MINISTÉRIO DA CIÊNCIA E TECNOLOGIA
INSTITUTO NACIONAL DE PESQUISAS ESPACIAIS

INPE-5616-RPQ/671

CHARACTERIZATION OF TRANSLATION–INVARIANT ELEMENTARY MORPHOLOGICAL OPERATORS BETWEEN GRAY–LEVEL IMAGES

Gerald Jean Francis Banon

INPE
São José dos Campos
novembro de 1995

ACKNOWLEDGMENTS

We are grateful to the first readers of the draft of this paper for their comments, and specially to: Junior Barrera, Ana Lúcia Bezerra Candéias, João Ricardo de Freitas Oliveira, Ronei Marcos de Moraes, Robson Pequeno de Souza, Francisco de Assis Tavares Ferreira Silva and Sergio Donizete Faria. During the period of this work, the author has obtained support from CNPq (Conselho acional de Desenvolvimento Científico e Tecnológico) under contracts 300966/90–3 and 680067/94–9 and FAPESP (Fundação de Amparo à Pesquisa do Estado de São Paulo), under contracts 91/3532–2 and 95/2025–0.

RESUMO

As quatro classes de operadores elementares da Morfologia Matemática: dilatações, erosões, anti–dilatações e anti–erosões mostraram–se de fundamental importância para a decomposição/representação de qualquer mapeamento entre reticulados completos. Neste artigo, vamos considerar a caracterização dos operadores elementares de janela invariantes por translação (com janela W) que transformam uma imagem em níveis de cinza com contra–domínio K_1 finito numa imagem en níveis de cinza com contra–domínio K_2 finito. Baseada nas propriedades de conexões de Galois entre reticulados completos, uma primeira caracterização, chamada de "caracterização por confrontação", é estabelecida. Nesta caracterização, cada operador elementar depende de uma família de mapeamentos de W em K_1, chamada de elemento estruturante. Baseada na decomposição de uma imagem em níveis de cinza em termos de imagens impulsivas, uma segunda caracterização, chamada de "caracterização por seleção", é estabelecida. Nesta caracterização cada operador elementar depende de uma família de mapeamentos de W em K_2, chamada de resposta impulsiva. Finalmente, a partir desta segunda caracterização, uma terceira, chamada de "caracterização por decomposição" é estabelecida. Nesta caracterização cada operador elementar depende de uma família de mapeamentos de K_1 em K_2 chamados de tabelas de transformação elementares ("Elementary Look Up Tables"). A caracterização por confrontação é a mais natural dentro da teoria da decomposição dos operadores. A caracterização por seleção e a por decomposição correspondem, respectivamente, a implementações computacionais seriais e paralelas eficientes.

ABSTRACT

The four classes of Mathematical Morphology elementary operators: dilations, erosions, anti–dilations and anti–erosions have proved to be of fundamental importance to the decomposition/representation of any mapping between complete lattices. In this paper, we are concerned with the characterization of translation–invariant window elementary operators (with window W) that transform a gray–level image with finite range K_1 into a gray–level image with possibly different finite range K_2. Based on the properties of Galois connections between complete lattices, a first characterization, called "characterization by confrontation" is derived. In this characterization each elementary operator depends on a family of mappings from W to K_1, called structuring element. Based on the decomposition of a gray–level image in terms of pulse images, a second characterization, called "characterization by selection", is presented. In this characterization each elementary operator depends on a family of mappings from W to K_2, called impulse response. Finally, from this second characterization, a third one, called "characterization by decomposition", is derived. In this characterization each elementary operator depends on a family of mappings from K_1 to K_2, called Elementary Look Up Tables. The characterization by confrontation is the natural one within the theory of operator decomposition. The characterization by selection and the one by decomposition correspond, respectively, to efficient serial and parallel computational implementations.

SUMMARY

LIST OF FIGURES

Page

CHAPTER 1

INTRODUCTION

1.1 – OVERVIEW ON SOME CLASSES OF DILATIONS AND EROSIONS

The four classes of Mathematical Morphology elementary operators: dilations, erosions, anti–dilations and anti–erosions have been shown by Banon & Barrera (1), (2), to be of fundamental importance to the decomposition/representation of any operator between complete lattices. For practical reasons, here we are specially interested with the very important class of translation–invariant elementary operators. Besides this point, we focus our interest to the window operators, that is, the operators for which the output pixel values depend only on the values of the pixels inside given windows that can be derived from a unique window.

In the early age of Mathematical Morphology, dilations and erosions have been introduced (3) for black and white images (binary images). Let E (equipped with an addition $+$) be a set representing the white image and let B be a subset of E. Let X and Y be subsets of E representing any black and white images. The *dilation of Y with respect to B* and the *erosion of X with respect to B* are the subsets, respectively, given by means of the addition (4) and subtraction (5) of Minkowski:

$$\varDelta_B(Y) \stackrel{\Delta}{=} Y \oplus B \quad \text{and} \tag{1a}$$

$$E_B(X) \stackrel{\Delta}{=} X \ominus B, \tag{2a}$$

where, for any subsets A and B of E,

$$A \oplus B \stackrel{\Delta}{=} \{x \in E : \exists a \in A \text{ e } \exists b \in B,\ x = a + b\} \quad \text{and} \tag{1b}$$

$$A \ominus B \stackrel{\Delta}{=} \{y \in E : \forall b \in B,\ (\exists a \in A,\ y = a - b)\}. \tag{2b}$$

The underlying operator $\varDelta_B$ is an example of *translation–invariant window operator* (with window B^t or greater). The underlying operator E_B is another example of translation–invariant window operator (with window B or greater). Heijmans & Ronse (6, Theorem 3.6), Banon & Barrera (7, Proposition 4.15) and (8, Section 2), have shown that $\varDelta_B$ (resp., E_B) commutes with the union (resp., intersection) of any family of subsets of E and that any translation–invariant operator which commutes with union (resp., intersection) has this form. In other words, the collection of subsets of E (like the B's) characterized these classes of operators. Since the subset B is the impulsive response or the point spread subset of the dilation $\varDelta_B$ we are used to calling B the *blur* of the dilation.

There exist many equivalent ways to define $\varDelta_B(Y)$ and $E_B(X)$. Let X^c be the complementary set of X in E, X^t and $X + u$ be, respectively, the transpose of X and the translate, with respect to $+$, of X by a point u of E. We may rewrite Expressions (1) and (2) as

$$\varDelta_B(Y) = \{x \in E : (Y - x) \subset B^{tc}\}^c \quad \text{and} \tag{3}$$

$$E_B(X) = \{y \in E : B \subset (X - y)\}. \tag{4}$$

Expressions (3) and (4) correspond to what we call, in Section 3.2, the dilation and erosion characterization by confrontation because to know if x belongs or not to the dilation of Y with respect to B, we have to confront $Y - x$ with B^{tc}, and to know if y belongs or not to the erosion of X with respect to B, we have to confront $X - y$ with B. Since the subset B plays the role of a probe in the computation of $E_B(X)$, it is used to calling B the *structuring element* of the erosion E_B.

In another way, we may rewrite Expression (1) and (2) as

$$\mathit{\Delta}_B(Y) = \{x \in E : \exists v \in B^{t}, v \in (Y - x)\} \quad \text{and} \tag{5}$$

$$E_B(X) = \{y \in E : \forall u \in B, u \in (X - y)\}, \tag{6}$$

these expressions correspond to what we call, in Section 3.3, the dilation and erosion characterization by selection.

For a visual construction of the dilation of Y with respect to B and the erosion of X with respect to B, it is interesting to use the equivalent expressions

$$\mathit{\Delta}_B(Y) = \bigcup_{y \in Y} (B + y) \quad \text{and} \tag{7}$$

$$E_B(X) = \bigcap_{x \in X^{c}} (B^{tc} + x). \tag{8}$$

From Expressions (5) and (6), we get another equivalent expressions

$$\mathit{\Delta}_B(Y) = \bigcup_{u \in B} (Y + u) \quad \text{and} \tag{9}$$

$$E_B(X) = \bigcap_{v \in B^{t}} (X + v). \tag{10}$$

Expression (9) corresponds to the decomposition of a dilation in terms of union of dilations reduced to translations. Expression (10) corresponds to the decomposition of an erosion in terms of intersection of erosions reduced to translations. In Section 3.4 of this work, we refer to them as the dilation and erosion characterization by decomposition.

The translation–invariant black and white elementary operators have been extended to gray–level images by introducing the notion of *flat* operators (9). Let f and g be functions from E to K representing gray–level images (with gray–scale K). The *flat dilation of g with respect to B* and the *flat erosion of f with respect to B*, are the functions from E to K, respectively, given by, for any x and y in E,

$$\mathit{\Delta}_B(g)(x) \triangleq \max g(B^{t} + x) \quad \text{and} \tag{11}$$

$$E_B(f)(y) \triangleq \min f(B + y), \tag{12}$$

where $f(A)$ is the image of A through f. Expressions (11) and (12) reduce to Expression (1) and (2) when the gray–scale K reduces to two gray–levels.

Expressions (11) and (12) can be rewritten as, for any x and y in E,

$$\mathit{\Delta}_B(g)(x) = \bigvee_{u \in B} g(x - u) \quad \text{and} \tag{13}$$

$$E_B(f)(y) = \bigwedge_{u \in B} f(y + u). \tag{14}$$

More general classes of dilations and erosions for gray–level images with *finite* gray–scale are the ones proposed by Heijmans (9, Section 11.8) and (10).

Let $K = [0, k] \subset \mathbf{Z}$ (the set of intergers) and let $\dot{+}$ be the operation from $K \times \mathbf{Z}$ to K defined by, for any $t \in K$ and $z \in \mathbf{Z}$,

$$t \dot{+} z \overset{\Delta}{=} \begin{cases} 0 & \text{if } t = 0, \\ 0 & \text{if } t > 0 \text{ and } t + z < 0, \\ t + z & \text{if } t > 0 \text{ and } 0 \le t + z \le k, \\ k & \text{if } t > 0 \text{ and } t + z > k. \end{cases}$$

Similarly, let $\dot{-}$ be the operation from $K \times \mathbf{Z}$ to K defined by, for any $s \in K$ and $z \in \mathbf{Z}$,

$$s \dot{-} z \overset{\Delta}{=} \begin{cases} 0 & \text{if } s < k \text{ and } s - z < 0, \\ s - z & \text{if } s < k \text{ and } 0 \le s - z \le k, \\ k & \text{if } s < k \text{ and } s - z > k, \\ k & \text{if } s = k. \end{cases}$$

Let b be a function defined from $B \subset E$ to $\mathbf{Z}$. The *Heijmans' dilation of g with respect to b* is the function from E to K, given by, for any x in E,

$$\mathit{\Delta}_b(g)(x) \overset{\Delta}{=} \bigvee_{u \in B} g(x - u) \dot{+} b(u) \tag{15}$$

and the *Heijmans' erosion of f with respect to b* is the function from E to K, given by, for any y in E,

$$E_b(f)(y) \overset{\Delta}{=} \bigwedge_{u \in B} f(y + u) \dot{-} b(u). \tag{16}$$

The underlying operators $\mathit{\Delta}_b$ and E_b are called here *Heijmans' elementary operators*.

Actually, based on other binary operations we can define other classes of dilations and erosions with respect to a function b from $B \subset E$ to $K = [0, k] \subset \mathbf{Z}$.

Let $\mathcal{C}^+$ and $\mathcal{C}^-$ be the classes of binary operations from K^2 to K defined by, respectively,

$$+ \in \mathcal{C}^+ \overset{\Delta}{\Leftrightarrow} \begin{cases} (t_1 \le t_2 \Rightarrow t_1 + z \le t_2 + z \ \ (t_1, t_2, z \in K)) \text{ and} \\ (0 + z = 0 \ \ (z \in K)) \end{cases}$$

and

$$- \in \mathcal{C}^- \overset{\Delta}{\Leftrightarrow} \begin{cases} (s_1 \le s_2 \Rightarrow s_1 - z \le s_2 - z \ \ (s_1, s_2, z \in K)) \text{ and} \\ (k - z = k \ \ (z \in K)). \end{cases}$$

Some examples of binary operations belonging to $\mathcal{C}^+$ are:

$$+^1\colon (t, z) \mapsto t +^1 z \overset{\Delta}{=} \begin{cases} 0 & \text{if } t = 0, \\ t + z & \text{if } t > 0 \text{ and } t + z \le k, \\ k & \text{if } t + z > k, \end{cases} \qquad \text{(Heijmans' addition for } z \text{ positive)}$$

$$+^2\colon (t, z) \mapsto t +^2 z \overset{\Delta}{=} 0 \vee (t - z) = \begin{cases} 0 & \text{if } t - z < 0, \\ t - z & \text{if } 0 \le t - z, \end{cases} \qquad \text{(Heijmans' addition for } z \text{ negtive)}$$

$$+^3\colon (t, z) \mapsto t +^3 z \overset{\Delta}{=} t \wedge z, \qquad \text{(Bloch and Maître; Baets et al.)}$$

$$+^4\colon (t, z) \mapsto t +^4 z \overset{\Delta}{=} \begin{cases} 0 & \text{if } t - z \le 0, \\ t & \text{if } 0 < t - z, \end{cases}$$

we will denote by $\mathfrak{D}^+$ the set of the above four binary operations.

Some examples of binary operations belonging to $\mathcal{C}^-$ are:

$$-^1\colon\ (s,z) \mapsto s -^1 z \overset{\Delta}{=} \begin{cases} 0 & \text{if } s < k \text{ and } s - z < 0, \\ s - z & \text{if } s < k \text{ and } 0 \le s - z \le k, \\ k & \text{if } s = k, \end{cases}$$ (Heijmans' subtraction for z positive)

$$-^2\colon\ (s,z) \mapsto s -^2 z \overset{\Delta}{=} k \wedge (s + z) = \begin{cases} s + z & \text{if } s + z \le k, \\ k & \text{if } s + z > k, \end{cases}$$

(Sinha and Dougherty, or Heijmans' subtraction for z negative)

$$-^3\colon\ (s,z) \mapsto s -^3 z \overset{\Delta}{=} \begin{cases} s & \text{if } s - z < 0, \\ k & \text{if } 0 \le s - z, \end{cases}$$

$$-^4\colon\ (s,z) \mapsto s -^4 z \overset{\Delta}{=} s \vee z,$$ (Bloch and Maître; Baets et al.)

we will denote by $\mathfrak{D}^-$ the set of the above four binary operations. The operations $+^3$ and $-^4$ have been introduced independently by Bloch and Maître (11) and Baets et al. (12). The operation $-^2$ has been introduced by Sinha and Dougherty (13).

We say that $+ \in \mathcal{C}^+$ and $- \in \mathcal{C}^-$ are *companions* iff for any $s, t, z \in K$,

$$t + z \le s \Leftrightarrow t \le s - z.$$

For example, the above operations $+^i$ and $-^i$ are companions for any $i = 1, 2, 3, 4$.

For a given $z \in K$, we say that $t \mapsto t + z$ and $s \mapsto s - z$ are *companions* iff $+ \in \mathcal{C}^+$ and $- \in \mathcal{C}^-$ are companions.

For example, for a given $i \in \{1, 2, 3, 4\}$ and $z \in K$, the mappings $t \mapsto t +^i z$ and $s \mapsto s -^i z$ are companions.

Figure 1 shows some companion mappings $t \mapsto t +^i z$ and $s \mapsto s -^i z$ for a given i and z.

Let b be a function defined from $B \subset E$ to $K = [0, k] \subset \mathbf{Z}$, and let $(+_u)_{u \in B}$ and $(-_u)_{u \in B}$ be, respectively, families of operations in $\mathfrak{D}^+$ and $\mathfrak{D}^-$, such that $+_u$ and $-_u$ are companions for any $u \in B$. We define the *dilation of g with respect to b and $(+_u)$*, as being the function from E to K, given by, for any x in E,

$$\Delta_{b,+}(g)(x) \overset{\Delta}{=} \bigvee_{u \in B} g(x - u) +_u b(u), \tag{17}$$

and the *erosion of f with respect to b and $(-_u)$*, as being the function from E to K, given by, for any y in E,

$$E_{b,-}(f)(y) \overset{\Delta}{=} \bigwedge_{u \in B} f(y + u) -_u b(u). \tag{18}$$

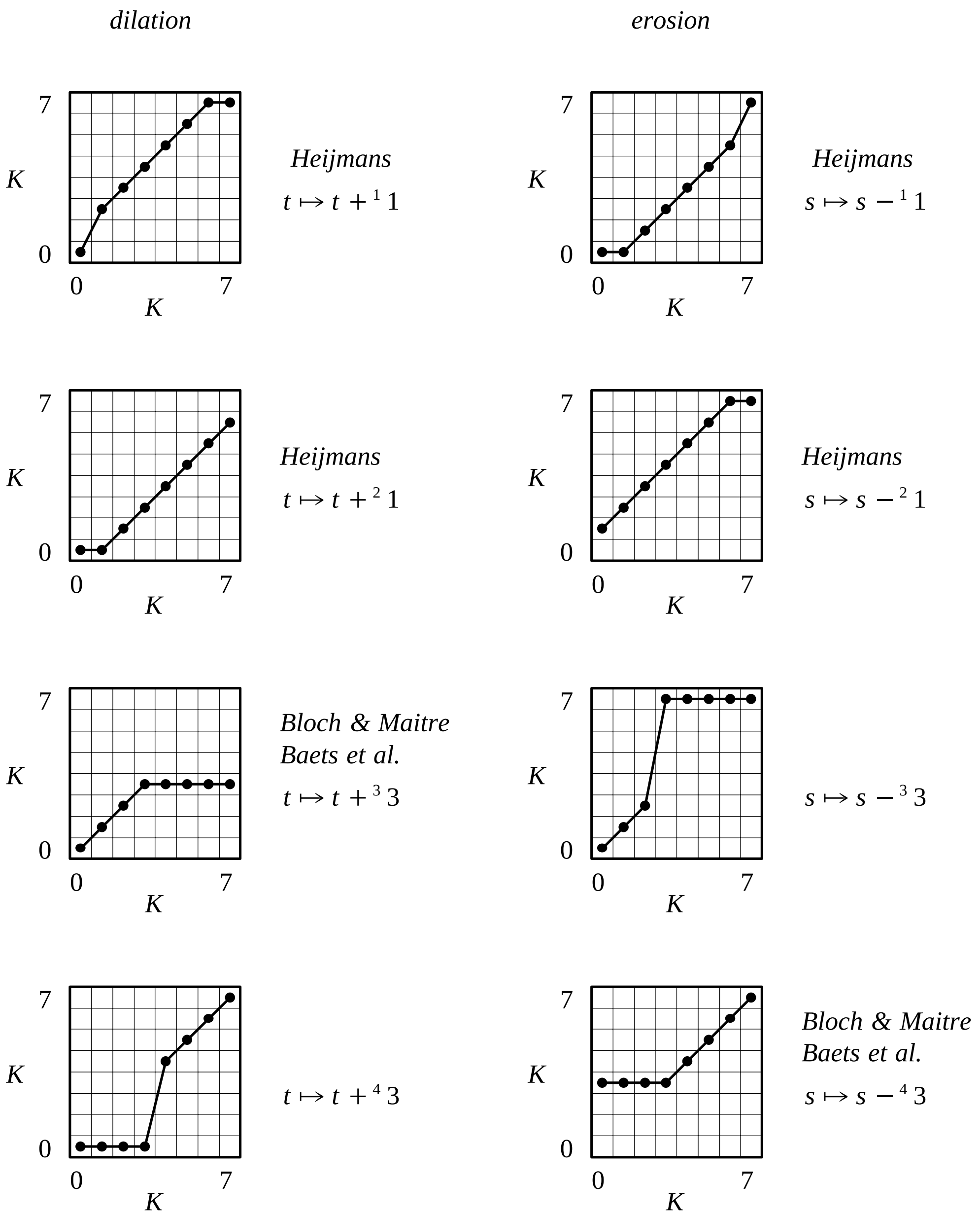

Fig. 1 – Some companion mappings .

Following the neuronal approach (14), (15) even more general classes of dilations and erosions for gray–level images with *finite* gray–scale can be derived just by composing the above elementary operators with an activation function. Let a be an increasing function from K to K such that $a(0) = 0$ and $a(k) = k,$ then we define the *dilation of g with respect to a, b* (in this order) *and* $(+_u)$, as the function from E to K, given by, for any x in E,

$$\mathit{\Delta}_{a,b,+}(g)(x) \triangleq a(\bigvee_{u \in B} g(x - u) +_u b(u)), \tag{19}$$

and the *erosion of* f *with respect to* a*,* b (in this order) *and* $(-_u)$, as the function from E to K, given by, for any y in E,

$$E_{a,b,-}(f)(y) \stackrel{\Delta}{=} \bigwedge_{u \in B} \overline{a}(f(y+u)) -_u b(u), \tag{20}$$

where, $\overline{a}$ from K to K is the companion function of a defined by $\overline{a}(s) \stackrel{\Delta}{=} \max\{t \in K : a(t) \leq s\}$ $(s \in K)$.

We can also define the dilation of g with respect to b and a (in this order) as the function from E to K, given by, for any x in E,

$$\varDelta_{b,a,+}(g)(x) \stackrel{\Delta}{=} \bigvee_{u \in B} \underline{a}(g(x-u)) +_u b(u), \tag{21}$$

the erosion of f with respect to a and b (in this order) is the function from E to K, given by, for any y in E,

$$E_{b,a,-}(f)(y) \stackrel{\Delta}{=} a(\bigwedge_{u \in B} f(y+u) -_u b(u)), \tag{22}$$

where, $\underline{a}$ from K to K is the companion function of a defined by $\underline{a}(t) \stackrel{\Delta}{=} \min\{s \in K : t \leq a(s)\}$ $(t \in K)$.

