial_B((i + 0)B_i) \Rightarrow x \in \partial_B((i + j)B_j) \)

(natural number property)

\( \Rightarrow x \in (\partial_B((i + j)B_j), \ y \in \{y\}) \)

(Property (ii) of Proposition 47)

\( \Rightarrow y \in \partial_B((i + j)B_j) \)

(Proposition 28 (properties of a ball with center at origin) and Property (iii) of Proposition 47)

\( \Rightarrow y \in \partial_B((i + 0)B_i) \)

(Property (ii) of Proposition 47 - B-border properties)

\( \Rightarrow \partial_B((i + 0)B_i) \)

(Definition 34)

\( \Rightarrow (\partial_B(iB_i), \ y \in \partial_B((i + 0)B_i) \neq \emptyset. \) (intersection and empty set definitions)

If \( j \neq 0 \),

\( x \in \partial_B((i + j)B_j) \Rightarrow x \in ((i + j)B_j) - \varepsilon_B(((i + j)B_j)) \) (Definition 46)

\( \Rightarrow x \notin \varepsilon_B((i + j)B_j) \) (def. of –)

\( \Rightarrow x \notin ((i + j - 1)B_i) \) (Property (i) of Proposition 49)

\( \Rightarrow (iB_i) \subset ((i + j - 1)B_i) \) (Property 44 - generated balls versus intersection)

\( \Rightarrow (iB_i) \subset (\varepsilon_B((i + j - 1)B_i))^c \) (property of \( \subset \))

That is, \( x \in \partial_B(((i + j)B_j) \Rightarrow x \in (\varepsilon_B(X))^c \).

Furthermore,

\( x \in \partial_B(((i + j)B_j) \Rightarrow x \in ((i + j)B_j) - \varepsilon_B(((i + j)B_j)) \) (Definition 46)

\( \Rightarrow x \in ((i + j)B_j) \)

(Proposition 44 - generated balls versus intersection)

\( \Rightarrow (iB_i) \subset ((i + j - 1)B_i) \) (Property (i) of Proposition 49)

\( \Rightarrow (iB_i) \subset (\varepsilon_B((i + j - 1)B_i))^c \) (property of \( \subset \))

\( \Rightarrow (iB_i) \subset (\varepsilon_B((i + j - 1)B_i))^c. \) (as above)

We observe that the implication in Property (i) of Proposition 50 may be false when \( B_k \) is not \( B \)-closed, for example, let \( B = (2B_k - 1B_k) + 0B_k \), let \( x \in 1B_k - 0B_k \) and let \( y = a \), then \( x \in \partial_B(2B_k) \), but \( \partial_B(3B_k) \cap \partial_B(1B_k) = \emptyset. \)

Figure 7 illustrates this counter example. On this figure we see that the point \( x \) belongs to the dark gray ball border \( \partial_B(2B_k) \), nevertheless we see that the light gray ball border \( \partial_B(3B_k) \) has no intersection with the ball border \( \partial_B(1B_k) \) (depicted as a set of squares).

In the last proposition of this section we present some properties of the radii of generated balls. Part of these properties will be used in the proof of Proposition 59. Properties (ii) and (iii) will be used in future work.

**Proposition 51** (radius properties of nested generated balls)

Let \( B \) be a finite symmetric ball (i.e., \( B \in S \), such that \( o \in B \) and \( B \neq \{0\} \). For any balls \( X \) and \( Y \) in \( B_k \),

(i) \( X \subset Y \Rightarrow \text{radius}_B(X) \leq \text{radius}_B(Y) \);

(ii) \( X \neq Y \wedge X \subset Y \Rightarrow \text{radius}_B(X) < \text{radius}_B(Y) \);
(a) Let us prove (i):
Proof
For any balls $X$ and $Y$ in $B_h$, let $x = \text{center}(X)$, let $y = \text{center}(Y)$, let $i = \text{radius}_h(X)$ and let $j = \text{radius}_h(Y)$.

We are now ready to begin the study of the relationship between metrics and symmetric balls.

6. From metric to symmetric ball

With a t.i. metric, we can associate a ball with center at origin. Let $(\mathbb{Z}^2, d)$ be a t.i. metric space, the unit ball of $(\mathbb{Z}^2, d)$, denoted by $B_d$, is the set of all the points at a distance less than or equal to one from the origin, that is,

$$B_d = \{ u \in \mathbb{Z}^2 : d(u, o) \leq 1 \}.$$

An exhaustive inspection of the nine points of $B_8$ shows that $B_{d_8} = B_8$, in other words, the $3 \times 3$ discrete square is the unit ball of the chessboard metric space $(\mathbb{Z}^2, d_8)$.