The underlying operators $\varDelta_{a,b,+}$ and $E_{b,a,-}$ are typically *elementary parametric neural networks* with activation function a and synaptic function b. The underlying operators $\varDelta_{b,a,+}$ and $E_{a,b,-}$ could be called elementary anti–neural networks because they were not derived from a physiological observation but are simply mathematical constructions.

Figure 2 shows a neural network with five neurones forming a dilation $\varDelta_{a,b,+}$.

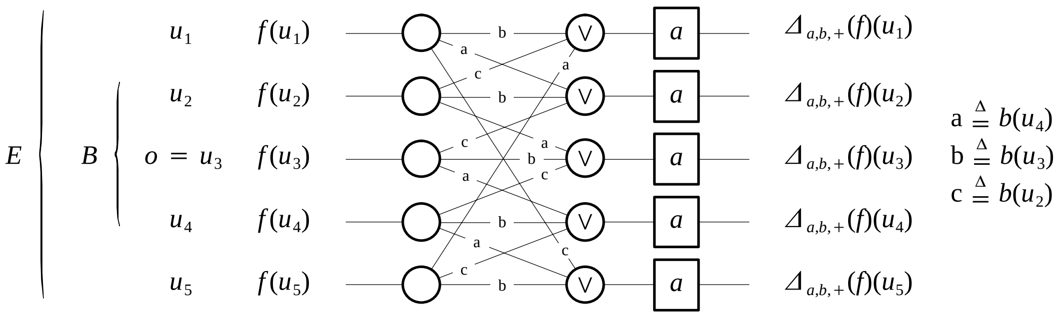

Fig. 2 – A neural network forming a dilation.

Despite the fact that Expressions (19) to (22) are more general than the previous ones, we cannot say, as in the black and white case, that *all* translation–invariant dilations and erosions on gray level–images (with finite gray–scale) have this form.

In this paper, we are precisely concerned with the problem of finding out mechanisms to construct *all* such translation–invariant dilations and erosions on gray level–images (with finite gray–scale). More precisely, we are concerned with the characterization of translation–invariant window elementary operators (with window W) that transform a gray–level image with finite range K_1 into a gray–level image with possibly different finite range K_2. This assumption corresponds to the framework of Computational Mathematical Morphology introduced by E.R. Dougherty and D. Sinha (16) in which the operators may not preserve the image range.

In Section 1.2, we recall the axiomatic definition of the four classes of elementary operators.

In Section 1.3, we present the characterization of the elementary operators defined on a bounded chain. We pay a special attention to the case of elementary operators between bounded chains since they have a great importance in image processing as transformation tables or Look Up Tables (LUTs).

In Chapter 2, we give three equivalent ways to characterize the so–called elementary measures, that is, operators between a gray–level image and a gray–scale. Based on the properties of Galois connections between complete lattices, a first characterization, called "characterization by confrontation" is derived. Based on the decomposition of a gray–level image in terms of pulse images, a second characterization, called "characterization by selection", is presented. Finally, from this second characterization, a third one, called "characterization by decomposition", is derived.

The measures are very important because they can be used to characterize the translation–invariant window operators as it is shown in Section 3.1.

From the characterizations given in Chapter 2 and Section 3.1, the so–called characterization by confrontation for the translation–invariant elementary operators is derived in Section 3.2. In this characterization each elementary operator depends on a family of mappings from W to K_1, called structuring element.

In the same way, the so–called characterization by selection is presented in Section 3.3. In this characterization each elementary operator depends on a family of mappings from W to K_2, called impulse response or point spread function or blur.

Finally, the so–called characterization by decomposition is derived in Section 3.4. In this characterization each elementary operator depends on a family of mappings from K_1 to K_2 called Elementary Look Up Tables. Actually, this last characterization is similar to the one presented by H.J.A.M. Heijmans and C. Ronse in their Algebraic Basis of Mathematical Morphology (6).

The above three types of mappings: the structuring elements, the impulse responses and the Elementary Look Up Tables are called *characteristic functions*. In the last chapter, we show how the characteristic functions simplified in the case of the Heijmans' elementary operators and in the the case of the flat elementary operators.

1.2 – AXIOMATIC DEFINITION OF ELEMENTARY OPERATORS

Let $(L, \underset{L}{\leq})$, or simply $(L, \leq)$ or L, be a complete lattice (17). The supremum and the infimum of a subset X of L are denoted, respectively, by $\underset{L}{\sup} X$ and $\underset{L}{\inf} X$, or simply, by $\sup X$ and $\inf X$. The union and intersection of a family $(a_i)_{i \in I}$ of elements of L, indexed by a set I, are denoted, respectively by $\bigvee_{i \in I} a_i$ and $\bigwedge_{i \in I} a_i$.

We are used to calling an *operator* a mapping between two complete lattices. Let L_1 and L_2 be two complete lattices. A operator ψ from L_1 to L_2 is *increasing* (or *isotone*) iff

$$a \leq b \Rightarrow \psi(a) \leq \psi(b) \quad (a, b \in L_1).$$

As usual, we denote by $\psi(X)$ the image of a subset X of L_1, that is $\psi(X) = \{y \in l_2 : \exists x \in X, y = \psi(x)\}$.

Proposition 1.1 (equivalent definitions of the increasing operators) – Let L_1 and L_2 be two complete lattices. A operator ψ from L_1 to L_2 is increasing iff

$$\sup\psi(X) \leq \psi(\sup X) \quad (X \subset L_1) \tag{23}$$

or equivalently, iff

$$\psi(\inf X) \leq \inf\psi(X) \quad (X \subset L_1). \tag{24}$$

□

Proof – See Lemma 2.1, p. 260 of Heijmans & Ronse (6) for the case where L_1 and L_2 are identical or Proposition 3.1, p. 33 of Banon & Barrera (7) for the case where L_1 and L_2 are two Boolean lattices. □

Following Serra (18), if the equality holds in (23), then ψ is called a *dilation*. If it holds in (24), then ψ is called an *erosion*. We denote, respectively, by $\Delta(L_1, L_2)$ and $\mathrm{E}(L_1, L_2)$ the classes of dilations and erosions from $(L_1, \leq)$ to $(L_2, \leq)$. We observe that an erosion (resp., dilation) from $(L_1, \leq)$ to $(L_2, \leq)$ is a dilation (resp., erosion) from $(L_1, \geq)$ to $(L_2, \geq)$.

A operator ψ from L_1 to L_2 is *decreasing* (or *antitone*) iff

$$a \leq b \;\Rightarrow\; \psi(b) \leq \psi(a) \quad (a, b \in L_1).$$

Proposition 1.2 (equivalent definitions of the decreasing operators) – Let L_1 and L_2 be two complete lattices. A operator ψ from L_1 to L_2 is decreasing iff

$$\psi(\sup X) \leq \inf\psi(X) \quad (X \subset L_1) \tag{25}$$

or equivalently, iff

$$\sup\psi(X) \leq \psi(\inf X) \quad (X \subset L_1). \tag{26}$$

□

Proof – The result follows from Proposition 1.1 by duality. □

Following Serra (19), if the equality holds in (25), then ψ is called an *anti–dilation*. If it holds in (26), then ψ is called an *anti–erosion*. We denote, respectively, by $\Delta^a(L_1, L_2)$ and $\mathrm{E}^a(L_1, L_2)$ the classes of anti–dilations and anti–erosions from $(L_1, \leq)$ to $(L_2, \leq)$. We observe that an anti–erosion (resp., anti–dilation) from $(L_1, \leq)$ to $(L_2, \leq)$ is an anti–dilation (resp., anti–erosion) from $(L_1, \geq)$ to $(L_2, \geq)$. Furthermore, an erosion (resp., dilation) from $(L_1, \leq)$ to $(L_2, \leq)$ is an anti–erosion (resp., anti–dilation) from $(L_1, \leq)$ to $(L_2, \geq)$.

The dilations, erosions, anti–dilations and anti–erosions are said to be the *elementary operators* of Mathematical Morphology since any operator between complete lattices can be decomposed from them as shown by Banon & Barrera (2). Expressed in an equivalent way,

$$\psi \text{ from } (L_1, \leq) \text{ to } (L_2, \leq) \text{ is a dilation} \Leftrightarrow \psi(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} \psi(a_i) \text{ for any family } (a_i)_{i \in I} \text{ in } L_1$$

$$\psi \text{ from } (L_1, \leq) \text{ to } (L_2, \leq) \text{ is an erosion} \Leftrightarrow \psi(\bigwedge_{i \in I} a_i) = \bigwedge_{i \in I} \psi(a_i) \text{ for any family } (a_i)_{i \in I} \text{ in } L_1$$

$$\psi \text{ from } (L_1, \leq) \text{ to } (L_2, \leq) \text{ is an anti–dilation} \Leftrightarrow \psi(\bigvee_{i \in I} a_i) = \bigwedge_{i \in I} \psi(a_i) \text{ for any family } (a_i)_{i \in I} \text{ in } L_1$$

$$\psi \text{ from } (L_1, \leq) \text{ to } (L_2, \leq) \text{ is an anti–erosion} \Leftrightarrow \psi(\bigwedge_{i \in I} a_i) = \bigvee_{i \in I} \psi(a_i) \text{ for any family } (a_i)_{i \in I} \text{ in } L_1.$$

From the duality principle (17), we observe that the classes of elementary operators from $(L_{1,} \leq)$ to $(L_{2,} \leq)$ have the same number of operators.

Let i_1 (resp., i_2) and o_1 (resp., o_2) be, respectively the greatest and least elements of L_1 (resp., L_2), that is, $i_1 = \sup L_1$ (resp., $i_2 = \sup L_2$) and $o_1 = \inf L_1$ (resp., $o_2 = \inf L_2$).

Proposition 1.3 (property of the elementary operators) – Let L_1 and L_2 be two complete lattices. We have the following statements:

$$\delta \in \Delta(L_1, L_2) \Rightarrow \delta \text{ is increasing and } \delta(o_1) = o_2$$

$$\epsilon \in \mathrm{E}(L_1, L_2) \Rightarrow \epsilon \text{ is increasing and } \epsilon(i_1) = i_2$$

$$\delta^a \in \Delta^a(L_1, L_2) \Rightarrow \delta^a \text{ is decreasing and } \delta^a(o_1) = i_2$$

$$\epsilon^a \in \mathrm{E}^a(L_1, L_2) \Rightarrow \epsilon^a \text{ is decreasing and } \epsilon^a(i_1) = o_2.$$

□

Proof – The increase and decrease properties derive from the definition of the four classes of elementary operators and from Propositions 1.1 and 1.2. The remaining properties derive from the definition of the four classes of elementary operators and recalling that $o_1 = \sup\emptyset_1$ (resp., $o_2 = \sup\emptyset_2$) and $i_1 = \inf\emptyset_1$ (resp., $i_2 = \inf\emptyset_2$), where $\emptyset_1$ (resp., $\emptyset_2$) denotes the empty subset of L_1 (resp., L_2). □

In the following section, we will study the characterization of elementary operators on a bounded chain.

1.3 – CHARACTERIZATION OF ELEMENTARY OPERATORS ON A BOUNDED CHAIN

In this section, we consider the case where the domain L_1 of the elementary operators is a bounded chain (i.e., a bounded totally ordered set). In this case, L_1 is a complete lattice, sup and inf with respect to L_1 can be replaced, respectively, with max and min, and we have $i_1 = \max L_1$ and $o_1 = \min L_1$.

Proposition 1.4 (characterization of the elementary operators defined on a bounded chain) – Let L_1 be a bounded chain and let L_2 be a complete lattice. Let ψ a operator from L_1 to L_2, then we have the following statements:

$$\psi \text{ is a dilation} \Leftrightarrow \psi \text{ is increasing and } \psi(o_1) = o_2$$

$$\psi \text{ is an erosion} \Leftrightarrow \psi \text{ is increasing and } \psi(i_1) = i_2$$

$$\psi \text{ is an anti–dilation} \Leftrightarrow \psi \text{ is decreasing and } \psi(o_1) = i_2$$

$$\psi \text{ is an anti–erosion} \Leftrightarrow \psi \text{ is decreasing and } \psi(i_1) = o_2.$$

□

Proof – Let us prove the first statement. Let us prove $\Rightarrow$. If ψ is a dilation then, by Proposition 1.3, ψ is increasing and $\psi(o_1) = o_2$. Let us prove $\Leftarrow$. For any $X \subset L_1$ and $X \neq \emptyset$,

$$\begin{aligned} \psi(\max X) &\leq \sup\psi(X) && (\max X \in X \text{ and } \sup\psi(X) \text{ is an u.b. for } \psi(X)) \\ &\leq \psi(\max X). && (\psi \text{ is increasing and Proposition 1.1}) \end{aligned}$$

In other words, by transitivity of $\leq$, for any $X \subset L_1$ and $X \neq \emptyset$, $\psi(\max X) = \sup\psi(X)$. For $X = \emptyset$,

$$\begin{aligned} \psi(\max\emptyset) &= \psi(o_1) && (\text{definition of the supremum of an empty set}) \\ &= o_2 && (\text{hypothesis}) \end{aligned}$$

$$= \sup\emptyset \qquad \text{(definition of the supremum of an empty set)}$$

$$= \sup\psi(\emptyset). \qquad \text{(definition of the image of a mapping)}$$

In other words, for any $X \subset L_1$, $\psi(\max X) = \sup\psi(X)$, that is ψ is a dilation.

The three other statements follow from the first one by duality. □

May be the simplest elementary operators are those defined between two finite chains, let's say K_1 and K_2. From Proposition 1.4, a mapping p from K_1 to K_2 is a dilation (resp., erosion) iff p is increasing and $p(\min K_1) = \min K_2$ (resp., $p(\max K_1) = \max K_2$). It is an anti–dilation (resp., anti–erosion) iff it is decreasing and $p(\min K_1) = \max K_2$ (resp., $p(\max K_1) = \min K_2$). Figure 3 shows two samples in each class of elementary operators from $K_1 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ to $K_2 = \{0, 1, 2, 3\}$. We see that the graph of the dilation (resp., erosion, anti–dilation, anti–erosion) must contain the point $(0, 0)$ (resp., $(7, 3)$, $(0, 3)$, $(7, 0)$).

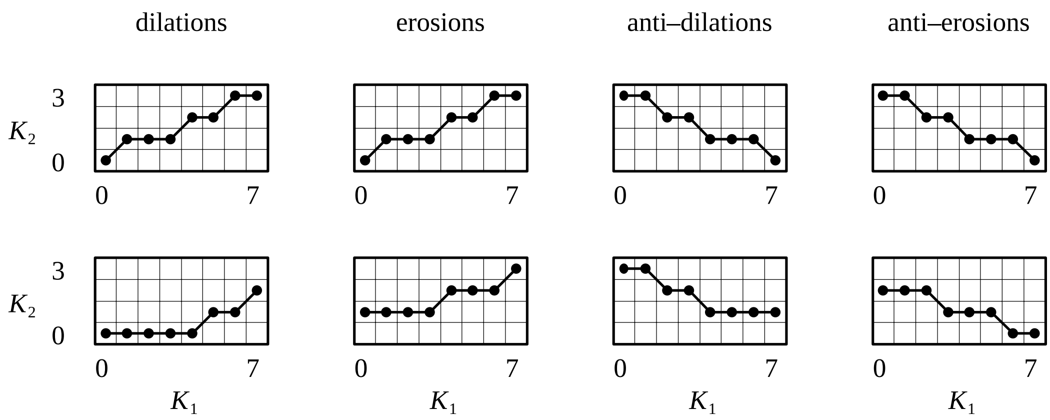

Fig. 3 – Examples of elementary operators.

As we have observed before, the classes of elementary operators from L_1 to L_2 have the same number of operators. The next proposition gives such number in the case where both lattices are bounded chains K_1 and K_2. Let's denote by $\#X$ the number of elements of a set X.

Proposition 1.5 (combination number property) – For any $i = 0, 1, \ldots$ and any $k = 1, 2, \ldots$, we have

$$\binom{k+i}{i} = \sum_{j=0}^{i} \binom{k+j-1}{j}. \qquad □$$

Proof – Let $P(i)$ be the property $\binom{k+i}{i} = \sum_{j=0}^{i} \binom{k+j-1}{j}$ for any $i = 0, 1, \ldots$ The properrty $P(0)$ is true since $\binom{k}{0} = \binom{k-1}{0} = 1$. Assuming that $P(i)$ is true, let us prove that $P(i+1)$ is also true. We have

$$\binom{k+i+1}{i+1} = \binom{k+i}{i} + \binom{k+i}{i+1} \qquad \text{(combination number property)}$$

$$= \sum_{j=0}^{i} \binom{k+j-1}{j} + \binom{k+i}{i+1} \qquad (P(i) \text{ is true})$$

$$= \sum_{j=0}^{i+1} \binom{k+j-1}{j}, \qquad \text{(definition of the sum)}$$

that is, $P(i+1)$ is true. Therefore, $P(i)$ is true for any $i = 0, 1, ...$ $\square$

Proposition 1.6 (number of elementary operators between bounded chains) – Let K_1 and K_2 be two bounded chains, and let $k_1 = (\#K_1) - 1$ and $k_2 = (\#K_2) - 1$. Let $N(k_1, k_2)$ be the number of operators in a given class of elementary operators from K_1 to K_2, then for any $k_1 \geq 1$ and $k_2 \geq 1$,

$$N(k_1, k_2) = \frac{(k_1 + k_2)!}{k_1! k_2!}.$$

$\square$

Proof – We consider for example the case of the dilations, that is, $N(k_1, k_2) \stackrel{\Delta}{=} \#\Delta(K_1, K_2)$. For any $k_1 \geq 1$ and $k_2 \geq 1$, let

$$N_i(k_1) \stackrel{\Delta}{=} \{\psi \in \Delta(K_1, K_2) : \psi(k_1) = i\} \quad (i \in K_2).$$

Let $P(k_1)$ be the property $N_i(k_1) = \binom{k_1 + i - 1}{i}$ for any $i \in K_2$. Let us prove that $P(k_1)$ is true for any $k_1 \geq 1$. First we note that $N_i(1) = 1$ for any $i \in K_2$, that is $P(1)$ is true. Second, for any $i \in K_2$, we have

$$N_i(k_1 + 1) = \sum_{j=0}^{i} N_j(k_1) \qquad \text{(definition of } N_i(k_1)\text{)}$$

$$= \sum_{j=0}^{i} \binom{k_1 + j - 1}{j} \qquad \text{(assuming that } P(k_1) \text{ is true)}$$

$$= \binom{k_1 + i}{i}, \qquad \text{(Proposition 1.5)}$$

that is $P(k_1 + 1)$ is true. Therefore, $P(k_1)$ is true for any $k_1 \geq 1$. Furthermore, for any $k_1 \geq 1$ and $k_2 \geq 1$,

$$N(k_1, k_2) = \sum_{i=0}^{k_2} N_i(k_1) \qquad \text{(definition of } N(k_1, k_2) \text{ and } N_i(k_1)\text{)}$$

$$= \sum_{i=0}^{k_2} \binom{k_1 + i - 1}{i} \qquad (P(k_1) \text{ is true})$$

$$= \binom{k_1 + k_2}{k_2}. \qquad \text{(Proposition 1.5)}$$

$\square$

Let W be a nonempty set. We denote by ${K_2}^W$ the set of mappings from W to K_2. As another example of elementary operators defined on a bounded chain, let us consider the mappings from K_1 to ${K_2}^W$ or, equivalently, the families of functions of ${K_2}^W$, indexed by K_1. The relation $\leq$ on K_1^W given by

$$g_1 \leq g_2 \Leftrightarrow g_1(y) \leq g_2(y) \quad (y \in W)$$

defines a partial ordering on K_1^W. The greatest and least elements of ${K_2}^W$ are, respectively, the constant functions g^* and g_* given by, for any $y \in W$,

$$g^*(y) \stackrel{\Delta}{=} \max K_2 \quad \text{and} \quad g_*(y) \stackrel{\Delta}{=} \min K_2.$$

From Proposition 1.4, a mapping p from K_1 to $K_2{}^W$ is a dilation (resp., erosion) iff p is increasing and $p(\min K_1) = g_*$ (resp., $p(\max K_1) = g^*$). It is an anti–dilation (resp., anti–erosion) iff it is decreasing and $p(\min K_1) = g^*$ (resp., $p(\max K_1) = g_*$).

Figure 4 shows a dilation d from K_1 to $K_2{}^W$, where $K_1 = \{0, 1, 2, 3\}$,

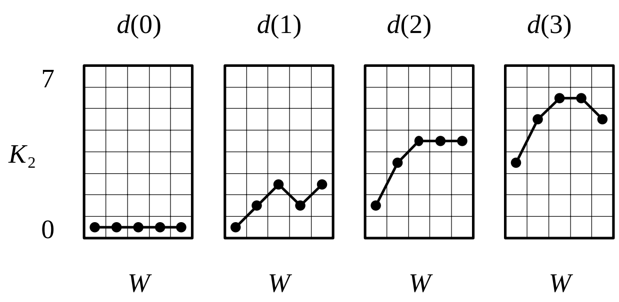

Fig. 4 – A family of functions forming a dilation.

$K_2 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and W is a set with 5 elements. This dilation is an increasing family of 4 functions from W to K_2 with the first one being g_*, that is, the constant function assuming value 0.

Figure 5 shows an erosion e from K_1 to $K_2{}^W$, where $K_1 = \{0, 1, 2, 3, 4, 5, 6, 7\}$,

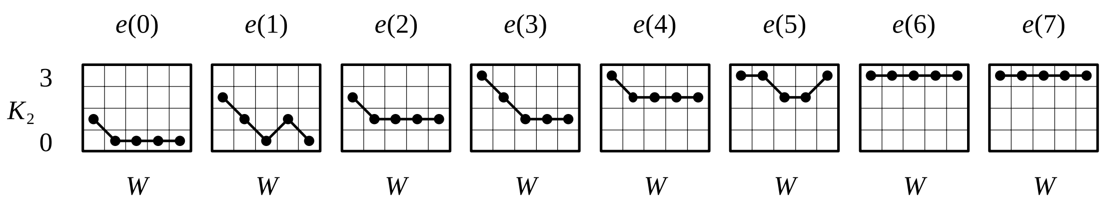

Fig. 5 – A family of functions forming an erosion.