In the next proposition, we show three properties of the unit ball.

Proposition 52 (properties of the unit ball) – For any metric $d$ on $\mathbb{Z}^2$:

(i) $B_d \subseteq S$.
(ii) $o \in B_d$ and $-o \in B_d$.
(iii) $B_d \neq \{ o \}$.

Proof
(a) Let us prove (i). For any $u \in \mathbb{Z}^2$,

- $u \in B_d \iff d(u, o) \leq 1$ (def. of $B_d$)
- $-u \in B_d$, $d(-u, o) = d(\epsilon, o) \leq 1$ (symmetry of $d$)
- $B_d \subseteq S$.

(b) Let us prove (ii):

- $d(u, o) = 0$ (def. of $B_d$) (natural number property)
- $B_d \neq \{ o \}$.

Property (i) of the previous proposition shows that the unit ball is really a ball in the sense of Definition 22 and that its center is the origin.

In the next proposition, we show a relationship between a t.i. lower regular metric and the Minkowski product of its unit ball.

(d) Let us prove (iv):

- true $\iff \text{center}(Y) = \text{center}(X)$ and $\text{radius}_h(X) \leq \text{radius}_h(Y)$
- $x = y$ and $i \leq j$ (hypothesis)
- $x = y$ and $(iB)_x \subseteq (jB)_y$ (the mapping $j \mapsto (jB)_y$ is a poset isomorphism (Proposition 43))
- $\iff (iB)_x \subseteq (jB)_y$ (logical derivation)
- $(\text{radius}_h(X)B)_{\text{center}(X)} \subseteq (\text{radius}_h(Y)B)_{\text{center}(Y)}$ (def. of $x$, $y$, $i$ and $j$)
- $\iff X \subseteq Y.$ (Proposition 41)
Proposition 53 (property of the generated balls in a lower regular metric space) – Let \( (Z^2, d) \) be a t.i. metric space, for any \( x \) and \( y \in Z^2 \) and any \( j \in N \),

(i) \( x - y \in jB_d \Rightarrow d(x, y) \leq j \)

(ii) if \( d \) is lower regular, then \( d(x, y) \leq j \Rightarrow x - y \in jB_d \).

\[ \iff x - y \in jB_d \]

\[ \iff d(x, y) \leq j \]

(\( \Rightarrow \) by Property (i) of Proposition 53 (property of the generated balls in a lower regular metric space); \( \Leftarrow \) by Property (ii) of Proposition 53 and hypothesis)

\[ d(x, y) \leq j \hspace{1cm} (\text{symmetry of } d) \]

\[ x \in B_d(y, j). \hspace{1cm} (\text{def. of } B_d(y, j)) \]

We can apply Proposition 53 to the chessboard distance. In this way, we see how to obtain the chessboard balls by using the Minkowski product.

Corollary 55 (property of the Minkowski product of the unit ball of the chessboard metric space) – If \( d_8 \) is the chessboard distance, then for any \( x \) and \( y \in Z^2 \), and any \( j \in N \),

\[ x - y \in jB_8 \iff d_8(x, y) \leq j. \]

\[ \iff x \in B_d(y, j). \hspace{1cm} (\text{symmetry of } d) \]

\[ \iff d(x, y) \leq j \hspace{1cm} (\text{def. of } B_d(y, j)) \]

Next proposition, which is consequence Proposition 53, will be used in Section 8 to characterize the regular metric.

Proposition 56 (closure property of the unit ball of a regular metric space) – Let \( (Z^2, d) \) be a t.i. metric space. If \( d \) is regular, then \( B_d \) has the closure property.

\[ B_d \]

Proof

Let assume that \( d \) is regular. We divide the proof in four parts.