$K_2 = \{0, 1, 2, 3\}$ and W is a set with 5 elements. This erosion is an increasing family of 8 functions from W to K_2 with the last one being g^*, that is, the constant function assuming value 3.

CHAPTER 2

ELEMENTARY MEASURES

2.1 – CHARACTERIZATION BY CONFRONTATION

We call a *measure*, an operator from a power lattice or function lattice to a chain. A power lattice or a function lattice is a set of mappings with a complete lattice as range (9, p. 22 and p. 26).

Let W be a nonempty set and let $(K_1, \le)$ be a bounded chain. The relation $\le$ on K_1^W given by

$$f_1 \le f_2 \Leftrightarrow f_1(x) \le f_2(x) \quad (x \in W) \tag{27}$$

defines a partial ordering on K_1^W. Furthermore, $(K_1^W, \le)$ is a complete lattice, with the supremum $\sup F$ and the infimum $\inf F$ of a subset F of K_1^W, respectively, given by

$$(\sup F)(x) = \max \{s \in K_1 : \exists f \in F,\ s = f(x)\} \quad (x \in W) \tag{28}$$

and

$$(\inf F)(x) = \min \{s \in K_1 : \exists f \in F,\ s = f(x)\} \quad (x \in W). \tag{29}$$

Relatively to $(K_1^W, \le)$, the union $\bigvee_{i \in I} f_i$ and intersection $\bigwedge_{i \in I} f_i$ of a family $(f_i)_{i \in I}$ of elements of K_1^W, indexed by a set I, are given by, respectively,

$$(\bigvee_{i \in I} f_i)(x) = \bigvee_{i \in I} f_i(x) \quad (x \in W) \tag{30}$$

and

$$(\bigwedge_{i \in I} f_i)(x) = \bigwedge_{i \in I} f_i(x) \quad (x \in W), \tag{31}$$

where the union and intersection on the right handside are given by, for any family $(s_i)_{i \in I}$ of elements of K_1,

$$\bigvee_{i \in I} s_i = \max \{s \in K_1 : \exists i \in I,\ s = s_i\} \tag{32}$$

and

$$\bigwedge_{i \in I} s_i = \min \{s \in K_1 : \exists i \in I,\ s = s_i\}. \tag{33}$$

The complete lattice $(K_1^W, \le)$ is an example of function lattice. It is interesting to note that, for any $x \in W$ the mappings $f \mapsto f(x)$ from K_1^W to K_1 are, at the same time, dilations and erosions.

We now give three different ways to characterize the elementary measures from ${K_1}^W$ to K_2, where K_1 and K_2 are two bounded chains.

Let us first introduce the so–called characterization by confrontation. In this characterization, the elementary measures from ${K_1}^W$ to K_2 will depend on elementary operators from K_2 to ${K_1}^W$ that we will call *structuring elements*. The characterization will be a direct application of the notion of Galois connection (17).

We define the following four pairs of useful expressions.

For any dilation $\delta \in \Delta({K_1}^W, K_2)$ and any erosion $e \in \mathrm{E}(K_2, {K_1}^W)$, let

$$e_\delta(t) \overset{\Delta}{=} \sup\{f \in {K_1}^W : \ \delta(f) \leq t\} \quad (t \in K_2) \tag{34}$$

$$_e\delta(f) \overset{\Delta}{=} \min\{t \in K_2 : \ f \leq e(t)\} \quad (f \in {K_1}^W), \tag{35}$$

e_δ is called the *structuring element of the dilation* δ, and $_e\delta(f)$ is called the *dilation of* f *by the structuring element* e.

For any erosion $\epsilon \in \mathrm{E}({K_1}^W, K_2)$ and any dilation $d \in \mathrm{D}(K_2, {K_1}^W)$, let

$$_\epsilon d(t) \overset{\Delta}{=} \inf\{f \in {K_1}^W : \ t \leq \epsilon(f)\} \quad (t \in K_2) \tag{36}$$

$$\epsilon_d(f) \overset{\Delta}{=} \max\{t \in K_2 : \ d(t) \leq f\} \quad (f \in {K_1}^W), \tag{37}$$

$_\epsilon d$ is called the *structuring element of the erosion* ϵ, and $\epsilon_d(f)$ is called the *erosion of* f *by the structuring element* d.

For any anti–dilation $\delta^a \in \Delta^a({K_1}^W, K_2)$ and any anti–dilation $d^a \in \mathrm{D}^a(K_2, {K_1}^W)$, let

$$_{\delta^a}d^a(t) \overset{\Delta}{=} \sup\{f \in {K_1}^W : \ t \leq \delta^a(f)\} \quad (t \in K_2) \tag{38}$$

$$_{d^a}\delta^a(f) \overset{\Delta}{=} \max\{t \in K_2 : \ f \leq d^a(t)\} \quad (f \in {K_1}^W), \tag{39}$$

$_{\delta^a}d^a$ is called the *structuring element of the anti–dilation* δ^a, and $_{d^a}\delta^a(f)$ is called the *anti–dilation of* f *by the structuring element* d^a.

For any anti–erosion $\epsilon^a \in \mathrm{E}^a({K_1}^W, K_2)$ and any anti–erosion $e^a \in \mathrm{E}^a(K_2, {K_1}^W)$, let

$$e^a{}_{\epsilon^a}(t) \overset{\Delta}{=} \inf\{f \in {K_1}^W : \ \epsilon^a(f) \leq t\} \quad (t \in K_2) \tag{40}$$

$$\epsilon^a{}_{e^a}(f) \overset{\Delta}{=} \min\{t \in K_2 : \ e^a(t) \leq f\} \quad (f \in {K_1}^W), \tag{41}$$

$e^a{}_{\epsilon^a}$ is called the *structuring element of the anti–erosion* ϵ^a, and $\epsilon^a{}_{e^a}(f)$ is called the *anti–erosion of* f *by the structuring element* e^a.

Proposition 1.7 (characterization by confrontation of the elementary measures) – Let W be a nonempty set, and let K_1 and K_2 be two bounded chains. We have the following statements:

$\delta \mapsto e_\delta$ (Exp. (34)) from $\Delta({K_1}^W, K_2)$ to $\mathrm{E}(K_2, {K_1}^W)$ is a bijection, its inverse is $e \mapsto {}_e\delta$ (Exp. (35));

$\epsilon \mapsto {}_\epsilon d$ (Exp. (36)) from $\mathrm{E}({K_1}^W, K_2)$ to $\mathrm{D}(K_2, {K_1}^W)$ is a bijection, its inverse is $d \mapsto \epsilon_d$ (Exp. (37));

$\delta^a \mapsto {}_{\delta^a}d^a$ (Exp. (38)) from $\Delta^a({K_1}^W, K_2)$ to $\mathrm{D}^a(K_2, {K_1}^W)$ is a bijection, its inverse is $d^a \mapsto {}_{d^a}\delta^a$ (Exp. (39));

$\epsilon^a \mapsto e^a{}_{\epsilon^a}$ (Exp. (40)) from $\mathrm{E}^a({K_1}^W, K_2)$ to $\mathrm{E}^a(K_2, {K_1}^W)$ is a bijection, its inverse is $e^a \mapsto \epsilon^a{}_{e^a}$ (Exp. (41)).

□

Proof – The first statement is a direct application of Propositions 3.15 and 3.14 of Heijmans (9). The three other statements follow from the first one by duality. The third statement is a direct application of Proposition 2 of Achache (20). □

Figure 6 shows an elementary measure ϵ_d which is an erosion from $K_1{}^W$ to K_2, where $K_1 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $K_2 = \{0, 1, 2, 3\}$ and W is a set with 5 elements. In its characterization by confrontation, this erosion depends on the dilation d from K_2 to $K_1{}^W$ given in Figure 4. In Figure 6, a particular function f is shown; the measure of f through ϵ_d is 1. This value is obtained through Expression (37). Figure 6 shows that we have to *confront* f with each function in the family defined by d in order to find the greatest function smaller or equal to f.

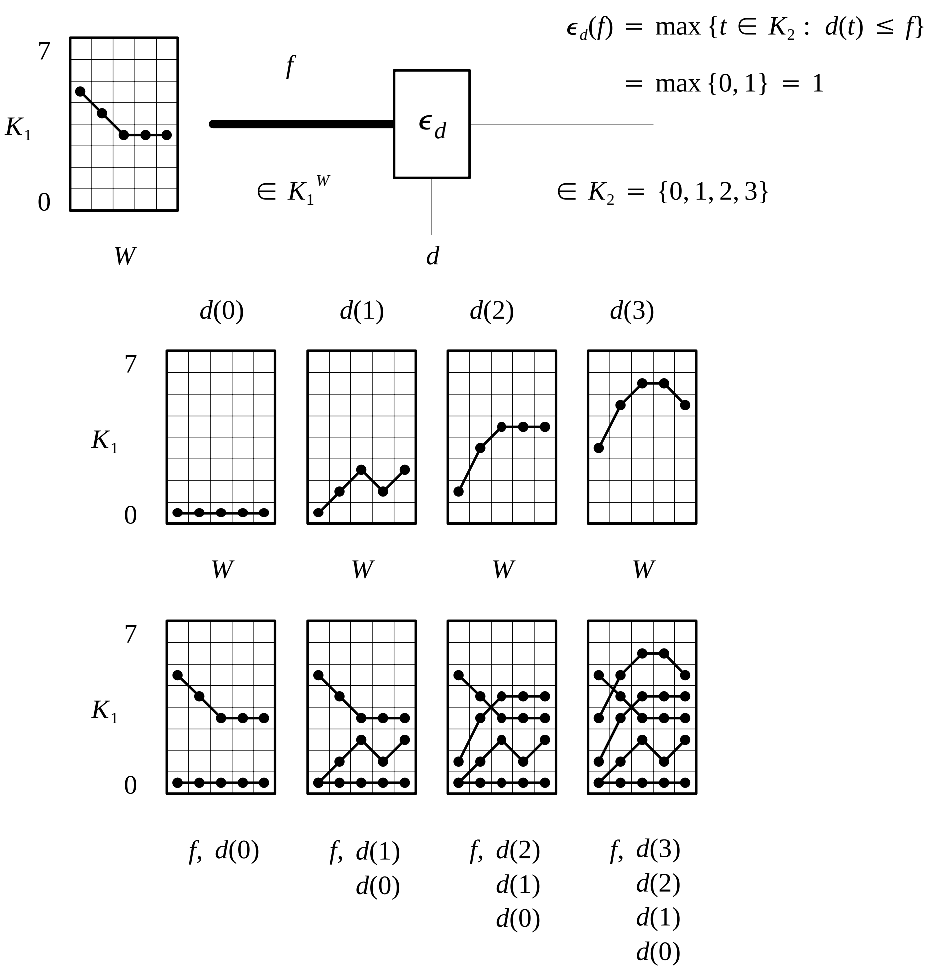

Fig. 6 – Erosion characterization by confrontation.

2.2 – CHARACTERIZATION BY SELECTION

Let us introduce our second way to characterize an elementary measure, the so–called characterization by selection. In this characterization, the elementary measures from $K_1{}^W$ to K_2 will depend on elementary operators from K_1 to $K_2{}^W$ that we call *distribution functions*. The characterization will be based on the decomposition of the functions in $K_1{}^W$ in terms of pulse functions.

Let $x \in W$ and $s \in K_1$. The functions $f_{x,s}$ and $f^{x,s}$ from W to K_1, respectively, given by

$$f_{x,s}(u) \triangleq \begin{cases} s & \text{if } u = x \\ \min K_1 & \text{otherwise} \end{cases} \quad (u \in W) \tag{42}$$

and

$$f^{x,s}(u) \triangleq \begin{cases} s & \text{if } u = x \\ \max K_1 & \text{otherwise} \end{cases} \quad (u \in W), \tag{43}$$

are called *pulse functions*.

Proposition 1.8 (sup and inf generating families of the pulse functions) – Let W be a nonempty set and let K_1 be a bounded chain. For any $f \in K_1{}^W$,

$$\bigvee_{x \in W} f_{x,f(x)} = f \quad \text{and} \quad \bigwedge_{x \in W} f^{x,f(x)} = f.$$

□

Proof – Let us prove the first statement. For any $f \in K_1{}^W$ and $u \in W$,

$$\begin{aligned}
(\bigvee_{x \in W} f_{x,f(x)})(u) &= \bigvee_{x \in W} f_{x,f(x)}(u) && \text{(Expression (30))} \\
&= f_{u,f(u)}(u) \vee (\bigvee_{\substack{x \in W \\ x \neq u}} f_{x,f(x)}(u)) && \text{(property of the union)} \\
&= f(u) \vee \max\{\min K_1\} && \text{(Expressions (42) and (32))} \\
&= f(u) \vee \min K_1 && \text{(definition of the max)} \\
&= f(u). && \text{(property of the union)}
\end{aligned}$$

That is, $\bigvee_{x \in W} f_{x,f(x)} = f$. The second statement follows in a similar way. □

Proposition 1.9 (union and intersection of some families of pulse functions) – Let W be a nonempty set and let K_1 be a bounded chain. For any $x \in W$ and any family $(s_i)_{i \in I}$ of elements of K_1,

$$\bigvee_{i \in I} f_{x,s_i} = f_{x, \vee s_i} \quad \text{and} \quad \bigwedge_{i \in I} f_{x,s_i} = f_{x, \wedge s_i}.$$

□

Proof – Let us prove the first expression. For any $x \in W$,

$$\begin{aligned}
(\bigvee_{i \in I} f_{x,s_i})(x) &= \bigvee_{i \in I} f_{x,s_i}(x) && \text{(Expression (30))} \\
&= \bigvee_{i \in I} s_i && \text{(Expression (42))} \\
&= f_{x, \vee s_i}(x); && \text{(Expression (42))}
\end{aligned}$$

for any $x \in W$ and $u \in W$, with $x \neq u$,

$$(\bigvee_{i \in I} f_{x,s_i})(u) = \bigvee_{i \in I} f_{x,s_i}(u) \qquad \text{(Expression (30))}$$

$$= \min K_1 \qquad \text{(Expression (42))}$$

$$= f_{x, \vee s_i}(u). \qquad \text{(Expression (42))}$$

That is, $\bigvee_{i \in I} f_{x,s_i} = f_{x, \vee s_i}$. The second expression follows in a similar way. □

We define the following four pairs of useful expressions.

For any dilation $\delta \in \Delta({K_1}^W, K_2)$ and any dilation $d \in \mathrm{D}(K_1, {K_2}^W)$, let

$$d_\delta(s)(x) \stackrel{\Delta}{=} \delta(f_{x,s}) \quad (s \in K_1, x \in W) \tag{44}$$

$$\delta_d(f) \stackrel{\Delta}{=} \bigvee_{x \in W} d(f(x))(x) \quad (f \in {K_1}^W), \tag{45}$$

d_δ is called the *distribution function of the dilation* δ, and $\delta_d(f)$ is called the *dilation of* f *by the dilation which distribution function is* d, or, in short, the *dilation of* f *w.r.t.* d.

For any erosion $\epsilon \in \mathrm{E}({K_1}^W, K_2)$ and any erosion $e \in \mathrm{E}(K_1, {K_2}^W)$, let

$${}_\epsilon e(s)(x) \stackrel{\Delta}{=} \epsilon(f^{x,s}) \quad (s \in K_1, x \in W) \tag{46}$$

$${}_e\epsilon(f) \stackrel{\Delta}{=} \bigwedge_{x \in W} e(f(x))(x) \quad (f \in {K_1}^W), \tag{47}$$

${}_\epsilon e$ is called the *distribution function of the erosion* ϵ, and ${}_e\epsilon(f)$ is called the *erosion of* f *by the erosion which distribution function is* e, or, in short, the *erosion of* f *w.r.t.* e.

For any anti–dilation $\delta^a \in \Delta^a({K_1}^W, K_2)$ and any anti–dilation $d^a \in \mathrm{D}^a(K_1, {K_2}^W)$, let

$$d^a{}_{\delta^a}(s)(x) \stackrel{\Delta}{=} \delta^a(f_{x,s}) \quad (s \in K_1, x \in W) \tag{48}$$

$$\delta^a{}_{d^a}(f) \stackrel{\Delta}{=} \bigwedge_{x \in W} d^a(f(x))(x) \quad (f \in {K_1}^W), \tag{49}$$

$d^a{}_{\delta^a}$ is called the *distribution function of the anti–dilation* δ^a, and $\delta^a{}_{d^a}(f)$ is called the *anti–dilation of* f *by the anti–dilation which distribution function is* d^a, or, in short, the *anti–dilation of* f *w.r.t.* d^a.

For any anti–erosion $\epsilon^a \in \mathrm{E}^a({K_1}^W, K_2)$ and any anti–erosion $e^a \in \mathrm{E}^a(K_1, {K_2}^W)$, let

$${}_{\epsilon^a}e^a(s)(x) \stackrel{\Delta}{=} \epsilon^a(f^{x,s}) \quad (s \in K_1, x \in W) \tag{50}$$

$${}_{e^a}\epsilon^a(f) \stackrel{\Delta}{=} \bigvee_{x \in W} e^a(f(x))(x) \quad (f \in {K_1}^W), \tag{51}$$

${}_{\epsilon^a}e^a$ is called the *distribution function of the anti–erosion* ϵ^a, and ${}_{e^a}\epsilon^a(f)$ is called the *anti–erosion of* f *by the anti–erosion which distribution function is* e^a, or, in short, the *anti–erosion of* f *w.r.t.* e^a.

Proposition 1.10 (characterization by selection of the elementary measures) – Let W be a nonempty set, and let K_1 and K_2 be two bounded chains. We have the following statements:

$\delta \mapsto d_\delta$ (Exp. (44)) from $\Delta({K_1}^W, K_2)$ to $\mathrm{D}(K_1, {K_2}^W)$ is a bijection, its inverse is $d \mapsto \delta_d$ (Exp. (45));

$\epsilon \mapsto {}_\epsilon e$ (Exp. (46)) from $\mathrm{E}({K_1}^W, K_2)$ to $\mathrm{E}(K_1, {K_2}^W)$ is a bijection, its inverse is $e \mapsto {}_e\epsilon$ (Exp. (47));

$\delta^{\mathrm{a}} \mapsto {d^{\mathrm{a}}}_{\delta^{\mathrm{a}}}$ (Exp. (48)) from $\Delta^{\mathrm{a}}({K_1}^W, K_2)$ to $\mathrm{D}^{\mathrm{a}}(K_1, {K_2}^W)$ is a bijection, its inverse is $d^{\mathrm{a}} \mapsto {\delta^{\mathrm{a}}}_{d^{\mathrm{a}}}$ (Exp. (49));

$\epsilon^{\mathrm{a}} \mapsto {}_{\epsilon^{\mathrm{a}}}e^{\mathrm{a}}$ (Exp. (50)) from $\mathrm{E}^{\mathrm{a}}({K_1}^W, K_2)$ to $\mathrm{E}^{\mathrm{a}}(K_1, {K_2}^W)$ is a bijection, its inverse is $e^{\mathrm{a}} \mapsto {}_{e^{\mathrm{a}}}\epsilon^{\mathrm{a}}$ (Exp. (51)).

□

Proof – Let us prove the first statement.

Let us prove that, for any $\delta \in \Delta({K_1}^W, K_2)$, we have $d_\delta \in \mathrm{D}(K_1, {K_2}^W)$. For any $x \in W$,

$$\begin{aligned} d_\delta(\min K_1)(x) &= \delta(f_{x,\min K_1}) && \text{(Expression (44))} \\ &= \delta(\inf({K_1}^W)) && \text{(Expression(42))} \\ &= \min K_2. && (\delta \text{ is a dilation and Proposition 1.3}) \end{aligned}$$

That is, $d_\delta(\min K_1) = \inf({K_2}^W)$. For any $x \in W$ and any family $(s_i)_{i\in I}$ of elements of K_1,

$$\begin{aligned} d_\delta(\bigvee_{i \in I} s_i)(x) &= \delta(f_{x,\vee s_i}) && \text{(Expression (44))} \\ &= \delta(\bigvee_{i \in I} f_{x,s_i}) && \text{(Proposition 1.9)} \\ &= \bigvee_{i \in I} \delta(f_{x,s_i}) && (\delta \text{ is a dilation}) \\ &= \bigvee_{i \in I} d_\delta(s_i)(x) && \text{(Expression (44))} \\ &= (\bigvee_{i \in I} d_\delta(s_i))(x). && \text{(Expression (30))} \end{aligned}$$

That is, $d_\delta(\bigvee_{i \in I} s_i) = \bigvee_{i \in I} d_\delta(s_i)$. In other words, $d_\delta \in \mathrm{D}(K_1, {K_2}^W)$.

Let us prove that, for any $d \in \mathrm{D}(K_1, {K_2}^W)$, we have $\delta_d \in \Delta({K_1}^W, K_2)$.

$$\begin{aligned} \delta_d(\inf({K_1}^W)) &= \bigvee_{x \in W} d(\min K_1)(x) && \text{(Expression (45))} \\ &= \bigvee_{x \in W} (\inf({K_2}^W))(x). && (d \text{ is a dilation and Proposition 1.3}) \\ &= \max\{\min K_2\}. && \text{(Expressions (32))} \\ &= \min K_2. && \text{(definition of the max)} \end{aligned}$$

For any family $(f_i)_{i\in I}$ of elements of ${K_1}^W$,

$$\delta_d(\bigvee_{i\in I} f_i) = \bigvee_{x\in W} d((\bigvee_{i\in I} f_i)(x))(x) \qquad \text{(Expression (45))}$$

$$= \bigvee_{x\in W} d(\bigvee_{i\in I} f_i(x))(x) \qquad \text{(Expression (30))}$$

$$= \bigvee_{x\in W}\bigvee_{i\in I} d(f_i(x))(x) \qquad \text{(d is a dilation)}$$

$$= \bigvee_{i\in I}\bigvee_{x\in W} d(f_i(x))(x) \qquad \text{(property of the union)}$$

$$= \bigvee_{i\in I} \delta_d(f_i). \qquad \text{(Expression (45))}$$

In other words, $\delta_d \in \Delta({K_1}^W, K_2)$.