(a) Let us prove that for any \( j \in N \) and any \( x \in Z^2 \), \( (B_d)_j \subset (j + 1)B_d \Rightarrow d(x, o) \leq j \),

\[ (B_d)_j \subset (j + 1)B_d \iff (1B_d)_j \subset (j + 1)B_d \]

(\( \text{Definition 34 - Minkowski product} \))

\[ \iff (1B_d(o, 1))_j \subset (j + 1)B_d \]

(definitions of \( B_d(y, j) \) and \( B_d(o, 1) \))

\[ \iff (1B_d(o, 1))_j \subset (j + 1)B_d(o, 1)+ \]

(o is unit element of +)

\[ \iff B_d(x, 1) \subset B_d(o, j + 1) \]

(Corollary 54 (property of the generated balls in a lower regular metric space), \( \iff \) is true under the hypothesis that \( d \) is lower regular)

\[ 1 + d(x, o) \leq j + 1. \]

(d is upper regular and Property (ii) of Proposition 18 - ball inclusion in an upper regular metric space)

\[ \iff d(x, o) \leq j + 1. \]

(b) Let us prove that, for any \( j \in N \), \( ((jB_d) \oplus B_d) \subset jB_d \).

\[ (jB_d) \oplus B_d \iff x \in ((j + 1)B_d) \oplus B_d \]

(\( \text{product def.} \))

\[ (Minkowski's \ subtraction \ property) \]

\[ \Rightarrow d(x, o) \leq j \]

(Part (a))
that is, for any \(j \in \mathbb{N}\), \(((jB_d) \oplus B_d) \oplus B_d \subset jB_d\).
(c) For any \(j \in \mathbb{N}\), \(((jB_d) \oplus B_d) \oplus B_d = jB_d\) by \(B_d\), therefore by \(\partial B_d\) (Property (3) of Prop. 6.21 of [2]) \(jB_d \subset \{(jB(d) \oplus B(d)) \subset B_d\) \nolimits.
Hence, from Parts (b) and (c), if \(d\) is regular then, for any \(j \in \mathbb{N}\), \(((jB_d) \oplus B_d) \oplus B_d = jB_d\), in other words, \(jB_d\) is \(B_d\) closed.
(d) For any \(X \in B_d\), let \(x = \text{center}(X)\) and \(i = \text{radius}R(X)\), \([X \oplus B_d) \oplus B_d = r = \text{radius}(X)\), \(X \subset B_d\).

By Proposition 15 (example of geometrical metric space) \(\text{center}(X) \oplus B_d \oplus B_d = jB_d\) (def. of \(x\) and \(i\))
\(\text{(Property (5) of Prop. 4.12 of [2])} = ((jB_d) \oplus B_d) \oplus B_d\) \(\text{(Part (c))} = (\text{radius}(X)B_d) \text{center}(X)\) \(\text{center}(X)\) \(= X\) \(\text{radius}(M')\) (Property of (i). First, \(x \in X \Rightarrow f_B(x) \leq \text{radius}R(X)\)), \(x \in \partial B_d \Rightarrow f_B(x) = \text{radius}R(X)\),
(iii) if \(X\) has the closure property, then \(f_B(x) = \text{radius}R(X) \Rightarrow x \in \partial B_d\).

By Proposition 43 (chain of generated balls with center at a given point), \(x \in X \subset B_d\) and it is the smallest ball with center at origin that contains the point \(x\).

Let \(f_B\) be the mapping from \(B_d\) to \(\mathbb{N}^+\) given by, for any \(x \in B_d\), \(f_B(x) = \text{radius}R(M')\).

In the next proposition, we give the relationship between the ball border and its radius. Property (i) will be useful to prove the subadditivity of \(f_B\), and Property (ii) to prove its regularity.

**Proposition 59** (ball border versus ball radius) -- Let \(B\) be a finite symmetric ball (i.e., \(B \in \mathbb{S}\)) such that \(o \in B\) and \(B \neq \{o\}\). For any \(x \in B_d\) and any finite \(X \in B_d\),
(i) \(x \in X \Leftrightarrow f_B(x) \leq \text{radius}R(X)\),
(ii) \(x \in \partial B_d \Rightarrow f_B(x) = \text{radius}R(X)\),
(iii) if \(X\) has the closure property, then \(f_B(x) = \text{radius}R(X) \Rightarrow x \in \partial B_d\).

**Proof**
Let \(x \in B_d\), let \(M' = \bigcap \{U \in B_d(o) : x \in U\}\) and let \(X \in B_d\).
(a) Let us prove \(\Rightarrow\) of (i). First, \(X \subset X \subset X \subset X\) \(\text{definition of } M'\) (property of \(\cap\))
\(\Rightarrow M' \subset X\).
Second, \(f_B(x) = \text{radius}R(M')\) \(f_B(x) \leq \text{radius}R(X)\), \(M' \subset X\) and Property (i) of Proposition 51 - radius properties of nested generated balls)
(b) Let us prove \(\Leftarrow\) of (i). First, \(\text{radius}R(M') = f_B(x) \leq \text{radius}R(X)\), (property of \(\cap\))
\(\Rightarrow f_B(x) \leq \text{radius}R(X)\), (hypothesis)
that is, under the hypothesis, \(\text{radius}R(M') \leq \text{radius}R(X)\).
Second, \(x \in M\) \(\subset X\), (property of \(\cap\))
\(\Rightarrow \text{radius}R(M') \leq \text{radius}R(X)\) and Property (iv) of Proposition 51
\(\Rightarrow \text{radius}R(X)\), (hypothesis)
\(\Rightarrow \text{radius}R(X)\), (property of \(\cap\))
\(\Rightarrow \text{radius}R(X)\), (hypothesis)
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\(\Rightarrow \text{radius}R(X)\), (property of \(\cap\))
\(\Rightarrow \text{radius}R(X)\), (hypothesis)
Let us prove (i). Since the radius of a ball is a natural number

\[ x \in jB - (j-1)B \]

(\( \Rightarrow \) by Property (i) of Proposition 49 (erosion of a generated ball) and \( \Leftarrow \) by Property (ii) of Proposition 49 since \( B \) has the closure property)

\[ x \in jB \quad \text{and} \quad x \notin (j-1)B \]

(def. of the set difference)

\[ f_B(x) \leq \text{radius}_B(jB) \quad \text{and} \quad \text{radius}_B((j-1)B) < f_B(x) \]

(Property (i))

\[ f_B(x) \leq j \quad \text{and} \quad (j-1) < f_B(x) \]

(Proposition 41)

\[ j \leq f_B(x) \leq j \quad \text{(property of \( \leq \) on \( \mathbb{N} \))} \]

\[ f_B(x) = j \quad \text{(anti-symmetry of \( \leq \))} \]

\[ f_B(x) = \text{radius}_B(X). \quad \text{(definition of \( j \))} \]

\[ \therefore \]

The mapping \( f_B \) has almost all the properties of a norm as shown in the next proposition. Nevertheless, it doesn’t verify the property: \( f_B(jx) = |j|f_B(x) \) for any \( j \in \mathbb{Z} \) (for example, if \( B \) is a five by five square and \( x = (1, 1) \), then \( f_B(2x) = 1 \) and \( 2f_B(x) = 2 \), instead it has a weaker property: the symmetry.

**Proposition 60** (properties of \( f_B \) – Let \( B \) be a finite symmetric ball (i.e., \( B \in S \)), such that \( o \in B \) and \( B \neq \{o\} \). For any \( x \) and \( y \in \mathbb{Z}^2 \),

(i) \( f_B(x) \geq 0 \),

(ii) \( x = o \Leftrightarrow f_B(x) = 0 \),

(iii) \( f_B(-x) = f_B(x) \),

(iv) \( f_B(x + y) \leq f_B(x) + f_B(y) \). (subaditivity)

\[ \therefore \]

**Proof**

Let us prove (i). Since the radius of a ball is a natural number (see Definition 40 - radius of a generated ball), for any \( x \in \mathbb{Z}^2 \), \( f_B(x) \geq 0 \).

Let us prove (ii). For any \( x \in \mathbb{Z}^2 \),

\[ x = o \Leftrightarrow f_B(x) = \text{radius}_B(\{X \in B_B(o) : o \in X\}) \]

(def. of \( f_B(x) \))

\[ \therefore f_B(x) = \text{radius}_B(\{o\}) \]

(Proposition 43 - chain of generated balls with center at a given point)

\[ \therefore f_B(x) = 0. \]

(Property (i) of Proposition 42 - radius property)

Let us prove (iii). For any \( x \in \mathbb{Z}^2 \),

\[ f_B(-x) = \text{radius}_B(\{X \in B_B(o) : -x \in X\}) \]

(def. of \( f_B(x) \))

\[ = \text{radius}_B(\{X \in B_B(o) : x \in X\}) \]

\[ = f_B(x). \]

(Property (i) of Proposition 42 - radius property)

Let us prove (iv). Let us divide the proof in three parts.