Let us prove that $\delta \mapsto d_\delta$ is injective. For any $\delta \in \Delta({K_1}^W, K_2)$ and $f \in {K_1}^W$,

$$\delta_{d_\delta}(f) = \bigvee_{x\in W} d_\delta(f(x))(x) \qquad \text{(Expression (45))}$$

$$= \bigvee_{x\in W} \delta(f_{x,f(x)}) \qquad \text{(Expression (44))}$$

$$= \delta(\bigvee_{x\in W} f_{x,f(x)}) \qquad \text{(δ is a dilation)}$$

$$= \delta(f) \qquad \text{(Proposition 1.8)}$$

That is, $\delta_{d_\delta} = \delta$ and $\delta \mapsto d_\delta$ is injective.

Let us prove that $\delta \mapsto d_\delta$ is onto. For any $d \in \mathrm{D}(K_1, {K_2}^W)$, $s \in K_1$ and $x \in W$,

$$d_{\delta_d}(s)(x) = \delta_d(f_{x,s}) \qquad \text{(Expression (44))}$$

$$= \bigvee_{u\in W} d(f_{x,s}(u))(u) \qquad \text{(Expression (45))}$$

$$= d(f_{x,s}(x))(x) \vee (\bigvee_{\substack{u\in W\\ u\neq x}} d(f_{x,s}(u))(u)) \qquad \text{(property of the union)}$$

$$= d(f_{x,s}(x))(x) \vee (\bigvee_{\substack{u\in W\\ u\neq x}} d(\min K_1)(u)) \qquad \text{(Expression (42))}$$

$$= d(f_{x,s}(x))(x) \vee (\bigvee_{\substack{u\in W\\ u\neq x}} (\inf {K_1}^W)(u)) \qquad \text{(d is a dilation and Proposition 1.3)}$$

$$= d(f_{x,s}(x))(x) \vee \max\{\min K_2\} \qquad \text{(Expressions (32))}$$

$$= d(f_{x,s}(x))(x) \vee \min K_2 \qquad \text{(definition of the max)}$$

$$= d(f_{x,s}(x))(x) \qquad \text{(property of the union)}$$

$$= d(s)(x). \qquad \text{(Expression (42))}$$

That is, $d_{\delta_d} = d$ and $\delta \mapsto d_\delta$ is onto. In other words, $\delta \mapsto d_\delta$ is a bijection and its inverse is $d \mapsto \delta_d$.

The three other statements follow from the first one by duality. □

Figure 7 shows an elementary measure ${}_e\epsilon$ which is an erosion from ${K_1}^W$ to K_2, where $K_1 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $K_2 = \{0, 1, 2, 3\}$ and W is a set with 5 elements. In its characterization by selection, this erosion depends on the erosion e from K_1 to ${K_2}^W$ given in Figure 5. In Figure 7, a particular function f is shown; the measure of f through ${}_e\epsilon$ is 1. This value is obtained through Expression (47). Figure 7 shows that when x runs over W, we have to *select*, according to the value $f(x)$, the appropriate function in the family defined by e. Actually, the erosion e of Figure 5 has been chosen in such a way that the resulting measure ${}_e\epsilon$ is identical to the measure ϵ_d shown in Figure 6.

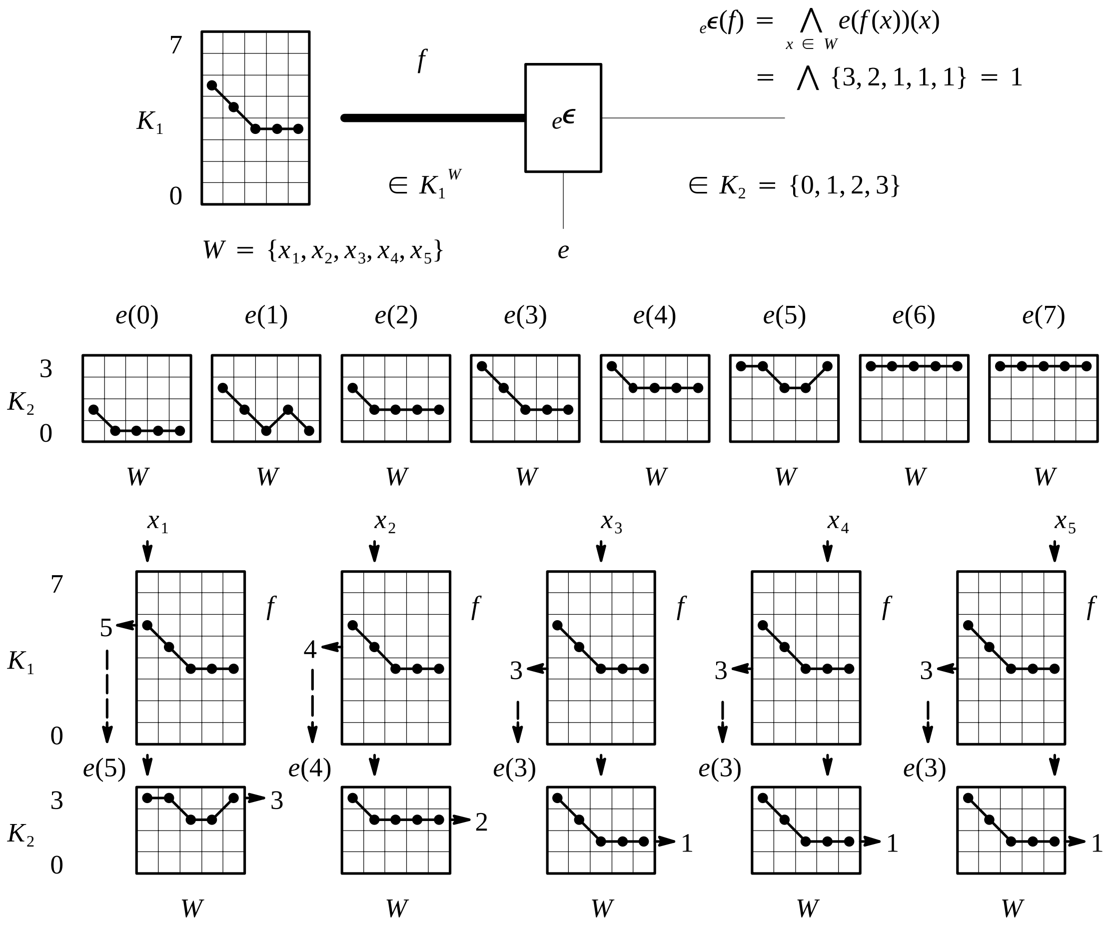

Fig. 7 – Erosion characterization by selection.

Let $K_1 = \{0, 1\}$ and $K_2 = [0, 1] \subset \mathbf{R}$ (the set of real numbers). A measure from $\{0, 1\}^W$ to $[0, 1]$, or equivalently from $\mathcal{P}(W)$ (the collection of subsets of W) to $[0, 1]$ such that $(\emptyset) = 0$ and $(W) = 1$ is called a *fuzzy measure* in the Theory of Fuzzy Set (21, 22). The dilations (resp., erosions) from $\mathcal{P}(W)$ to $[0, 1]$ are then called *possibility measures* (resp., *necessity measures*) (23). If is a possibility measure, then d (1) (Expression (44)) is then the so–called *possibility distribution of* (24). If is a necessity measure, then e (0) (Expression (46)) is then the so–called *necessity distribution of* (23).

2.3 – CHARACTERIZATION BY DECOMPOSITION

Let us introduce our last way to characterize a measure, the so–called characterization by decomposition. In this characterization, the elementary measures from $K_1{}^W$ to K_2 will depend on a family of elementary operators from K_1 to K_2 that we call elementary transformation tables. The characterization will be derived from the characterization by selection.

We define the following two useful expressions.

For any mapping p from K_1 to $K_2{}^W$, let, for any $x \in W$,

$$p_x(s) \stackrel{\Delta}{=} p(s)(x) \quad (s \in K_1); \tag{52}$$

For any family $(p_x)_{x \in W}$ of mappings from K_1 to K_2, let

$$p(s)(x) \stackrel{\Delta}{=} p_x(s) \quad (s \in K_1, x \in W). \tag{53}$$

Proposition 1.11 (bijection between the elementary distribution functions and the families of elementary transformation tables) – Let W be a nonempty set, and let K_1 and K_2 be two bounded chains. We have the following statements:

$p \mapsto (p_x)$ (Exp. (52)) from $\mathrm{D}(K_1, K_2{}^W)$ to $\mathrm{D}(K_1, K_2)^W$ is a bijection. Its inverse is $(p_x) \mapsto p$ (Exp. (53));

$p \mapsto (p_x)$ (Exp. (52)) from $\mathrm{E}(K_1, K_2{}^W)$ to $\mathrm{E}(K_1, K_2)^W$ is a bijection. Its inverse is $(p_x) \mapsto p$ (Exp. (53));

$p \mapsto (p_x)$ (Exp. (52)) from $\mathrm{D}^{\mathrm{a}}(K_1, K_2{}^W)$ to $\mathrm{D}^{\mathrm{a}}(K_1, K_2)^W$ is a bijection. Its inverse is $(p_x) \mapsto p$ (Exp. (53));

$p \mapsto (p_x)$ (Exp. (52)) from $\mathrm{E}^{\mathrm{a}}(K_1, K_2{}^W)$ to $\mathrm{E}^{\mathrm{a}}(K_1, K_2)^W$ is a bijection. Its inverse is $(p_x) \mapsto p$ (Exp. (53)).

□

Proof – Let us prove the first statement. Expressions (52) and (53) are symmetrical, therefore the mapping $p \mapsto (p_x)$ is a bijection.

Let us prove that, for any $p \in \mathrm{D}(K_1, K_2{}^W)$, we have $(p_x) \in \mathrm{D}(K_1, K_2)^W$. For any $x \in W$,

$$\begin{aligned} p_u(\min K_1) &= p(\min K_1)(x) && \text{(Expression (52))} \\ &= \inf(K_2{}^W)(x) && (p \text{ is a dilation and Proposition 1.3)} \\ &= \min K_2; && \text{(Expression (29))} \end{aligned}$$

for any family $(s_i)_{i\in I}$ of elements of K_1 and $x \in W$,

$$p_x(\bigvee_{i \in I} s_i) = p(\bigvee_{i \in I} s_i)(x) \qquad \text{(Expression (52))}$$

$$= (\bigvee_{i \in I} p(s_i))(x) \qquad (p \text{ is a dilation})$$

$$= \bigvee_{i \in I} p(s_i)(x) \qquad \text{(Expression (30))}$$

$$= \bigvee_{i \in I} p_x(s_i) \qquad \text{(Expression (52))}$$

That is, $p_x \in \mathrm{D}(K_1, K_2)$. In other words, $(p_x) \in \mathrm{D}(K_1, K_2)^W$. Let us prove that, for any $(p_x) \in \mathrm{D}(K_1, K_2)^W$, we have $p \in \mathrm{D}(K_1, {K_2}^W)$. For any $x \in W$,

$$p(\min K_1)(x) = p_x(\min K_1) \qquad \text{(Expression (53))}$$

$$= \max K_2; \qquad (p_x \text{ is a dilation and Proposition 1.3})$$

for any family $(s_i)_{i\in I}$ of elements of K_1 and $x \in W$,

$$p(\bigvee_{i \in I} s_i)(x) = p_x(\bigvee_{i \in I} s_i) \qquad \text{(Expression (53))}$$

$$= \bigvee_{i \in I} p_x(s_i) \qquad (p_u \text{ is a dilation})$$

$$= \bigvee_{i \in I} p(s_i)(x) \qquad \text{(Expression (53))}$$

$$= (\bigvee_{i \in I} p(s_i))(x). \qquad \text{(Expression (30))}$$

That is, $p(\bigvee_{i \in I} s_i) = \bigvee_{i \in I} p(s_i)$. In other words, $p \in \mathrm{D}(K_1, {K_2}^W)$.

The three other statements follow from the first one by duality. □

We define the following four pairs of useful expressions.

For any dilation $\delta \in \Delta({K_1}^W, K_2)$ and any family of dilations $(d_x) \in \mathrm{D}(K_1, K_2)^W$, let

$$(d_\delta)_x(s) \stackrel{\Delta}{=} \delta(f_{x,s}) \quad (s \in K_1, x \in W) \tag{54}$$

$$\delta_{(d)}(f) \stackrel{\Delta}{=} \bigvee_{x \in W} d_x(f(x)) \quad (f \in {K_1}^W), \tag{55}$$

$((d_\delta)_x)$ is called the *family of transformation tables of the dilation* δ, and $\delta_{(d)}(f)$ is called the *dilation of* f *w.r.t. the family of transformation tables* (d_x).

For any erosion $\epsilon \in \mathrm{E}({K_1}^W, K_2)$ and any family of erosions $(e_x) \in \mathrm{E}(K_1, K_2)^W$, let

$$({}_\epsilon e)_x(s) \stackrel{\Delta}{=} \epsilon(f^{x,s}) \quad (s \in K_1, x \in W) \tag{56}$$

$${}_{(e)}\epsilon(f) \stackrel{\Delta}{=} \bigwedge_{x \in W} e_x(f(x)) \quad (f \in {K_1}^W), \tag{57}$$

$(({}_\epsilon e)_x)$ is called the *family of transformation tables of the erosion* ϵ, and ${}_{(e)}\epsilon(f)$ is called the *erosion of* f *w.r.t. the family of transformation tables* (e_x).

For any anti–dilation $\delta^{\mathrm{a}} \in \Delta^{\mathrm{a}}({K_1}^W, K_2)$ and any family of anti–dilations $(d^{\mathrm{a}}{}_x) \in \mathrm{D}^{\mathrm{a}}(K_1, K_2)^W$, let

$$(d^{\mathrm{a}}{}_{\delta^{\mathrm{a}}})_x(s) \stackrel{\Delta}{=} \delta^{\mathrm{a}}(f_{x,s}) \quad (s \in K_1, x \in W) \tag{58}$$

$$\delta^{\mathrm{a}}{}_{(d^{\mathrm{a}})}(f) \stackrel{\Delta}{=} \bigwedge_{x \in W} d^{\mathrm{a}}{}_x(f(x)) \quad (f \in {K_1}^W), \tag{59}$$

$((d^{\mathrm{a}}{}_{\delta^{\mathrm{a}}})_x)$ is called the *family of transformation tables of the anti–dilation* δ^{a}, and $\delta^{\mathrm{a}}{}_{(d^{\mathrm{a}})}(f)$ is called the *anti–dilation of* f *w.r.t. the family of transformation tables* $(d^{\mathrm{a}}{}_x)$.

For any anti–erosion $\epsilon^{\mathrm{a}} \in \mathrm{E}^{\mathrm{a}}({K_1}^W, K_2)$ and any family of anti–erosions $(e^{\mathrm{a}}{}_x) \in \mathrm{E}^{\mathrm{a}}(K_1, K_2)^W$, let

$$({}_{\epsilon^{\mathrm{a}}}e^{\mathrm{a}})_x(s) \stackrel{\Delta}{=} \epsilon^{\mathrm{a}}(f^{x,s}) \quad (s \in K_1, x \in W) \tag{60}$$

$${}_{(e^{\mathrm{a}})}\epsilon^{\mathrm{a}}(f) \stackrel{\Delta}{=} \bigvee_{x \in W} e^{\mathrm{a}}{}_x(f(x)) \quad (f \in {K_1}^W), \tag{61}$$

$(({}_{\epsilon^{\mathrm{a}}}e^{\mathrm{a}})_x)$ is called the *family of transformation tables of the anti–erosion* ϵ^{a}, and ${}_{(e^{\mathrm{a}})}\epsilon^{\mathrm{a}}(f)$ is called the *anti–dilation of* f *w.r.t. the family of transformation tables* $(e^{\mathrm{a}}{}_x)$.

Proposition 1.12 (characterization by decomposition of the elementary measures) – Let W be a nonempty set, and let K_1 and K_2 be two bounded chains. We have the following statements:

$\delta \mapsto ((d_\delta)_x)$ (Exp. (54)) from $\Delta({K_1}^W, K_2)$ to $\mathrm{D}(K_1, K_2)^W$ is a bijection,
its inverse is $(d_x) \mapsto \delta_{(d)}$ (Exp. (55));

$\epsilon \mapsto (({}_\epsilon e)_x)$ (Exp. (56)) from $\mathrm{E}({K_1}^W, K_2)$ to $\mathrm{E}(K_1, K_2)^W$ is a bijection,
its inverse is $(e_x) \mapsto {}_{(e)}\epsilon$ (Exp. (57));

$\delta^{\mathrm{a}} \mapsto ((d^{\mathrm{a}}{}_{\delta^{\mathrm{a}}})_x)$ (Exp. (58)) from $\Delta^{\mathrm{a}}({K_1}^W, K_2)$ to $\mathrm{D}^{\mathrm{a}}(K_1, K_2)^W$ is a bijection,
its inverse is $(d^{\mathrm{a}}{}_x) \mapsto \delta^{\mathrm{a}}{}_{(d^{\mathrm{a}})}$ (Exp. (59));

$\epsilon^{\mathrm{a}} \mapsto (({}_{\epsilon^{\mathrm{a}}}e^{\mathrm{a}})_x)$ (Exp. (60)) from $\mathrm{E}^{\mathrm{a}}({K_1}^W, K_2)$ to $\mathrm{E}^{\mathrm{a}}(K_1, K_2)^W$ is a bijection,
its inverse is $(e^{\mathrm{a}}{}_x) \mapsto {}_{(e^{\mathrm{a}})}\epsilon^{\mathrm{a}}$ (Exp. (61)).

□

Proof – Let us prove the first statement. Let us prove that the mapping $\delta \mapsto ((d_\delta)_x)$ (Exp. (54)) from $\Delta({K_1}^W, K_2)$ to $\mathrm{D}(K_1, K_2)^W$ is the composition of the mapping $\delta \mapsto d_\delta$ (Exp. (44)) from $\Delta({K_1}^W, K_2)$ to $\mathrm{D}(K_1, {K_2}^W)$, with the mapping $p \mapsto (p_x)$ (Exp. (52)) from $\mathrm{D}(K_1, {K_2}^W)$ to $\mathrm{D}(K_1, K_2)^W$. For any $\delta \in \Delta({K_1}^W, K_2)$, $x \in W$ and $s \in K_1$,

$$\begin{aligned} (d_\delta)_x(s) &= d_\delta(s)(x) && \text{(Expression (52))} \\ &= \delta(f_{x,s}) && \text{(Expression (44))} \\ &= (d_\delta)_x(s). && \text{(Expression (54))} \end{aligned}$$

By Propositions 1.10 and 1.11 these two mappings are bijections, therefore their composition $\delta \mapsto ((d_\delta)_x)$ is also a bijection. Let us prove that the mapping $(d_x) \mapsto \delta_{(d)}$ (Exp. (55)) from $\mathrm{D}(K_1, K_2)^W$ to $\Delta({K_1}^W, K_2)$ is the composition of $(p_x) \mapsto p$ (Exp. (53)) from $\mathrm{D}(K_1, K_2)^W$ to $\mathrm{D}(K_1, {K_2}^W)$ with $d \mapsto \delta_d$ (Exp. (45)) from $\mathrm{D}(K_1, {K_2}^W)$ to $\Delta({K_1}^W, K_2)$. For any $(d_x) \in \mathrm{D}(K_1, K_2)^W$ and $f \in {K_1}^W$,

$$\delta_d(f) = \bigvee_{x \in W} d(f(x))(x) \qquad \text{(Expression (45))}$$

$$= \bigvee_{x \in W} d_x(f(x)) \qquad \text{(Expression (53))}$$

$$= \delta_{(d)}(f). \qquad \text{(Expression (55))}$$

Hence, the mapping $(d_x) \mapsto \delta_{(d)}$ (Exp. (55)) from $\mathrm{D}(K_1, K_2)^W$ to $\Delta({K_1}^W, K_2)$ is the inverse of $\delta \mapsto ((d_\delta)_x)$ since it is the composition of $(p_x) \mapsto p$ (Exp. (53)) from $\mathrm{D}(K_1, K_2)^W$ to $\mathrm{D}(K_1, {K_2}^W)$ with $d \mapsto \delta_d$ (Exp. (45)) from $\mathrm{D}(K_1, {K_2}^W)$ to $\Delta({K_1}^W, K_2)$ which are, by Propositions 1.11 and 1.10, the inverses of the above two bijections. The three other statements follow from the first one by duality. □

From the above characterization we can state the following proposition.