(a) For any \( x \in \mathbb{Z}^2 \), let us prove that \( x \in f_B(x)B \).

\[ \text{true} \Leftrightarrow f_B(x) \leq f_B(x) \] (reflexivity of \( \leq \))

\[ \Leftrightarrow f_B(x) \leq \text{radius}_B(f_B(x)B) \]

\[ (f_B(x)B \in B_B(o)) \] (definition of \( B_B(o) \)) and Proposition 41 - characterization of the generated balls)

\[ \Leftrightarrow x \in f_B(x)B. \]

(Property (i) of Proposition 59 - ball border versus ball radius)

(b) For any \( x, y \in \mathbb{Z}^2 \), let prove that \( x + y \in (f_B(x) + f_B(y))B \).

\[ \text{true} \Leftrightarrow x \in f_B(x)B \quad \text{and} \quad y \in f_B(y)B \quad \text{(Part (a))} \]

\[ \Rightarrow x + y \in (f_B(x)B) \oplus (f_B(y)B) \]

(Definition 31 - Minkowski’s addition)

\[ \Leftrightarrow x + y \in (f_B(x) + f_B(y))B \]

(Proposition 35 - mix distributivity of the product)

(c) For any \( x, y \in \mathbb{Z}^2 \),

\[ f_B(x + y) \leq \text{radius}_B((f_B(x) + f_B(y))B) \]

(Proposition 35 - mix distributivity of the product)

\[ = f_B(x) + f_B(y). \] (Part (b) and Property (i) of Proposition 59)

(Proposition 41)

\[ \therefore \]

With a mapping \( f_B \) we can associate the mapping \( d_B = d_{f_B} \). By Proposition 9 (characterization of translation-invariant metric) and Proposition 60 (properties of \( f_B \)) \( d_B \) is a t.i. metric; we call it the **metric induced by \( B \)** or simply **induced metric**. Let \( x \) and \( y \in \mathbb{Z}^2 \), from the definition of \( d_j \) in Section 2, we have

\[ d_B(x, y) = f_B(x - y). \]

It is interesting to note that by substituting the Minkowski product by the scaling operations: scaling or scaling2, of Section 4, the above construction doesn’t lead to a metric.

As we can see in Figure 8, the triangle inequality is not satisfied when the induced distance is obtained by using the scaling transformations.

Before closing this section, we verify that the image of \( B_c \) through \( B \mapsto d_B \) is contained in \( M_c \).

**Figure 8 - Induced distances using Minkowski product and scaling.**

**Proposition 61** (properties of the metric induced by a ball having the closure property) – \( (B \mapsto d_B)(B_c) \subset M_c \).

**Proof**

We must verify four properties.

(a) By Proposition 9 (characterization of translation-invariant metric) and Proposition 60 (properties of \( f_B \)) \( d_B \) is a t.i. metric.

(b) For any \( B \in B_c \), by construction, \( f_B \) is a mapping from \( \mathbb{Z}^2 \) to \( \mathbb{N}^2 \), therefore, by definition of induced metric, \( d_B(\mathbb{Z}^2 \times \mathbb{Z}^2) = \mathbb{N}^2 \).

(c) Let us prove the lower regularity. Let \( x \) and \( y \in \mathbb{Z}^2 \), let \( i \in \mathbb{N} \), and let \( j = d_B(x, y) \) such that \( i \leq j \),

\[ \text{true} \Leftrightarrow d_B(x, y) = j \] (definition of \( j \))

\[ \Leftrightarrow f_B(x - y) = j \] (definition of induced metric)

\[ \Leftrightarrow f_B(x - y) = \text{radius}_B(jB) \]

\[ \therefore \]
Let $M_2$ be the set of integer-valued t.i. lower regular metrics, that is, $M_2 = \{ d \in \mathbb{N}^{\mathbb{Z} \times \mathbb{Z}} : d \text{ is t.i. and lower regular} \}$.

**Proposition 62** (existence of a left inverse) – The mapping $B \mapsto d_B$ is a left inverse for the mapping $d \mapsto B_d$ from $M_2$ to the collection of finite symmetric balls $B$ (i.e., $B \in S$), such that $o \in B$ and $B \neq \{o\}$.

**Proof**

(a) From Proposition 52 (properties of the unit ball), for any metric $d$, $B_d \in S$, $o \in B_d$ and $B_d \neq \{o\}$.

(b) For any $d \in M_2$, any $x$ and $y \in \mathbb{Z}$ and any $j \in \mathbb{N}$,

$$d(x, y) \leq j \iff x - y \in jB_d$$

($d$ is lower regular and Proposition 53 - property of the generated balls in a lower regular metric space)

$$\iff f_{B_d}(x - y) \leq \text{radius } B_d(jB_d)$$

(Property (i) of Proposition 59 - ball border versus ball radius)

$$\iff f_{B_d}(x - y) \leq j$$

(Property 41 - characterization of the generated balls)

$$\iff d_{B_d}(x, y) \leq j$$

(definition of induced metric)

That is, by anti-symmetry of $\leq$, for any $x$ and $y \in \mathbb{Z}$, $d(x, y) = d_{B_d}(x, y)$ in other words, by mapping equality definition, $d = d_{B_d}$.