Proposition 1.13 (construction/decomposition of the elementary measures) – Let W be a nonempty set, and let K_1 and K_2 be two bounded chains. Let $(\delta_x)_{x\in W}$, $(\epsilon_x)_{x\in W}$, $({\delta^a}_x)_{x\in W}$ and $({\epsilon^a}_x)_{x\in W}$ be the families of elementary measures from ${K_1}^W$ to K_2 given by, respectively, for any $x \in W$,

$$\delta_x(f) = d_x(f(x)) \quad (f \in {K_1}^W)$$

$$\epsilon_x(f) = e_x(f(x)) \quad (f \in {K_1}^W)$$

$${\delta^a}_x(f) = {d^a}_x(f(x)) \quad (f \in {K_1}^W)$$

$${\epsilon^a}_x(f) = {e^a}_x(f(x)) \quad (f \in {K_1}^W),$$

where, $(d_x)_{x\in W}$, $(e_x)_{x\in W}$, $({d^a}_x)_{x\in W}$ and $({e^a}_x)_{x\in W}$ are, respectively, families of dilations, erosions, anti–dilations and anti–erosions from K_1 to K_2. For any $x \in W$, the measures δ_x, ϵ_x, ${\delta^a}_x$ and ${\epsilon^a}_x$ from ${K_1}^W$ to K_2 are, respectively, a dilation, an erosion, an anti–dilation and an anti–erosion. The measures δ, ϵ, δ^a and ϵ^a from ${K_1}^W$ to K_2 given by, respectively,

$$\delta = \bigvee_{x \in W} \delta_x$$

$$\epsilon = \bigwedge_{x \in W} \epsilon_x$$

$$\delta^a = \bigwedge_{x \in W} {\delta^a}_x$$

$$\epsilon^a = \bigvee_{x \in W} {\epsilon^a}_x.$$

are, respectively, a dilation, an erosion, an anti–dilation and an anti–erosion. Conversely, any measure δ, ϵ, δ^a and ϵ^a from ${K_1}^W$ to K_2 which is, respectively, a dilation, an erosion, an anti–dilation and an anti–erosion has this form. Furthermore, $(d_x)_{x\in W}$, $(e_x)_{x\in W}$, $({d^a}_x)_{x\in W}$ and $({e^a}_x)_{x\in W}$ are given by, respectively, for any $x \in W$,

$$d_x(s) = \delta(f_{x,s}) \quad (s \in K_1)$$

$$e_x(s) = \epsilon(f^{x,s}) \quad (s \in K_1)$$

$${d^a}_x(s) = \delta^a(f_{x,s}) \quad (s \in K_1)$$

$${e^a}_x(s) = \epsilon^a(f^{x,s}) \quad (s \in K_1).$$

□

Proof – Let us prove the result for the class of dilations. By construction, for any $x \in W$, δ_x is the composition of $f \mapsto f(x)$ with d_x; since both are dilations, δ_x is also a dilation. The rest of the result is Proposition 1.12. The results for the other classes of measures follow by duality. □

Figure 8 shows an elementary measure ${}_{(e)}\epsilon$ which is an erosion from $K_1{}^W$ to K_2, where $K_1 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $K_2 = \{0, 1, 2, 3\}$ and W is a set with 5 elements. In its characterization by decomposition, this erosion depends on a family (e_x) of 5 erosions from K_1 to K_2 which are derived from the erosion e given in Figure 5 by applying Expression (52). In this way the measure ${}_{(e)}\epsilon$ is identical to ${}_e\epsilon$ shown in Figure 7. In Figure 8, a particular function f is shown; the measure of f through ${}_{(e)}\epsilon$ is 1. This value is obtained through Expression (57). Figure 8 shows a *decomposition* of ${}_{(e)}\epsilon$. Each branch is a particular measure which is an erosion as stated in Proposition 1.13.

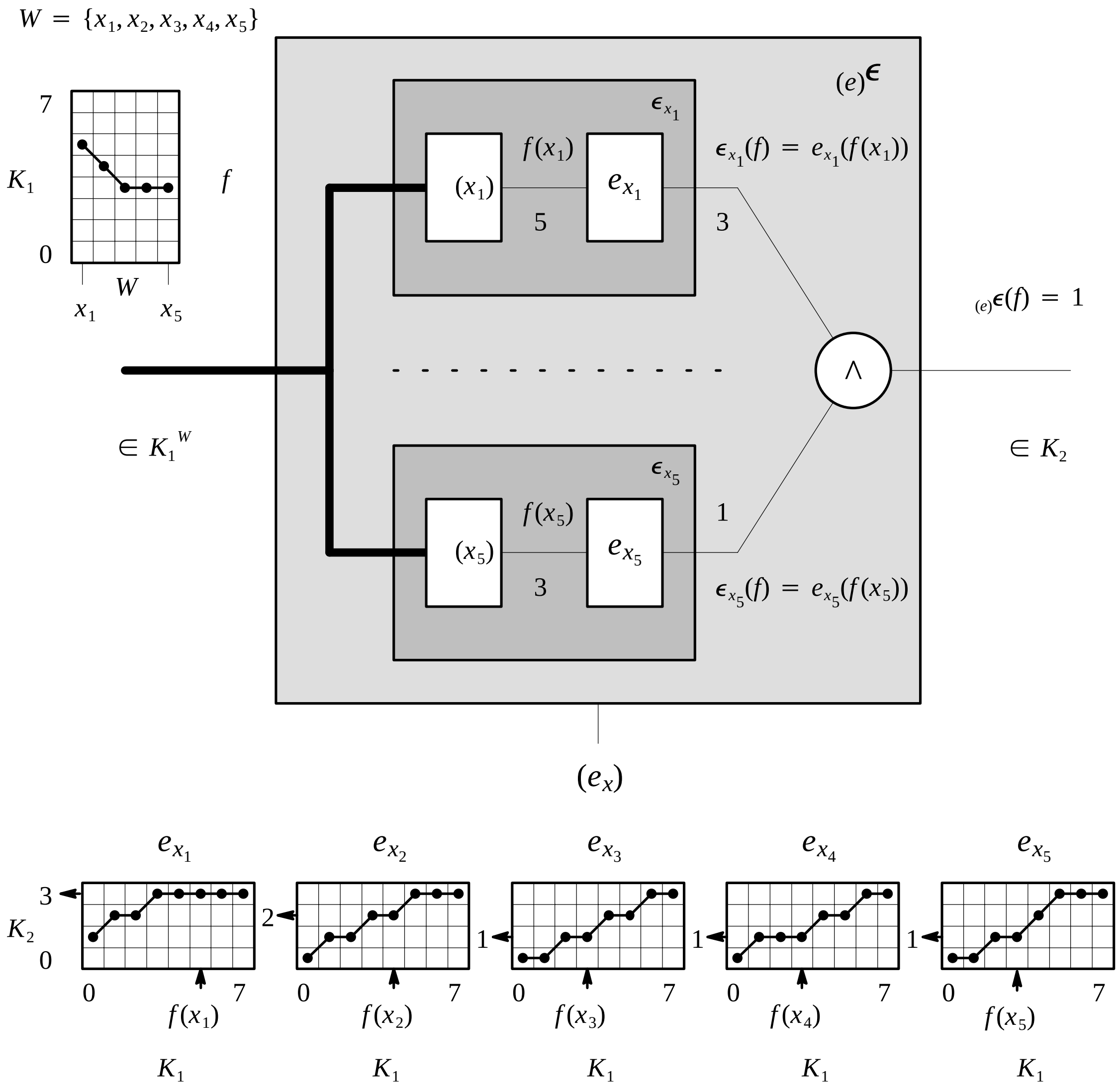

Fig. 8 – Erosion characterization by decomposition.

CHAPTER 3

ELEMENTARY TRANSLATION–INVARIANT OPERATORS

3.1 – CHARECTERIZATION IN TERMS OF MEASURES

Let $(E, +)$ be an Abelian group and let K be a bounded chain. The functions from E to K are the appropriate mathematical representation for the *gray–level images* with domain E (the set of pixel positions) and gray–scale K (the set of possible pixel values).

Let x be an element of E, we denote by $-x$ the *negative of* x. For any two elements x_1 and x_2 of E we denote by $x_1 - x_2$ the element of E given by $x_1 - x_2 \triangleq x_1 + (-x_2)$.

Let u be an element of E. Let W be a nonempty subset of E, we denote by $W + u$ the *translate of W by u*, that is, the subset of E given by

$$W + u \triangleq \{x \in E : x - u \in W\}.$$

Let f be an element of K^E, we denote by $f + u$ the *translate of f by u*, that is, the element of K^E given by

$$(f + u)(x) \triangleq f(x - u) \quad (x \in E). \tag{62}$$

We denote by $f - u$ the element of K^E given by $f - u \triangleq f + (-u)$.

We denote by τ_u the *translation by u*, that is the operator from K^E to K^E given by

$$\tau_u(f) \triangleq f + u \quad (f \in K^E).$$

Let K_1 and K_2 be two bounded chains sets representing possibly two different gray–scales. Any operator Ψ from ${K_1}^E$ to ${K_2}^E$ transforms any gray–level image into a gray–level image. In this chapter we focus our attention on the translation–invariant window operators. An operator Ψ from ${K_1}^E$ to ${K_2}^E$ is said *translation–invariant* (t.i.) iff, for any $f \in {K_1}^E$ and $u \in E$,

$$\Psi(f + u) = \Psi(f) + u.$$

translation–invariant operators from K^E to K^E, in the above sense, are called *H–operators* by Heijmans [9, p. 109 and p. 373].

In order to introduce the notion of window operator, we recall the definition of restriction and extension of a function. Let $f \in K^E$ and $W \subset E$, the *restriction of f to W*, denoted by f/W is the function from W to K (i.e., the element of K^W) given by

$$(f/W)(x) \triangleq f(x) \quad (x \in W). \tag{63}$$

From now on, we assume that $(K_1, \leq)$ and $(K_2, \leq)$ are two bounded chains which correspond to two possibly different gray–scales.

Let f be a function from W to K_1, we call the functions $\bar{f}$ and $\underline{f}$ from E to K_1, given by

$$\bar{f}(x) \triangleq \begin{cases} f(x) & \text{if } x \in W \\ \max K_1 & \text{otherwise} \end{cases} \quad (x \in E) \tag{64}$$

and

$$\underline{f}(x) \triangleq \begin{cases} f(x) & \text{if } x \in W \\ \min K_1 & \text{otherwise} \end{cases} \quad (x \in E), \tag{65}$$

respectively, the *upper* and *lower extensions of* f.

Proposition 1.14 (property of the restriction/extension) – Let E be a nonempty set and let K_1 be a bounded chain. For any nonempty subset W of E, and $f \in K_1{}^W$,

$$\bar{f}/W = f \quad \text{and} \quad \underline{f}/W = f.$$

□

Proof – Let us prove the first statement. For any $f \in K_1{}^W$ and $x \in W$,

$$\begin{aligned} (\overline{f}/W)(x) &= \bar{f}(x) && \text{(Expression (63))} \\ &= f(x). && \text{(Expression (64))} \end{aligned}$$

The second statement can be proved in a similar way. □

Proposition 1.15 (property of the restriction/extension) – Let $(E, +)$ be an Abelian group and let K_1 be a bounded chain. For any nonempty subset W of E, $f \in K_1{}^E$ and $x \in E$,

$$(\overline{f/(W+x)} - x)/W = (f - x)/W \quad \text{and} \quad (\underline{f/(W+x)} - x)/W = (f - x)/W.$$

□

Proof – For any $f \in K_1{}^E$, $x \in E$ and $u \in W$,

$$\begin{aligned} ((\overline{f/(W+x)} - x)/W)(u) &= (\overline{f/(W+x)} - x)(u) && \text{(Expression (63))} \\ &= (\overline{f/(W+x)})(u + x) && \text{(Expression (62))} \\ &= (f/(W+x))(u + x) && \text{(Expression (64))} \\ &= f(u + x) && \text{(Expression (63))} \\ &= (f - x)(u) && \text{(Expression (62))} \\ &= ((f - x)/W)(u). && \text{(Expression (63))} \end{aligned}$$

The second statement can be proved in a similar way. □

An operator Ψ from $K_1{}^E$ to $K_2{}^E$ is called *a window operator* (with window W ($W \subset E$)) iff, for any $f \in K_1{}^E$,

$$\Psi(f)(y) = \Psi(\overline{f/(W+y)})(y) \quad (y \in E), \tag{66a}$$

or equivalently,

$$\Psi(f)(y) = \Psi(\underline{f/(W+y)})(y) \quad (y \in E). \tag{66b}$$

We can prove, for example, that Expression (66a) implies Expression (66b). For any $f \in K_1{}^E$ and any $y \in E$,

$$\begin{aligned} \Psi(\underline{f/(W+y)})(y) &= \Psi(\overline{\underline{f/(W+y)}/(W+y)})(y) && \text{(Expression (66a))} \\ &= \Psi(\overline{f/(W+y)})(y) && \text{(Proposition 1.14, second statement)} \\ &= \Psi(f)(y). && \text{(Expression (66a))} \end{aligned}$$

Let o be the null element of $(E, +)$, called *origin*, we define the following two useful expressions, for any Ψ from $K_1{}^E$ to $K_2{}^E$ and ψ from $K_1{}^W$ to K_2:

$$\psi_\Psi(f) \triangleq \Psi(\bar{f})(o) \quad (f \in K_1{}^W) \tag{67a}$$

or equivalently in the case of window operators,

$$\psi_\Psi(f) \triangleq \Psi(\underline{f})(o) \quad (f \in K_1{}^W), \tag{67b}$$

and

$$\Psi_\psi(f)(y) \triangleq \psi((f - y)/W) \quad (f \in K_1{}^E, y \in E). \tag{68}$$

In the case of window operators, we can prove that $\Psi(\bar{f})(o) = \Psi(\underline{f})(o)$. For any $f \in K_1{}^W$,

$$\begin{aligned} \Psi(\bar{f})(o) &= \Psi(\overline{\bar{f}/W})(o) && \text{(Expression (66a))} \\ &= \Psi(\underline{\bar{f}/W})(o) && \text{(equivalency between Exp. (66a) and (66b))} \\ &= \Psi(\underline{f})(o). && \text{(Proposition 1.14)} \end{aligned}$$

Given all the above definitions we can now state the following proposition that shows how to characterize a translation–invariant window operator in terms of a measure.

Proposition 1.16 (characterization of the translation–invariant window operators) – Let $(E, +)$ be an Abelian group, let W be a nonempty subset of E, and let K_1 and K_2 be two bounded chains. The mapping $\Psi \mapsto \psi_\Psi$ (Expression (67)) from the set of translation–invariant window operators from $K_1{}^E$ to $K_2{}^E$, to the set of measures from $K_1{}^W$ to K_2 is a bijection. Its inverse is $\psi \mapsto \Psi_\psi$ (Expression (68)). □

Proof – By construction, for any operator Ψ from $K_1{}^E$ to $K_2{}^E$, ψ_Ψ is a operator from $K_1{}^W$ to K_2. Let us prove that, for any measure ψ from $K_1{}^W$ to K_2, Ψ_ψ is a t.i. window operator from $K_1{}^E$ to $K_2{}^E$. For any $f \in K_1{}^E$ and $y, v \in E$,

$$\begin{aligned} \Psi_\psi(f + v)(y) &= \psi(((f + v) - y)/W) && \text{(Expression (68))} \\ &= \psi((f + (v - y))/W) && \text{(property of the translate)} \\ &= \Psi_\psi(f)(y - v) && \text{(Expression (68))} \\ &= (\Psi_\psi(f) + v)(y). && \text{(Expression (62))} \end{aligned}$$

In other words, Ψ_ψ is t.i.. For any $f \in K_1{}^E$ and $y \in E$,

$$\begin{aligned} \Psi_\psi(\overline{f/(W + y)})(y) &= \psi((\overline{f/(W + y)} - y)/W) && \text{(Expression (68))} \\ &= \psi((f - y)/W) && \text{(Proposition 1.15)} \\ &= \Psi_\psi(f)(y). && \text{(Expression (68))} \end{aligned}$$

In other words, Ψ_ψ is a window operator (with window W).

Let us prove that $\Psi \mapsto \psi_\Psi$ is injective. For any t.i. window operator Ψ from $K_1{}^E$ to $K_2{}^E$, $f \in K_1{}^E$ and $y \in E$,

$$\begin{aligned} \Psi_{\psi_\Psi}(f)(y) &= \psi_\Psi((f - y)/W) && \text{(Expression (68))} \\ &= \Psi(\overline{(f - y)/W})(o) && \text{(Expression (67a))} \end{aligned}$$

$$= \Psi(f - y)(o) \qquad \text{(definition of window operator)}$$

$$= (\Psi(f) - y)(o) \qquad \text{(definition of t.i.)}$$

$$= \Psi(f)(y). \qquad \text{(Expression (62))}$$

That is, $\Psi_{\psi_\Psi} = \Psi$ and $\Psi \mapsto \psi_\Psi$ is injective.

Let us prove that $\Psi \mapsto \psi_\Psi$ is onto. For any operator ψ from $K_1{}^W$ to K_2, $f \in K_1{}^W$,

$$\psi_{\Psi_\psi}(f) = \Psi_\psi(\bar{f})(o) \qquad \text{(Expression (67a))}$$

$$= \psi(\bar{f}/W) \qquad \text{(Expression (68))}$$

$$= \psi(f). \qquad \text{(Proposition (1.14))}$$

That is, $\psi_{\Psi_\psi} = \psi$ and $\Psi \mapsto \psi_\Psi$ is onto. In other words, $\Psi \mapsto \psi_\Psi$ is a bijection and its inverse is $\psi \mapsto \Psi_\psi$.

□

Heijmans [9, p. 117 e p. 372] gives a similar expression to (68) for the class of flat function operators associated with (finite) window operators on $\mathcal{P}(E^d)$.

From the previous characterizations we can now derive the three characterizations of the translation–invariant elementary operators.

3.2 – CHARACTERIZATION BY CONFRONTATION

From the elementary measure characterizations we derive now the characterization by confrontation of the translation–invariant window elementary operators.

Let us first introduce two technical propositions.

Proposition 1.17 (extension of the union and intersection) – Let E be a nonempty set, let W be a nonempty subset of E, and let K_1 be a bounded chain. For any family $(f_i)_{i \in I}$ of elements of K_1^W,

$$\overline{\bigvee_{i \in I} f_i} = \bigvee_{i \in I} \overline{f_i} \quad \text{and} \quad \overline{\bigwedge_{i \in I} f_i} = \bigwedge_{i \in I} \overline{f_i}$$

and

$$\underline{\bigvee_{i \in I} f_i} = \bigvee_{i \in I} \underline{f_i} \quad \text{and} \quad \underline{\bigwedge_{i \in I} f_i} = \bigwedge_{i \in I} \underline{f_i}.$$

□

Proof – Let us prove the first statement. For any $x \in E$,

$$(\overline{\bigvee_{i \in I} f_i})(x) = \begin{cases} (\bigvee_{i \in I} f_i)(x) & \text{if } x \in W \\ \max K_1 & \text{otherwise} \end{cases} \qquad \text{(Expression (64))}$$

$$= \begin{cases} \bigvee_{i \in I} f_i(x) & \text{if } x \in W \\ \max K_1 & \text{otherwise} \end{cases} \qquad \text{(Expression (30))}$$

$$= \bigvee_{i \in I} \begin{cases} f_i(x) & \text{if } x \in W \\ \max K_1 & \text{otherwise} \end{cases} \qquad \text{(property of the union)}$$

$$= \bigvee_{i \in I} \overline{f}_i(x) \qquad \text{(Expression (64))}$$

$$= (\bigvee_{i \in I} \overline{f}_i)(x). \qquad \text{(Expression (30))}$$

In other words, $\overline{\bigvee_{i \in I} f_i} = \bigvee_{i \in I} \overline{f}_i$. The other statements follow in a similar way. □

Let $W \subset E$. Let's denote by $\Delta_W(K_1{}^E, K_2{}^E)$ (resp., $\mathrm{E}_W(K_1{}^E, K_2{}^E)$, $\Delta^a{}_W(K_1{}^E, K_2{}^E)$, $\mathrm{E}^a{}_W(K_1{}^E, K_2{}^E)$) the set of translation–invariant window dilations (resp., erosions, anti–dilations, anti–erosions) with window W.

Proposition 1.18 (bijection between the t.i. elementary operators and the elementary measures) – Let $(E, +)$ be an Abelian group, let W be a nonempty subset of E, and let K_1 and K_2 be two bounded chains. We have the following statements:

$\Psi \mapsto \psi_\Psi$ (Exp. (67)) from $\Delta_W(K_1{}^E, K_2{}^E)$ to $\Delta(K_1{}^W, K_2)$ is a bijection. Its inverse is $\psi \mapsto \Psi_\psi$ (Exp. (68));

$\Psi \mapsto \psi_\Psi$ (Exp. (67)) from $\mathrm{E}_W(K_1{}^E, K_2{}^E)$ to $\mathrm{E}(K_1{}^W, K_2)$ is a bijection. Its inverse is $\psi \mapsto \Psi_\psi$ (Exp. (68));

$\Psi \mapsto \psi_\Psi$ (Exp. (67)) from $\Delta^a{}_W(K_1{}^E, K_2{}^E)$ to $\Delta^a(K_1{}^W, K_2)$ is a bijection. Its inverse is $\psi \mapsto \Psi_\psi$ (Exp. (68));

$\Psi \mapsto \psi_\Psi$ (Exp. (67)) from $\mathrm{E}^a{}_W(K_1{}^E, K_2{}^E)$ to $\mathrm{E}^a(K_1{}^W, K_2)$ is a bijection. Its inverse is $\psi \mapsto \Psi_\psi$ (Exp. (68)).

□

Proof – Let us prove the first statement.

Let us prove that, for any $\Psi \in \Delta_W(K_1{}^E, K_2{}^E)$, we have $\psi_\Psi \in \Delta(K_1{}^W, K_2)$.

$$\psi_\Psi(\inf(K_1{}^W)) = \Psi(\underline{\inf(K_1{}^W)})(o) \qquad \text{(Expression (67b))}$$

$$= \Psi(\inf(K_1{}^E))(o) \qquad \text{(Expression (65))}$$

$$= (\inf(K_2{}^E))(o) \qquad (\Psi \text{ is a dilation and Proposition 1.3})$$

$$= \min K_2. \qquad \text{(Expression (28))}$$

That is, $\psi_\Psi(\inf(K_1{}^W)) = \min K_2$. For any family $(f_i)_{i \in I}$ of elements of K_1^W,

$$\psi_\Psi(\bigvee_{i \in I} f_i) = \Psi(\overline{\bigvee_{i \in I} f_i})(o) \qquad \text{(Expression (67a))}$$

$$= \Psi(\bigvee_{i \in I} \overline{f}_i)(o) \qquad \text{(Proposition 1.17)}$$

$$= (\bigvee_{i \in I} \Psi(\overline{f}_i))(o) \qquad (\Psi \text{ is a dilation})$$

$$= \bigvee_{i \in I} \Psi(\overline{f}_i)(o) \qquad \text{(Expression (30))}$$

$$= \bigvee_{i \in I} \psi_\Psi(f_i). \qquad \text{(Expression (67a))}$$

That is, $\psi_\Psi(\bigvee_{i \in I} f_i) = \bigvee_{i \in I} \psi_\Psi(f_i)$. In other words, $\psi_\Psi \in \Delta(K_1{}^W, K_2)$.

Let us prove that, for any $\psi \in \Delta(K_1{}^W, K_2)$, we have $\Psi_\psi \in \Delta_W(K_1{}^E, K_2{}^E)$. For any $y \in E$,

$$
\begin{aligned}
\Psi_\psi(\inf(K_1{}^E))(y) &= \psi((\inf(K_1{}^E) - y)/W) && \text{(Expression (68))}\\
&= \psi((\inf(K_1{}^E))/W) && (\inf(K_2{}^E) \text{ is translation–invariant})\\
&= \psi(\inf(K_1{}^W)) && \text{(Expression (63))}\\
&= \min K_2. && (\psi \text{ is a dilation and Proposition 1.3})
\end{aligned}
$$

That is, $\Psi_\psi(\inf(K_1{}^E)) = \inf(K_2{}^E)$. For any family $(f_i)_{i \in I}$ of elements of K_1^E and $y \in E$,

$$
\begin{aligned}
\Psi_\psi(\bigvee_{i \in I} f_i)(y) &= \psi(((\bigvee_{i \in I} f_i) - y)/W) && \text{(Expression (68))}\\
&= \psi((\bigvee_{i \in I} (f_i - y))/W) && \text{(property of the translate)}\\
&= \psi(\bigvee_{i \in I} (f_i - y)/W) && \text{(Expression (30) twice)}\\
&= \bigvee_{i \in I} \psi((f_i - y)/W) && (\psi \text{ is a dilation})\\
&= \bigvee_{i \in I} \Psi_\psi(f_i)(y). && \text{(Expression (68))}
\end{aligned}
$$

That is, $\Psi_\psi(\bigvee_{i \in I} f_i) = \bigvee_{i \in I} \Psi_\psi(f_i)$. In other words, $\Psi_\psi \in \Delta_W(K_1{}^E, K_2{}^E)$.

The three other statements follow from the first one by duality. □

We can now introduce the so–called characterization by confrontation. In this characterization, the t.i. window elementary operators (with window W) from $K_1{}^E$ to $K_2{}^E$ will depend on elementary operators from K_2 to $K_1{}^W$ that we will call *structuring elements*.

We define the following four pairs of useful expressions.

For any dilation $\varDelta \in \Delta_W(K_1{}^E, K_2{}^E)$ and any erosion $e \in \mathrm{E}(K_2, K_1{}^W)$, let

$$e_\varDelta(t) \triangleq \sup\{f \in K_1{}^W : \varDelta(\underline{f})(o) \le t\} \quad (t \in K_2) \tag{69}$$

$$\boxed{{}_e\varDelta(f)(y) \triangleq \min\{t \in K_2 : (f - y)/W \le e(t)\} \quad (f \in K_1{}^E, y \in E),} \tag{70}$$

$e_\varDelta$ is called the *structuring element of the dilation* $\varDelta$, and ${}_e\varDelta(f)$ is called the *dilation of* f *by the structuring element* e.

For any erosion $E \in \mathrm{E}_W(K_1{}^E, K_2{}^E)$ and any dilation $d \in \mathrm{D}(K_2, K_1{}^W)$, let

$${}_Ed(t) \triangleq \inf\{f \in K_1{}^W : t \le E(\bar{f})(o)\} \quad (t \in K_2) \tag{71}$$

$$\boxed{E_d(f)(y) \triangleq \max\{t \in K_2 : d(t) \le (f - y)/W\} \quad (f \in K_1{}^E, y \in E),} \tag{72}$$

${}_Ed$ is called the *structuring element of the erosion* E, and $E_d(f)$ is called the *erosion of* f *by the structuring element* d.

For any anti–dilation $\delta^a \in \Delta^a{}_W(K_1{}^E, K_2{}^E)$ and any anti–dilation $d^a \in \mathrm{D}^a(K_2, K_1{}^W)$, let

$$_{\Delta^a}d^a(t) \triangleq \sup\{f \in K_1{}^W : \; t \leq \Delta^a(\underline{f})(o)\} \quad (t \in K_2) \tag{73}$$

$$\boxed{_{d^a}\Delta^a(f)(y) \triangleq \max\{t \in K_2 : \; (f - y)/W \leq d^a(t)\} \quad (f \in K_1{}^E, y \in E),} \tag{74}$$

$_{\Delta^a}d^a$ is called the *structuring element of the anti–dilation* Δ^a, and $_{d^a}\Delta^a(f)$ is called the *anti–dilation of* f *by the structuring element* d^a.

For any anti–erosion $\epsilon^a \in \mathrm{E}^a{}_W(K_1{}^E, K_2{}^E)$ and any anti–erosion $e^a \in \mathrm{E}^a(K_2, K_1{}^W)$, let

$$e^a{}_{E^a}(t) \triangleq \inf\{f \in K_1{}^W : \; E^a(\bar{f})(o) \leq t\} \quad (t \in K_2) \tag{75}$$

$$\boxed{E^a{}_{e^a}(f)(y) \triangleq \min\{t \in K_2 : \; e^a(t) \leq (f - y)/W\} \quad (f \in K_1{}^E, y \in E),} \tag{76}$$

$e^a{}_{E^a}$ is called the *structuring element of the anti–erosion* E^a, and $E^a{}_{e^a}(f)$ is called the *anti–erosion of* f *by the structuring element* e^a.

In the next proposition, we state a result relative to the class of dilations, similar results could be stated for the other classes of elementary operators.

Proposition 1.19 (composition properties) – Let $(E, +)$ be an Abelian group, let W be a nonempty subset of E, and let K_1 and K_2 be two bounded chains. We have the following statements:

$\Delta \mapsto e_\Delta$ (Exp. (69)) is the composition of $\Psi \mapsto \psi_\Psi$ (Exp. (67)) with $\delta \mapsto e_\delta$ (Exp. (34))

$e \mapsto {}_e\Delta$ (Exp. (70)) is the composition of $e \mapsto {}_e\delta$ (Exp. (35)) with $\psi \mapsto \Psi_\psi$ (Exp. (68)).

□

Proof – Let us prove that the mapping $\Delta \mapsto e_\Delta$ (Exp. (69)) from $\Delta_W(K_1{}^E, K_2{}^E)$ to $\mathrm{E}(K_2, K_1{}^W)$ is the composition of the mapping $\Psi \mapsto \psi_\Psi$ (Exp. (67)) from $\Delta_W(K_1{}^E, K_2{}^E)$ to $\Delta(K_1{}^W, K_2)$, with the mapping $\delta \mapsto e_\delta$ (Exp. (34)) from $\Delta(K_1{}^W, K_2)$ to $\mathrm{E}(K_2, K_1{}^W)$. For any $\Delta \in \Delta_W(K_1{}^E, K_2{}^E)$ and $t \in K_2$,

$$e_{\psi_\Delta}(t) = \sup\{f \in K_1{}^W : \; \psi_\Delta(f) \leq t\} \qquad \text{(Expression (34))}$$

$$= \sup\{f \in K_1{}^W : \; \Delta(\underline{f})(o) \leq t\} \qquad \text{(Expression (67b))}$$

$$= e_\Delta(t). \qquad \text{(Expression (69))}$$

Let us prove that the mapping $e \mapsto {}_e\Delta$ (Exp. (70)) from $\mathrm{E}(K_2, K_1{}^W)$ to $\Delta_W(K_1{}^E, K_2{}^E)$ is the composition of the mapping $e \mapsto {}_e\delta$ (Exp. (35)) from $\mathrm{E}(K_2, K_1{}^W)$ to $\Delta(K_1{}^W, K_2)$, with the mapping $\psi \mapsto \Psi_\psi$ (Exp. (68)) from $\Delta(K_1{}^W, K_2)$ to $\Delta_W(K_1{}^E, K_2{}^E)$. For any $e \in \mathrm{E}(K_2, K_1{}^W)$, $f \in K_1{}^E$ and $y \in E$,

$$\Psi_{{}_e\delta}(f)(y) = {}_e\delta((f - y)/W) \qquad \text{(Expression (68))}$$

$$= \min\{t \in K_2 : \; (f - y)/W \leq e(t)\} \qquad \text{(Expression (35))}$$

$$= {}_e\Delta(f)(y). \qquad \text{(Expression (70))}$$

□

In Proposition 7.2, Barrera (25) gives the caracterization by confrontation for the erosion in the case of lattices having sup–generating families. Here we give a similar result for all the elementary operators between gray–level images.

Proposition 1.20 (characterization by confrontation of the t.i. elementary operators) – Let $(E, +)$ be an Abelian group, let W be a nonempty subset of E, and let K_1 and K_2 be two bounded chains. We have the following statements:

$\mathit{\Delta} \mapsto e_{\mathit{\Delta}}$ (Exp. (69)) from $\Delta_W({K_1}^E, {K_2}^E)$ to $\mathrm{E}(K_2, {K_1}^W)$ is a bijection,
its inverse is $e \mapsto {}_e\mathit{\Delta}$ (Exp. (70));

$E \mapsto {}_E d$ (Exp. (71)) from $\mathrm{E}_W({K_1}^E, {K_2}^E)$ to $\mathrm{D}(K_2, {K_1}^W)$ is a bijection,
its inverse is $d \mapsto E_d$ (Exp. (72));

$\mathit{\Delta}^{\mathrm{a}} \mapsto {}_{\mathit{\Delta}^{\mathrm{a}}}d^{\mathrm{a}}$ (Exp. (73)) from $\Delta^{\mathrm{a}}{}_W({K_1}^E, {K_2}^E)$ to $\mathrm{D}^{\mathrm{a}}(K_2, {K_1}^W)$ is a bijection,
its inverse is $d^{\mathrm{a}} \mapsto {}_{d^{\mathrm{a}}}\mathit{\Delta}^{\mathrm{a}}$ (Exp. (74));

$E^{\mathrm{a}} \mapsto e^{\mathrm{a}}{}_{E^{\mathrm{a}}}$ (Exp. (75)) from $\mathrm{E}^{\mathrm{a}}{}_W({K_1}^E, {K_2}^E)$ to $\mathrm{E}^{\mathrm{a}}(K_2, {K_1}^W)$ is a bijection,
its inverse is $e^{\mathrm{a}} \mapsto E^{\mathrm{a}}{}_{e^{\mathrm{a}}}$ (Exp. (76)). □

Proof – Let us prove the first statement. By Proposition 1.19, $\mathit{\Delta} \mapsto e_{\mathit{\Delta}}$ (Exp. (69)) is the composition of $\mathit{\Psi} \mapsto \psi_{\mathit{\Psi}}$ (Exp. (67)) with $\delta \mapsto e_\delta$ (Exp. (34)). By Propositions 1.18 and 1.7, these two mappings are bijections, therefore their composition $\mathit{\Delta} \mapsto e_{\mathit{\Delta}}$ is also a bijection. Furthermore, the mapping $e \mapsto {}_e\mathit{\Delta}$ (Exp. (70)) is its inverse since it is, by Proposition 1.19, the composition of $e \mapsto {}_e\delta$ (Exp. (35)) with $\psi \mapsto \mathit{\Psi}_\psi$ (Exp. (68)) which are, by Propositions 1.7 and 1.18, the inverses of the above two bijections. The three other statements follow from the first one by duality. □

Figure 9 shows an elementary operator E_d which is a t.i. window erosion (with window W) from ${K_1}^E$ to ${K_2}^E$, where $E = [-5, 5] \subset \mathbf{Z}$ (the set of integers), $W = [-2, 2] \subset E$, $K_1 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $K_2 = \{0, 1, 2, 3\}$ and E is equipped with the addition $+$ modulus 11. In its characterization by confrontation, this erosion depends on the dilation d from K_2 to ${K_1}^W$ given in Figure 4. In Figure 9, a particular function f is shown; the erosion of f through E_d at point 3 of E is 1. This value is obtained through Expression (72). Figure 9 shows that we have to *confront* $f' \triangleq (f - 3)/W$ with each function in the family defined by d in order to find the greatest function smaller or equal to f'.

3.3 – CHARACTERIZATION BY SELECTION

Let W be a subset of the Abelian group $(E, +)$, we denote by W^{t} the *transpose of* W, that is, the subset of E given by

$$W^{\mathrm{t}} \triangleq \{y \in E : -y \in W\}. \tag{77}$$

We now introduce our second way to characterize an elementary operator, the so–called characterization by selection. In this characterization, the t.i. window elementary operators (with window W) from ${K_1}^E$ to ${K_2}^E$ will depend on elementary operators from K_1 to ${K_2}^{W^{\mathrm{t}}}$ that we will call *impulsive responses*.

Let p be a mapping from K_1 to ${K_2}^W$, we define the *transpose of* p as the mapping p^{t} from K_1 to ${K_2}^{W^{\mathrm{t}}}$ given by

$$p^{\mathrm{t}}(s)(y) \triangleq p(s)(-y) \quad (s \in K_1,\ y \in W^{\mathrm{t}}). \tag{78}$$

In the same way, let p be a mapping from K_1 to ${K_2}^{W^{\mathrm{t}}}$, we define the *transpose of* p as the mapping p^{t} from K_1 to ${K_2}^W$ given by

$$p^{\mathrm{t}}(s)(x) \triangleq p(s)(-x) \quad (s \in K_1,\ x \in W). \tag{79}$$

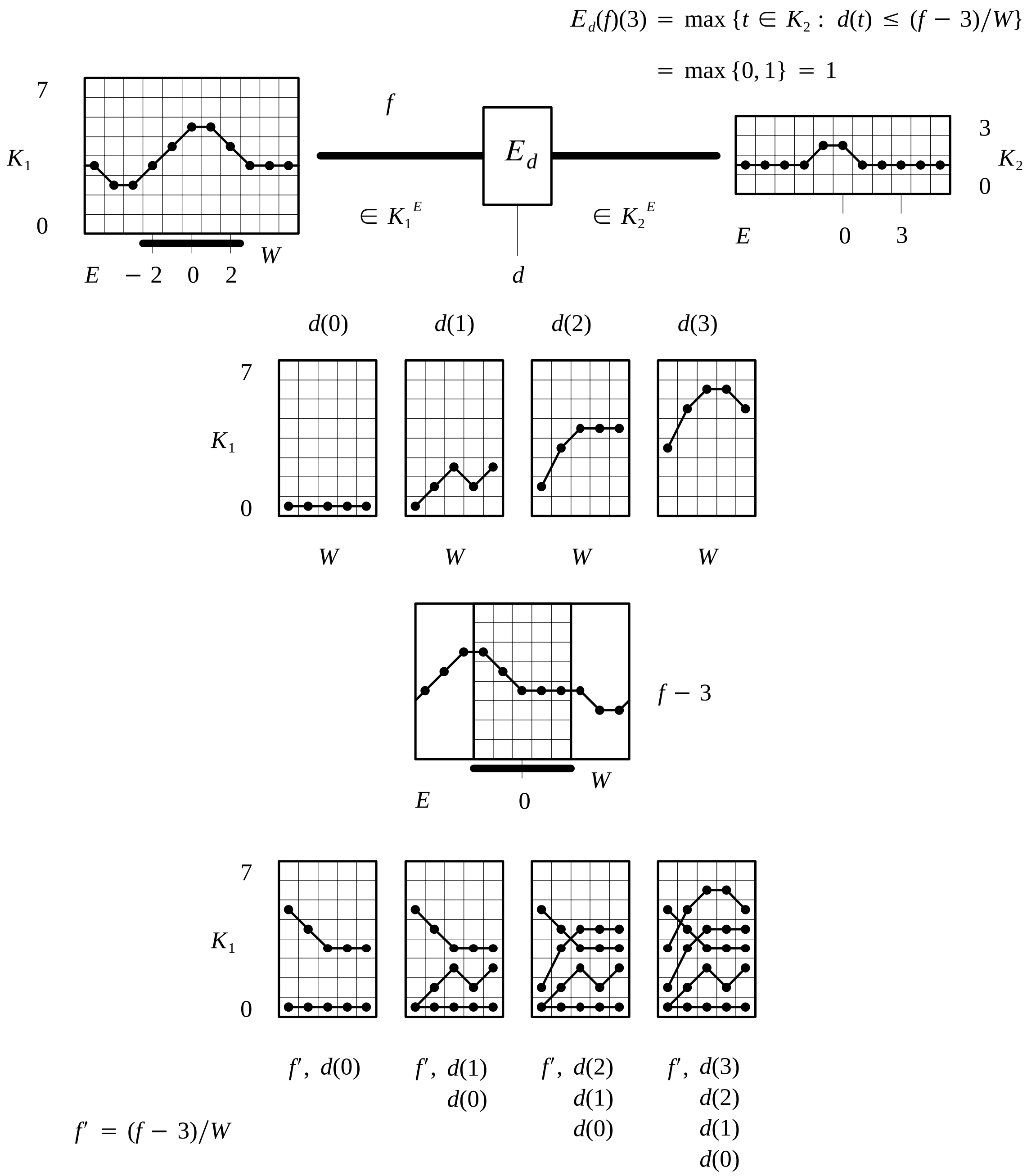

Fig. 9 – Erosion characterization by confrontation.

Proposition 1.21 (property of the transposition) – Let $(E, +)$ be an Abelian group, let W be a nonempty subset of E, and let K_1 and K_2 be two bounded chains. The mapping $p \mapsto p^t$ (Expression (78)) from the set of mappings from K_1 to $K_2{}^W$, to the set of mappings from K_1 to $K_2{}^{W^t}$ is a bijection. Its inverse is $p \mapsto p^t$ (Expression (79)).

□

Proof – For any $W \subset E$, for any mapping p from K_1 to $K_2{}^W$, $s \in K_1$ and $x \in W$,

$$\begin{aligned}(p^{\mathrm{t}})^{\mathrm{t}}(s)(x) &= p^{\mathrm{t}}(s)(-x) && \text{(Expression (79))}\\ &= p(s)(-(-x)) && \text{(Expression (78), with } y = -x)\\ &= p(s)(x). && \text{(property of } +)\end{aligned}$$

That is, $(p^{\mathrm{t}})^{\mathrm{t}} = p$. For any mapping p from K_1 to $K_2{}^{W\mathrm{t}}$, $s \in K_1$ and $y \in W^{\mathrm{t}}$,

$$\begin{aligned}(p^{\mathrm{t}})^{\mathrm{t}}(s)(y) &= p^{\mathrm{t}}(s)(-y) && \text{(Expression (78))}\\ &= p(s)(-(-y)) && \text{(Expression (79) with } x = -y)\\ &= p(s)(y). && \text{(property of } +)\end{aligned}$$

That is, $(p^{\mathrm{t}})^{\mathrm{t}} = p$. In other words $p \mapsto p^{\mathrm{t}}$ is a bijection. □

Proposition 1.22 (bijection between the elementary operators and their transposes) – Let $(E, +)$ be an Abelian group, let W be a nonempty subset of E, and let K_1 and K_2 be two bounded chains. We have the following statements:

$p \mapsto p^{\mathrm{t}}$ (Exp. (78)) from $\mathrm{D}(K_1, K_2{}^W)$ to $\mathrm{D}(K_1, K_2{}^{W\mathrm{t}})$ is a bijection. Its inverse is $p \mapsto p^{\mathrm{t}}$ (Exp. (79));

$p \mapsto p^{\mathrm{t}}$ (Exp. (78)) from $\mathrm{E}(K_1, K_2{}^W)$ to $\mathrm{E}(K_1, K_2{}^{W\mathrm{t}})$ is a bijection. Its inverse is $p \mapsto p^{\mathrm{t}}$ (Exp. (79));

$p \mapsto p^{\mathrm{t}}$ (Exp. (78)) from $\mathrm{D}^{\mathrm{a}}(K_1, K_2{}^W)$ to $\mathrm{D}^{\mathrm{a}}(K_1, K_2{}^{W\mathrm{t}})$ is a bijection. Its inverse is $p \mapsto p^{\mathrm{t}}$ (Exp. (79));

$p \mapsto p^{\mathrm{t}}$ (Exp. (78)) from $\mathrm{E}^{\mathrm{a}}(K_1, K_2{}^W)$ to $\mathrm{E}^{\mathrm{a}}(K_1, K_2{}^{W\mathrm{t}})$ is a bijection. Its inverse is $p \mapsto p^{\mathrm{t}}$ (Exp. (79)).
□

Proof – Let us prove the first statement.

Let us prove that, for any $p \in \mathrm{D}(K_1, K_2{}^W)$, we have $p^{\mathrm{t}} \in \mathrm{D}(K_1, K_2{}^{W\mathrm{t}})$. For any $y \in W^{\mathrm{t}}$,

$$\begin{aligned}p^{\mathrm{t}}(\min K_1)(y) &= p(\min K_1)(-y) && \text{(Expression (78))}\\ &= \inf(K_2{}^W)(-y) && (p \text{ is a dilation and Proposition 1.3})\\ &= \inf(K_2{}^{W\mathrm{t}})(y). && \text{(Expression (77))}\end{aligned}$$

That is, $p^{\mathrm{t}}(\min K_1) = \inf(K_2{}^{W\mathrm{t}})$. For any family $(s_i)_{i\in I}$ of elements of K_1 and $y \in W^{\mathrm{t}}$,

$$\begin{aligned}p^{\mathrm{t}}\Big(\bigvee_{i \in I} s_i\Big)(y) &= p\Big(\bigvee_{i \in I} s_i\Big)(-y) && \text{(Expression (78))}\\ &= \Big(\bigvee_{i \in I} p(s_i)\Big)(-y) && (p \text{ is a dilation})\\ &= \bigvee_{i \in I} p(s_i)(-y) && \text{(Expression (30))}\\ &= \bigvee_{i \in I} p^{\mathrm{t}}(s_i)(y) && \text{(Expression (78))}\\ &= \Big(\bigvee_{i \in I} p^{\mathrm{t}}(s_i)\Big)(y). && \text{(Expression (30))}\end{aligned}$$

That is, $p^{\mathrm{t}}(\bigvee_{i \in I} s_i) = \bigvee_{i \in I} p^{\mathrm{t}}(s_i)$. In other words, $p^{\mathrm{t}} \in \mathrm{D}(K_1, K_2^{W^{\mathrm{t}}})$. In the same way we could prove that for any $p \in \mathrm{D}(K_1, K_2^{W^{\mathrm{t}}})$, we have $p^{\mathrm{t}} \in \mathrm{D}(K_1, K_2^{W})$.