Proposition 62 shows that every integer-valued t.i. lower regular metric can be reconstructed from its unit ball using the Minkowski product. Since $M_i \subset M_2$, this is also true for the regular metric. In other words, its unit ball uniquely defines any (lower) regular metric space.

**Corollary 63** (chessboard distance as a derived distance) – If $d_{\mathbb{Z}}$ is the chessboard distance, then $d_{\mathbb{Z}} = d_{B_{\mathbb{Z}}}$.

**Proof**

Since $B_{\mathbb{Z}}$ is the unit ball of the chessboard metric space, $B_{d_{\mathbb{Z}}} = B_{\mathbb{Z}}$. By Proposition 15 (example of regular metric space) ($d_{\mathbb{Z}}$, $\mathbb{Z}$) is a regular metric space, therefore by Proposition 62 (existence of a left inverse) $d_{\mathbb{Z}} = d_{B_{\mathbb{Z}}}$.

**Proposition 64** (existence of a right inverse) – The mapping $B \mapsto d_B$ is a right inverse for the mapping $d \mapsto B_d$ from $M_2$ to $B_c$.

**Proof**

(a) By Proposition 61 (properties of the metric induced by a ball having the closure property) for any $B \in B_c, d_B \in M_i$.

(b) For any $B \in B_c$, $B_{d_B} = \{ u \in \mathbb{Z} : d_B(u, o) \leq 1 \}$

$$= \{ u \in \mathbb{Z} : f_B(u) \leq 1 \}$$

(definition of induced metric)

$$= \{ u \in \mathbb{Z} : \text{radius}_B(\bigcap \{X \in B_B(o) : u \in X\}) \leq 1 \}$$

(definition of $f_B$)

$$= \{ u \in \mathbb{Z} : u \in \{o\} \text{ or } u \in B \}$$

...
We are now ready to state our characterization theorem.

**Theorem 65** (characterization of integer-valued translation-invariant regular metrics) — The mapping \( d \mapsto B_d \) from \( M \) to \( B_c \) is a bijection. Its inverse is the mapping \( B \mapsto d_B \).

**Proof**
Let us divide the proof into two parts.

(a) By Proposition 58 (properties of the Minkowski product of the unit ball of a regular metric space) for any \( d \in M \), \( B_d \in B_c \). By Proposition 62 (existence of a left inverse), \( d \mapsto B_d \) from \( M \subseteq M_2 \) to \( B_c \) is one-to-one.

(b) By Proposition 64 (existence of a right inverse), \( d \mapsto B_d \) from \( M \) to \( B_c \) is onto.
Hence, from (a) and (b), \( d \mapsto B_d \) from \( M \) to \( B_c \) is a bijection and its inverse is \( B \mapsto d_B \).

**9. Conclusion**

In the first part of this work we have introduced a definition of regular metric space and showed its relation to the Kiselman’s regularity axioms for translation-invariant metrics. We have shown, in particular, that the lower regularity of type 1 is a redundant axiom in the definition of regular metrics.

In the second part, we have established a one-to-one relationship between the set of integer-valued and translation-invariant regular metrics defined on the discrete plane, and the set of symmetric balls satisfying a special closure property.

From this result we now know how to construct a regular metric on the discrete plane.

To this end, we choose in the discrete plane a symmetric ball \( B \) that has the closure property, i.e., that induces, through the Minkowski product, a chain of generated balls that are morphologically closed with respect to \( B \).

Then the distance of a point \( x \) to the origin is given by the radius (in the sense of the Minkowski product) of the smallest ball of the chain, that contains \( x \).

This construction shows that to preserve in the discrete plane the regularity property of the Euclidean metric on the continuous plane, we have to reach a compromise between a good approximation of a continuous ball and thin contours. “Closer” \( B \) from an Euclidean continuous ball, bigger \( B \) and thicker the borders in the discrete plane.

Actually it will be interesting in a future work to give a proof that if \( B \) is the intersection of an Euclidean continuous ball with the discrete plane then the generated balls are morphologically closed with respect to \( B \).

The proof should be based on a closure property of the convex subsets of the continuous plane [3, Proposition 9.8].