The three other statements follow from the first one by duality. □

We define the following four pairs of useful expressions.

For any dilation $\Delta \in \Delta_W(K_1{}^E, K_2{}^E)$ and any dilation $d \in \mathrm{D}(K_1, K_2^{W^{\mathrm{t}}})$, let

$$d_{\Delta}(s) \triangleq \Delta(f_{o,s})/W^{\mathrm{t}} \quad (s \in K_1) \tag{80}$$

$$\boxed{\Delta_d(f)(y) \triangleq \bigvee_{u \in W} d^{\mathrm{t}}((f - y)(u))(u) \quad (f \in K_1{}^W, y \in E),} \tag{81}$$

d_{Δ} is called the *impulse response of the dilation* Δ, and $\Delta_d(f)$ is called the *dilation of* f *by the dilation which impulse response is* d, or, in short, the *dilation of* f *w.r.t.* d.

For any erosion $E \in \mathrm{E}_W(K_1{}^E, K_2{}^E)$ and any erosion $e \in \mathrm{E}(K_1, K_2^{W^{\mathrm{t}}})$, let

$$_{E}e(s) \triangleq E(f^{o,s})/W^{\mathrm{t}} \quad (s \in K_1) \tag{82}$$

$$\boxed{_{e}E(f)(y) \triangleq \bigwedge_{u \in W} e^{\mathrm{t}}((f - y)(u))(u) \quad (f \in K_1{}^W, y \in E),} \tag{83}$$

$_{E}e$ is called the *impulse response of the erosion* E, and $_{e}E(f)$ is called the *erosion of* f *by the erosion which impulse response is* e, or, in short, the *erosion of* f *w.r.t.* e.

For any anti–dilation $\delta^{\mathrm{a}} \in \Delta^{\mathrm{a}}{}_W(K_1{}^E, K_2{}^E)$ and any anti–dilation $d^{\mathrm{a}} \in \mathrm{D}^{\mathrm{a}}(K_1, K_2^{W^{\mathrm{t}}})$, let

$$d^{\mathrm{a}}{}_{\Delta^{\mathrm{a}}}(s) \triangleq \Delta^{\mathrm{a}}(f_{o,s})/W^{\mathrm{t}} \quad (s \in K_1) \tag{84}$$

$$\boxed{\Delta^{\mathrm{a}}{}_{d^{\mathrm{a}}}(f)(y) \triangleq \bigwedge_{u \in W} d^{\mathrm{at}}((f - y)(u))(u) \quad (f \in K_1{}^W, y \in E),} \tag{85}$$

$d^{\mathrm{a}}{}_{\Delta^{\mathrm{a}}}$ is called the *impulse response of the anti–dilation* Δ^{a}, and $\Delta^{\mathrm{a}}{}_{d^{\mathrm{a}}}(f)$ is called the *anti–dilation of* f *by the anti–dilation which impulse response is* d^{a}, or, in short, the *anti–dilation of* f *w.r.t.* d^{a}.

For any anti–erosion $\epsilon^{\mathrm{a}} \in \mathrm{E}^{\mathrm{a}}{}_W(K_1{}^E, K_2{}^E)$ and any anti–erosion $e^{\mathrm{a}} \in \mathrm{E}^{\mathrm{a}}(K_1, K_2^{W^{\mathrm{t}}})$, let

$$_{E^{\mathrm{a}}}e^{\mathrm{a}}(s) \triangleq E^{\mathrm{a}}(f^{o,s})/W^{\mathrm{t}} \quad (s \in K_1) \tag{86}$$

$$\boxed{_{e^{\mathrm{a}}}E^{\mathrm{a}}(f)(y) \triangleq \bigvee_{u \in W} e^{\mathrm{at}}((f - y)(u))(u) \quad (f \in K_1{}^W, y \in E),} \tag{87}$$

$_{E^{\mathrm{a}}}e^{\mathrm{a}}$ is called the *impulse response of the anti–erosion* E^{a}, and $_{e^{\mathrm{a}}}E^{\mathrm{a}}(f)$ is called the *anti–erosion of* f *by the anti–erosion which impulse response is* e^{a}, or, in short, the *anti–erosion of* f *w.r.t.* e^{a}.

Instead of the expression “impulse response” we can use *point spread function* or *blur*.

In the next proposition, we state a result relative to the class of dilations, similar results could be stated for the other classes of elementary operators.

Proposition 1.23 (composition properties) – Let $(E, +)$ be an Abelian group, let W be a nonempty subset of E, and let K_1 and K_2 be two bounded chains. We have the following statements:

$\Delta \mapsto d_{\Delta}$ (Exp. (80)) is the composition of $\Psi \mapsto \psi_{\Psi}$ (Exp. (67)) with $\delta \mapsto d_{\delta}$ (Exp. (44)) and with $p \mapsto p^{\mathrm{t}}$ (Exp. (78)),

$d \mapsto \Delta_d$ (Exp. (81)) is the composition of $p \mapsto p^{\mathrm{t}}$ (Exp. (79)) with $d \mapsto \delta_d$ (Exp. (45)) and with $\psi \mapsto \Psi_{\psi}$ (Exp. (68)). □

Proof – Let us prove that the mapping $\mathit{\Delta} \mapsto d_{\mathit{\Delta}}$ (Exp. (80)) from $\Delta_W({K_1}^E, {K_2}^E)$ to $\mathrm{D}(K_1, {K_2}^{W^{\mathrm{t}}})$ is the composition of the mapping $\Psi \mapsto \psi_\Psi$ (Exp. (67)) from $\Delta_W({K_1}^E, {K_2}^E)$ to $\Delta({K_1}^W, K_2)$, with the mapping $\delta \mapsto d_\delta$ (Exp. (44)) from $\Delta({K_1}^W, K_2)$ to $\mathrm{D}(K_1, {K_2}^W)$, and with the mapping $p \mapsto p^{\mathrm{t}}$ (Exp. (78)) from $\mathrm{D}(K_1, {K_2}^W)$ to $\mathrm{D}(K_1, {K_2}^{W^{\mathrm{t}}})$. For any $\mathit{\Delta} \in \Delta_W({K_1}^E, {K_2}^E)$, $s \in K_1$ and $y \in W^{\mathrm{t}}$,

$$
\begin{aligned}
{d_{\delta_{\mathit{\Delta}}}}^{\mathrm{t}}(s)(y) &= d_{\delta_{\mathit{\Delta}}}(s)(-y) && \text{(Expression (78))}\\
&= \delta_{\mathit{\Delta}}(f_{-y,s}) && \text{(Expression (44))}\\
&= \mathit{\Delta}(\underline{f_{-y,s}})(o) && \text{(Expression (67b))}\\
&= \mathit{\Delta}(f_{-y,s})(o) && \text{(Exp. (65) and (42) with } f_{-y,s} \text{ defined on } E)\\
&= \mathit{\Delta}(f_{o,s} - y)(o) && \text{(Expressions (43) and (62))}\\
&= \mathit{\Delta}(f_{o,s})(y) && (\mathit{\Delta} \text{ is a t.i.})\\
&= (\mathit{\Delta}(f_{o,s})/W^{\mathrm{t}})(y) && \text{(Expression (63))}\\
&= d_{\mathit{\Delta}}(s)(y). && \text{(Expression (80))}
\end{aligned}
$$

Let us prove that $d \mapsto \mathit{\Delta}_d$ (Exp. (81)) from $\mathrm{D}(K_1, {K_2}^{W^{\mathrm{t}}})$ to $\Delta_W({K_1}^E, {K_2}^E)$ is the composition of the mapping $p \mapsto p^{\mathrm{t}}$ (Exp. (79)) from $\mathrm{D}(K_1, {K_2}^{W^{\mathrm{t}}})$ to $\mathrm{D}(K_1, {K_2}^W)$, with the mapping $d \mapsto \delta_d$ (Exp. (45)) from $\mathrm{D}(K_1, {K_2}^W)$ to $\Delta({K_1}^W, K_2)$, and with the mapping $\psi \mapsto \Psi_\psi$ (Exp. (68)) from $\Delta({K_1}^W, K_2)$ to $\Delta_W({K_1}^E, {K_2}^E)$. For any $d \in \mathrm{D}(K_1, {K_2}^{W^{\mathrm{t}}})$, $f \in {K_1}^E$ and $y \in E$,

$$
\begin{aligned}
\mathit{\Delta}_{\delta_{d^{\mathrm{t}}}}(f)(y) &= \delta_{d^{\mathrm{t}}}((f-y)/W) && \text{(Expression (68))}\\
&= \bigvee_{u \in W} d^{\mathrm{t}}(((f-y)/W)(u))(u) && \text{(Expression (45))}\\
&= \bigvee_{u \in W} d^{\mathrm{t}}((f-y)(u))(u) && \text{(Expression (63))}\\
&= \mathit{\Delta}_d(f)(y). && \text{(Expression (81))}
\end{aligned}
$$

□

Proposition 1.24 (characterization by selection of the t.i. elementary operators) – Let $(E, +)$ be an Abelian group, let W be a nonempty subset of E, and let K_1 and K_2 be two bounded chains. We have the following statements:

$\mathit{\Delta} \mapsto d_{\mathit{\Delta}}$ (Exp. (80)) from $\Delta_W({K_1}^E, {K_2}^E)$ to $\mathrm{D}(K_1, {K_2}^{W^{\mathrm{t}}})$ is a bijection,
its inverse is $d \mapsto \mathit{\Delta}_d$ (Exp. (81));

$E \mapsto {}_E e$ (Exp. (82)) from $\mathrm{E}_W({K_1}^E, {K_2}^E)$ to $\mathrm{E}(K_1, {K_2}^{W^{\mathrm{t}}})$ is a bijection,
its inverse is $e \mapsto {}_e E$ (Exp. (83));

$\mathit{\Delta}^{\mathrm{a}} \mapsto d^{\mathrm{a}}_{\mathit{\Delta}^{\mathrm{a}}}$ (Exp. (84)) from $\Delta^{\mathrm{a}}_W({K_1}^E, {K_2}^E)$ to $\mathrm{D}^{\mathrm{a}}(K_1, {K_2}^{W^{\mathrm{t}}})$ is a bijection,
its inverse is $d^{\mathrm{a}} \mapsto \mathit{\Delta}^{\mathrm{a}}_{d^{\mathrm{a}}}$ (Exp. (85));

$E^{\mathrm{a}} \mapsto {}_{E^{\mathrm{a}}} e^{\mathrm{a}}$ (Exp. (86)) from $\mathrm{E}^{\mathrm{a}}_W({K_1}^E, {K_2}^E)$ to $\mathrm{E}^{\mathrm{a}}(K_1, {K_2}^{W^{\mathrm{t}}})$ is a bijection,
its inverse is $e^{\mathrm{a}} \mapsto {}_{e^{\mathrm{a}}} E^{\mathrm{a}}$ (Exp. (87)).

□

Proof – Let us prove the first statement. By Proposition 1.24, $\varDelta \mapsto d_{\varDelta}$ (Exp. (80)) is the composition of $\Psi \mapsto \psi_{\Psi}$ (Exp. (67)) with $\delta \mapsto d_{\delta}$ (Exp. (44)) and with $p \mapsto p^{\mathrm{t}}$ (Exp. (78)). By Propositions 1.18, 1.10 and 1.22, these three mappings are bijections, therefore their composition $\varDelta \mapsto d_{\varDelta}$ is also a bijection. Furthermore, the mapping $d \mapsto \varDelta_d$ (Exp. (81)) is its inverse since it is, by Proposition 1.24, the composition of $p \mapsto p^{\mathrm{t}}$ (Exp. (79)) with $d \mapsto \delta_d$ (Exp. (45)) and with $\psi \mapsto \Psi_{\psi}$ (Exp. (68)) which are, by Propositions 1.22, 1.10 and 1.18, the inverses of the above three bijections. The three other statements follow from the first one by duality. □

We observe that he impulse responses of the elementary t.i. operators are the distributions functions of the elementary measures that characterize them.

Figure 10 shows an elementary operator ${}_eE$ which is a t.i. window erosion (with window W) from ${K_1}^E$ to ${K_2}^E$, where $E = [-5, 5] \subset \mathbf{Z}$ (the set of integers), $W = [-2, 2] \subset E$, $K_1 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $K_2 = \{0, 1, 2, 3\}$ and E is equipped with the addition $+$ modulus 11. In its characterization by selection, this erosion depends on the erosion e from K_1 to ${K_2}^{W^{\mathrm{t}}}$ which is the transpose of the erosion given in Figure 5. In Figure 10, a particular function f is shown; the erosion of f through ${}_eE$ at point 3 of E is 1. This value is obtained through Expression (83). Figure 10 shows that when u runs over W, we have to *select*, according to the value $(f - 3)(u)$, the appropriate function in the family defined by e^{t}. Actually, the erosion e has been chosen in such a way that the resulting operator ${}_eE$ is identical to the operator E_d shown in Figure 9 (see Proposition 1.26 below).

We now give equivalent expressions for the characterization by selection of the elementary operators. These expressions are useful for a visual construction of the operator output.

Proposition 1.25 (equivalent expressions for the characterization by selection) – Let $(E, +)$ be an Abelian group, let W be a nonempty subset of E, and let K_1 and K_2 be two bounded chains. We have the following statements:

for any $d \in \mathrm{D}(K_1, {K_2}^{W^{\mathrm{t}}})$,

$$\varDelta_d(f) = \bigvee_{u \in E} (\underline{d(f(u))} + u) \quad (f \in {K_1}^W);$$

for any $e \in \mathrm{E}(K_1, {K_2}^{W^{\mathrm{t}}})$,

$${}_eE(f) = \bigwedge_{u \in E} (\overline{e(f(u))} + u) \quad (f \in {K_1}^W);$$

for any $d^{\mathrm{a}} \in \mathrm{D}^{\mathrm{a}}(K_1, {K_2}^{W^{\mathrm{t}}})$,

$$\varDelta^{\mathrm{a}}{}_{d^{\mathrm{a}}}(f) = \bigwedge_{u \in E} (\overline{d^{\mathrm{a}}(f(u))} + u) \quad (f \in {K_1}^W);$$

for any $e^{\mathrm{a}} \in \mathrm{E}^{\mathrm{a}}(K_1, {K_2}^{W^{\mathrm{t}}})$,

$${}_{e^{\mathrm{a}}}E^{\mathrm{a}}(f) = \bigvee_{u \in E} (\underline{e^{\mathrm{a}}(f(u))} + u) \quad (f \in {K_1}^W).$$ □

Proof – Let us prove the first statement. For any $d \in \mathrm{D}(K_1, {K_2}^{W^{\mathrm{t}}})$, any $f \in {K_1}^W$ and any $y \in E$,

$$\varDelta_d(f)(y) = \bigvee_{u \in W} d^{\mathrm{t}}((f - y)(u))(u) \qquad \text{(Expression (81))}$$

$$= \bigvee_{u \in W} d((f - y)(u))(-u) \qquad \text{(Expression (79))}$$

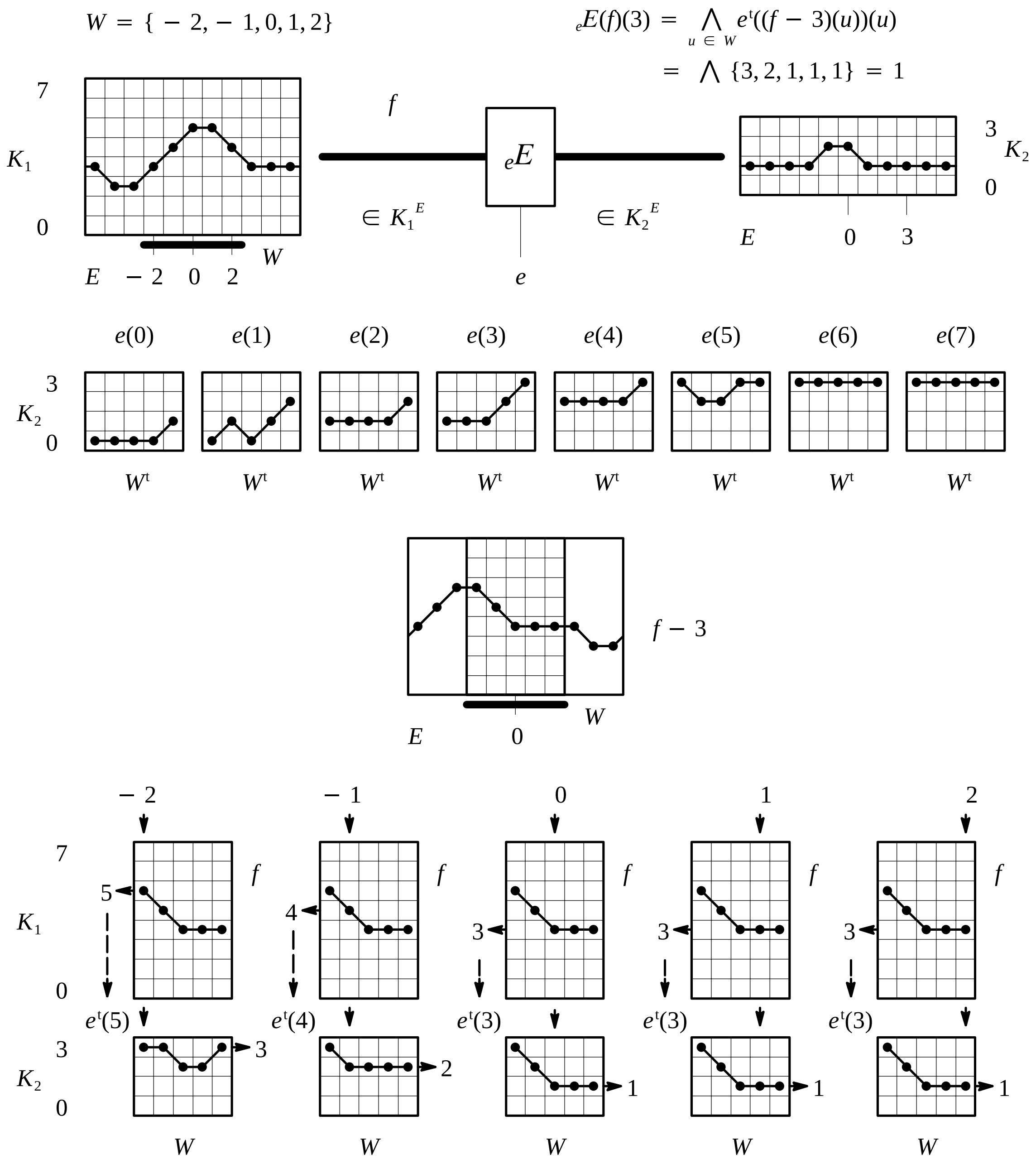

Fig. 10 – Erosion characterization by selection.

$$= \bigvee_{u \in W} d(f(u+y))(-u) \qquad \text{(Expression (62))}$$

$$= \bigvee_{v \in W^t} d(f(y-v))(v) \qquad \text{(Exp. (77), } v = -u \text{ and } u = -v\text{)}$$

$$= \bigvee_{v \in W^t} \underline{d(f(y-v))}(v) \qquad \text{(Proposition 1.14)}$$

$$= \bigvee_{v \in E} \underline{d(f(y - v))}(v) \qquad \text{(Exp. (65) and } \max K_2 \text{ is the null element for the union)}$$

$$= \bigvee_{u \in E} \underline{d(f(u))}(y - u) \qquad (u = y - v \text{ and } v = y - u)$$

$$= \bigvee_{u \in E} (\underline{d(f(u))} + u)(y). \qquad \text{(Expression (62))}$$

That is, for any $f \in K_1{}^W$, $\varDelta_d(f) = \bigvee_{u \in E} (\underline{d(f(u))} + u)$.

The other statements follow from the first one by duality. □

Figure 11 shows all the bijections previously defined with respect to the characterization of translation–invariant window dilations. This figure helps to establish the next proposition.

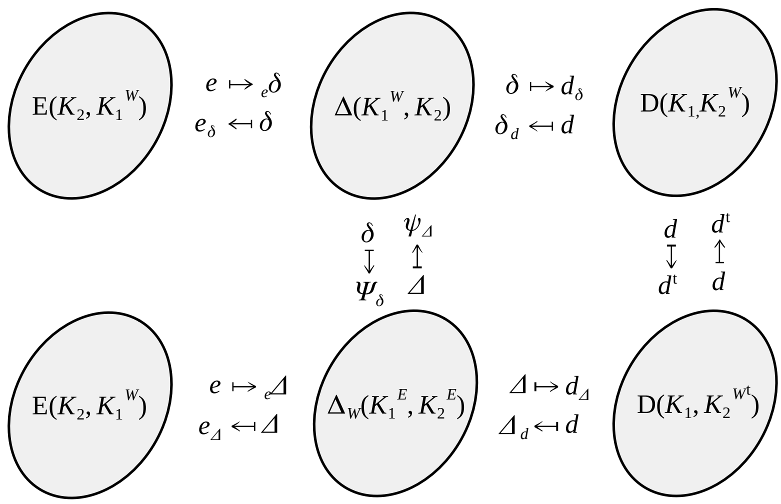

Fig. 11 – Bijections involved in dilation characterization.

In order to define the relationships between the characterization by confrontation and by selection, we define the following four pairs of useful expressions.

Let W_1 and W_2 be two subsets of E mutually transposed (i.e., $W_2 = W_1{}^{\mathrm{t}}$ or equivalently $W_1 = W_2{}^{\mathrm{t}}$), and let $P(x, y)$ be the property: $x \in W_1$ and $y \in W_2$ such that $x + y = o$ (the null element of $+$).

For any erosion $e \in \mathrm{E}(K_2, K_1{}^{W_1})$ and any dilation $d \in \mathrm{D}(K_1, K_2{}^{W_2})$, let

$$\underline{e}(s)(y) \triangleq \min\{t \in K_2 : \; s \le e(t)(x)\} \quad (s \in K_1,\; P(x, y)) \tag{88}$$

$$\overline{d}(t)(x) \triangleq \max\{s \in K_1 : \; d(s)(y) \le t\} \quad (t \in K_2, P(x, y)). \tag{89}$$

For any dilation $d \in \mathrm{D}(K_2, K_1{}^{W_1})$ and any erosion $e \in \mathrm{E}(K_1, K_2{}^{W_2})$, let

$$\overline{d}(s)(y) \triangleq \max\{t \in K_2 : \; d(t)(x) \le s\} \quad (s \in K_1,\; P(x, y)) \tag{90}$$

$$\underline{e}(t)(x) \triangleq \min\{s \in K_1 : \; t \le e(s)(y)\} \quad (t \in K_2, P(x, y)). \tag{91}$$

For any anti–dilation $d^{\mathrm{a}} \in \mathrm{D}^{\mathrm{a}}(K_2, K_1^{W_1})$ and any anti–dilation $d^{\mathrm{a}} \in \mathrm{D}^{\mathrm{a}}(K_1, K_2^{W_2})$, let respectively

$$\overline{d^{\mathrm{a}}}(s)(y) \triangleq \max\{t \in K_2 : \ s \leq d^{\mathrm{a}}(t)(x)\} \quad (s \in K_1, \ P(x, y)) \tag{92}$$

$$\overline{d^{\mathrm{a}}}(t)(x) \triangleq \max\{s \in K_1 : \ t \leq d^{\mathrm{a}}(s)(y)\} \quad (t \in K_2, P(x, y)). \tag{93}$$

For any anti–erosion $e^{\mathrm{a}} \in \mathrm{E}^{\mathrm{a}}(K_2, K_1^{W_1})$ and any anti–erosion $e^{\mathrm{a}} \in \mathrm{E}^{\mathrm{a}}(K_1, K_2^{W_2})$, let respectively

$$\underline{e^{\mathrm{a}}}(s)(y) \triangleq \min\{t \in K_2 : \ e^{\mathrm{a}}(t)(x) \leq s\} \quad (s \in K_1, \ P(x, y)) \tag{94}$$

$$\underline{e^{\mathrm{a}}}(t)(x) \triangleq \min\{s \in K_1 : \ e^{\mathrm{a}}(s)(y) \leq t\} \quad (t \in K_2, \ P(x, y)). \tag{95}$$

Proposition 1.26 (relationships between the characterization by confrontation and by selection) – Let $(E, +)$ be an Abelian group, let W be a nonempty subset of E, and let K_1 and K_2 be two bounded chains. We have the following statements:

if $e \in \mathrm{E}(K_2, K_1^{W_1})$ then ${}_{e}\mathit{\Delta}$ (Exp. (70)) is identical to $\mathit{\Delta}_{\underline{e}}$ (Exp. (81)), i.e., ${}_{e}\mathit{\Delta} = \mathit{\Delta}_{\underline{e}}$

if $d \in \mathrm{D}(K_1, K_2^{W_2})$ then $\mathit{\Delta}_d$ (Exp. (81)) is identical to ${}_{\overline{d}}\mathit{\Delta}$ (Exp. (70)), i.e., $\mathit{\Delta}_d = {}_{\overline{d}}\mathit{\Delta}$

if $d \in \mathrm{D}(K_2, K_1^{W_1})$ then E_d (Exp. (72)) is identical to ${}_{\overline{d}}E$ (Exp. (83)), i.e., $E_d = {}_{\overline{d}}E$

if $e \in \mathrm{E}(K_1, K_2^{W_2})$ then ${}_{e}E$ (Exp. (83)) is identical to $E_{\underline{e}}$ (Exp. (72)), i.e., ${}_{e}E = E_{\underline{e}}$

if $d^{\mathrm{a}} \in \mathrm{D}^{\mathrm{a}}(K_2, K_1^{W_1})$ then ${}_{d^{\mathrm{a}}}\mathit{\Delta}^{\mathrm{a}}$ (Exp. (74)) is identical to $\mathit{\Delta}^{\mathrm{a}}_{\overline{d^{\mathrm{a}}}}$ (Exp. (85)), i.e., ${}_{d^{\mathrm{a}}}\mathit{\Delta}^{\mathrm{a}} = \mathit{\Delta}^{\mathrm{a}}_{\overline{d^{\mathrm{a}}}}$

if $d^{\mathrm{a}} \in \mathrm{D}^{\mathrm{a}}(K_1, K_2^{W_2})$ then $\mathit{\Delta}^{\mathrm{a}}_{d^{\mathrm{a}}}$ (Exp. (85)) is identical to ${}_{\overline{d^{\mathrm{a}}}}\mathit{\Delta}^{\mathrm{a}}$ (Exp. (74)), i.e., $\mathit{\Delta}^{\mathrm{a}}_{d^{\mathrm{a}}} = {}_{\overline{d^{\mathrm{a}}}}\mathit{\Delta}^{\mathrm{a}}$

if $e^{\mathrm{a}} \in \mathrm{E}^{\mathrm{a}}(K_2, K_1^{W_1})$ then $E^{\mathrm{a}}_{e^{\mathrm{a}}}$ (Exp. (76)) is identical to ${}_{\underline{e^{\mathrm{a}}}}E^{\mathrm{a}}$ (Exp. (87)), i.e., $E^{\mathrm{a}}_{e^{\mathrm{a}}} = {}_{\underline{e^{\mathrm{a}}}}E^{\mathrm{a}}$

if $e^{\mathrm{a}} \in \mathrm{E}^{\mathrm{a}}(K_1, K_2^{W_2})$ then ${}_{e^{\mathrm{a}}}E^{\mathrm{a}}$ (Exp. (87)) is identical to $E^{\mathrm{a}}_{\underline{e^{\mathrm{a}}}}$ (Exp. (76)), i.e., ${}_{e^{\mathrm{a}}}E^{\mathrm{a}} = E^{\mathrm{a}}_{\underline{e^{\mathrm{a}}}}$.

□

Proof – Let us prove the first statement. For any erosion $e \in \mathrm{E}(K_2, K_1^{W})$ and any dilation $d \in \mathrm{D}(K_1, K_2^{W^{\mathrm{t}}})$,

$$
\begin{aligned}
{}_{e}\mathit{\Delta} = \mathit{\Delta}_d &\Leftrightarrow \Psi_{{}_{e}\delta} = \Psi_{\delta_{d^{\mathrm{t}}}} && \text{(Propositions 1.19 and 1.23)}\\
&\Leftrightarrow {}_{e}\delta = \delta_{d^{\mathrm{t}}} && \text{(Proposition 1.18)}\\
&\Leftrightarrow d^{\mathrm{t}} = d_{{}_{e}\delta} && \text{(Proposition 1.10)}\\
&\Leftrightarrow d = d_{{}_{e}\delta}{}^{\mathrm{t}} && \text{(Proposition 1.20)}\\
&\Leftrightarrow d(s)(y) = d_{{}_{e}\delta}{}^{\mathrm{t}}(s)(y) \ (s \in K_1, \ y \in W_2) && \text{(equality between mappings)}\\
&\Leftrightarrow d(s)(y) = d_{{}_{e}\delta}(s)(x) \ (s \in K_1, \ P(x, y)) && \text{(Expression (78))}\\
&\Leftrightarrow d(s)(y) = {}_{e}\delta(f_{x,s}) \ (s \in K_1, \ P(x, y)) && \text{(Expression (44))}\\
&\Leftrightarrow d(s)(y) = \min\{t \in K_2 : \ f_{x,s} \leq e(t)\} \ (s \in K_1, \ P(x, y)) && \text{(Expression (35))}
\end{aligned}
$$

$\Leftrightarrow\ d(s)(y) = \min\{t \in K_2 : f_{x,s}(u) \leq e(t)(u)\ (u \in W)\}\ (s \in K_1,\ P(x,y))$ (Exp. (27))

$\Leftrightarrow\ d(s)(y) = \min\{t \in K_2 : s \leq e(t)(x)\}\ (s \in K_1,\ P(x,y))$ (Expression (42))

$\Leftrightarrow\ d(s)(y) = \underline{e}(s)(y)\ (s \in K_1,\ y \in W_2)$ (Expression (88))

$\Leftrightarrow\ d = \underline{e}.$ (equality between mappings)

Let us prove the second statement. For any dilation $d \in \mathrm{D}(K_1, K_2{}^{W_2})$ and any erosion $e \in \mathrm{E}(K_2, K_1{}^{W_1})$,

$\varDelta_d = {}_e\varDelta\ \Leftrightarrow\ \varPsi_{\delta_{d^{\mathrm{t}}}} = \varPsi_{e\delta}$ (Propositions 1.19 and 1.23)

$\Leftrightarrow\ \delta_{d^{\mathrm{t}}} = {}_e\delta$ (Proposition 1.18)

$\Leftrightarrow\ e = e_{\delta_{d^{\mathrm{t}}}}$ (Proposition 1.7)

$\Leftrightarrow\ e(t) = e_{\delta_{d^{\mathrm{t}}}}(t)\ (t \in K_2)$ (equality between mappings)

$\Leftrightarrow\ e(t) = \sup\{f \in K_1{}^{W_1} : \delta_{d^{\mathrm{t}}}(f) \leq t\}\ (t \in K_2)$ (Expression (34))

$\Leftrightarrow\ e(t) = \sup\{f \in K_1{}^{W_1} : \bigvee_{u \in W_1} d^{\mathrm{t}}(f(u))(u) \leq t\}\ (t \in K_2)$ (Expression (45))

$\Leftrightarrow\ e(t) = \sup\{f \in K_1{}^{W_1} : d^{\mathrm{t}}(f(u))(u) \leq t\ (u \in W_1)\}\ (t \in K_2)$ (property of the union)

$\Leftrightarrow\ e(t)(x) = \max\{s \in K_1 : \exists f \in K_1{}^{W_1},\ d^{\mathrm{t}}(f(u))(u) \leq t\ (u \in W_1)$ and $s = f(x)\}$
$(t \in K_2,\ x \in W_1)$
(Expression (28))

$\Leftrightarrow\ e(t)(x) = \max\{s \in K_1 : d^{\mathrm{t}}(s)(x) \leq t\}\ (t \in K_2,\ x \in W_1)$ (one can chose f to be $f_{x,s}$)

$\Leftrightarrow\ e(t)(x) = \max\{s \in K_1 : d(s)(y) \leq t\}\ (t \in K_2,\ P(x,y))$ (Expression (79))

$\Leftrightarrow\ e(t)(x) = \overline{d}(t)(x)\ (t \in K_2,\ x \in W_1)$
(Expression (89))

$\Leftrightarrow\ e = \overline{d}.$
(equality between mappings)

The other three pairs of statement follow from the first pair by duality. □

We say that:

$e \in \mathrm{E}(K_2, K_1{}^{W_1})$ and $d \in \mathrm{D}(K_1, K_2{}^{W_2})$ are *companions* iff
$s \leq e(t)(u)\ \Leftrightarrow\ t \geq d(s)(v)\ ((s,t) \in K_1 \times K_2, P(u,v))$;

$d \in \mathrm{D}(K_2, K_1{}^{W_1})$ and $e \in \mathrm{E}(K_1, K_2{}^{W_2})$ are *companions* iff
$s \geq d(t)(u)\ \Leftrightarrow\ t \leq e(s)(v)\ ((s,t) \in K_1 \times K_2, P(u,v))$;

$d^{\mathrm{a}} \in \mathrm{D}^{\mathrm{a}}(K_2, K_1{}^{W_1})$ and $d^{\mathrm{a}} \in \mathrm{D}^{\mathrm{a}}(K_1, K_2{}^{W_2})$ are *companions* iff
$s \leq d^{\mathrm{a}}(t)(u) \Leftrightarrow\ t \leq d^{\mathrm{a}}(s)(v)\ ((s,t) \in K_1 \times K_2, P(u,v))$;

$e^{\mathrm{a}} \in \mathrm{E}^{\mathrm{a}}(K_2, K_1{}^{W_1})$ and $e^{\mathrm{a}} \in \mathrm{E}^{\mathrm{a}}(K_1, K_2{}^{W_2})$ are *companions* iff
$s \geq e^{\mathrm{a}}(t)(u) \Leftrightarrow t \geq e^{\mathrm{a}}(s)(v)$ $((s, t) \in K_1 \times K_2, P(u, v))$.

Equivalently (see Proposition 5.4 of Banon & Barrera [7] for a similar case), we say that:

$e \in \mathrm{E}(K_2, K_1{}^{W_1})$ and $d \in \mathrm{D}(K_1, K_2{}^{W_2})$ are *companions* iff
$\underline{e} = d$ (Exp. (88)) (or $\overline{d} = e$ (Exp. (89)));

$d \in \mathrm{D}(K_2, K_1{}^{W_1})$ and $e \in \mathrm{E}(K_1, K_2{}^{W_2})$ are *companions* iff
$\overline{d} = e$ (Exp. (90)) (or $\underline{e} = d$ (Exp. (91)));

$d^{\mathrm{a}} \in \mathrm{D}^{\mathrm{a}}(K_2, K_1{}^{W_1})$ and $d^{\mathrm{a}} \in \mathrm{D}^{\mathrm{a}}(K_1, K_2{}^{W_2})$ are *companions*
iff $\overline{d^{\mathrm{a}}} = d^{\mathrm{a}}$ (Exp. (92)) (or $\overline{d^{\mathrm{a}}} = d^{\mathrm{a}}$ (Exp. (93)));

$e^{\mathrm{a}} \in \mathrm{E}^{\mathrm{a}}(K_2, K_1{}^{W_1})$ and $e^{\mathrm{a}} \in \mathrm{E}^{\mathrm{a}}(K_1, K_2{}^{W_2})$ are *companions* iff
$\underline{e^{\mathrm{a}}} = e^{\mathrm{a}}$ (Exp. (94)) (or $\underline{e^{\mathrm{a}}} = e^{\mathrm{a}}$ (Exp. (95))).

The dilation d of Figure 9 (with $W = W_1$) and the erosion e of Figure 10 (with $W^{\mathrm{t}} = W_2$) are examples of companion mappings. For this reason, by Proposition 1.26, the erosions E_d in Figure 9 and ${}_eE$ in Figure 10 are identical.

We now give sufficient conditions on the structuring elements and the impulse responses in order to a pair of t.i. elementary operators be a Galois connection (G.c.) [17].

Proposition 1.27 (sufficient condition for Galois connections) – Let $(E, +)$ be an Abelian group, let W_1 and W_2 be two subsets of E, mutually transposed, and let K_1 and K_2 be two bounded chains. We have the following statements:

if $e \in \mathrm{E}(K_2, K_1{}^{W_1})$ and $d \in \mathrm{D}(K_1, K_2{}^{W_2})$ are companions then
$(\Delta_d, {}_eE)$ is a G.c. between $(K_1{}^E, \leq)$ and $(K_2{}^E, \geq)$;

if $d \in \mathrm{D}(K_2, K_1{}^{W_1})$ and $e \in \mathrm{E}(K_1, K_2{}^{W_2})$ are companions then
$({}_eE, \Delta_d)$ is a G.c. between $(K_1{}^E, \geq)$ and $(K_2{}^E, \leq)$;

if $d^{\mathrm{a}} \in \mathrm{D}^{\mathrm{a}}(K_2, K_1{}^{W_1})$ and $d^{\mathrm{a}} \in \mathrm{D}^{\mathrm{a}}(K_1, K_2{}^{W_2})$ are companions then
$(\Delta_{d^{\mathrm{a}}}, {}_{d^{\mathrm{a}}}\Delta)$ is a G.c. between $(K_1{}^E, \leq)$ and $(K_2{}^E, \leq)$;

if $e^{\mathrm{a}} \in \mathrm{E}^{\mathrm{a}}(K_2, K_1{}^{W_1})$ and $e^{\mathrm{a}} \in \mathrm{E}^{\mathrm{a}}(K_1, K_2{}^{W_2})$ are companions then
$({}_{e^{\mathrm{a}}}E, E_{e^{\mathrm{a}}})$ is a G.c. between $(K_1{}^E, \geq)$ and $(K_2{}^E, \geq)$.

□

Proof – Let us prove the first statement. For any $f \in K_1{}^E$ and $g \in K_2{}^E$,

$$f \leq {}_eE(g) \Leftrightarrow f(x) \leq {}_eE(g)(x) \; (x \in E) \qquad \text{(Expression (27))}$$

$$\Leftrightarrow f(x) \leq \bigwedge_{v \in W_2} e^{\mathrm{t}}((g - x)(v))(v) \; (x \in E) \qquad \text{(Expression (83))}$$

$$\Leftrightarrow f(x) \leq \bigwedge_{v \in W_2} e^{\mathrm{t}}((g)(v + x))(v) \; (x \in E) \qquad \text{(Expression (62))}$$

$$\Leftrightarrow (f(x) \leq e^{\mathrm{t}}((g)(v + x))(v) \; (v \in W_2)) \; (x \in E)$$
(property of the intersection)

$$\Leftrightarrow\ (f(x) \le e((g)(v+x))(u)\ (P(u,v)))\ (x \in E) \qquad \text{(Expression (78))}$$

$$\Leftrightarrow\ (g(y) \ge d((f)(u+y))(v)\ (P(u,v)))\ (y \in E)$$
$$\text{(hypothesis, } y = v + x \text{ and } x = u + y\text{)}$$

$$\Leftrightarrow\ (g(y) \ge d^{\mathrm{t}}((f)(u+y))(u)\ (u \in W_1))\ (x \in E) \qquad \text{(Expression (79))}$$

$$\Leftrightarrow\ g(y) \ge \bigvee_{u \in W_1} d^{\mathrm{t}}((f)(u+y))(u)\ (y \in E) \qquad \text{(property of the union)}$$

$$\Leftrightarrow\ g(y) \ge \bigvee_{u \in W_1} d^{\mathrm{t}}((f-y)(u))(u)\ (y \in E) \qquad \text{(Expression (62))}$$

$$\Leftrightarrow\ g(y) \ge \varDelta_d(f)(y)\ (y \in E) \qquad \text{(Expression (81))}$$

$$\Leftrightarrow\ g \ge \varDelta_d(f). \qquad \text{(Expression (27))}$$

That is, $(\varDelta_d, {}_{e}E)$ is a G.c. between $(K_1{}^E, \le)$ and $(K_2{}^E, \ge)$.

The other statements follow from the first one by duality. □

The next proposition shows how to easily get the different types of Galois connections.

Proposition 1.28 (Galois connections construction) – Let $(E, +)$ be an Abelian group, let W_1 and W_2 be two subsets of E, mutually transposed, and let K_1 and K_2 be two bounded chains. We have the following statements:

for any $e \in \mathrm{E}(K_2, K_1{}^{W_1})$, $(\varDelta_{\underline{e}}, {}_{e}E)$, $({}_{e}\varDelta, {}_{e}E)$ and $({}_{e}\varDelta, E_{\underline{e}})$ are G.c. between $(K_1{}^E, \le)$ and $(K_2{}^E, \ge)$;

for any $d \in \mathrm{D}(K_1, K_2{}^{W_2})$, $(\varDelta_d, {}_{\overline{d}}E)$, $(\varDelta_d, E_d)$ and $({}_{\overline{d}}\varDelta, E_d)$ are G.c. between $(K_1{}^E, \le)$ and $(K_2{}^E, \ge)$;

for any $d \in \mathrm{D}(K_2, K_1{}^{W_1})$, $({}_{\overline{d}}E, \varDelta_d)$, $(E_d, \varDelta_d)$ and $(E_d, {}_{\overline{d}}\varDelta)$ are G.c. between $(K_1{}^E, \ge)$ and $(K_2{}^E, \le)$;

for any $e \in \mathrm{E}(K_1, K_2{}^{W_2})$, $({}_{e}E, \varDelta_{\underline{e}})$, $({}_{e}E, {}_{e}\varDelta)$ and $(E_{\underline{e}}, {}_{e}\varDelta)$ are G.c. between $(K_1{}^E, \ge)$ and $(K_2{}^E, \le)$;

for any $d^{\mathrm{a}} \in \mathrm{D}^{\mathrm{a}}(K_2, K_1{}^{W_1})$, $(\varDelta_{\overline{d}^{\mathrm{a}}}, \varDelta_{d^{\mathrm{a}}})$, $({}_{d^{\mathrm{a}}}\varDelta, \varDelta_{d^{\mathrm{a}}})$ and $({}_{d^{\mathrm{a}}}\varDelta, {}_{\overline{d}^{\mathrm{a}}}\varDelta)$ are G.c. between $(K_1{}^E, \le)$ and $(K_2{}^E, \le)$;

for any $d^{\mathrm{a}} \in \mathrm{D}^{\mathrm{a}}(K_1, K_2{}^{W_2})$, $(\varDelta_{d^{\mathrm{a}}}, \varDelta_{\overline{d}^{\mathrm{a}}})$, $(\varDelta_{d^{\mathrm{a}}}, {}_{d^{\mathrm{a}}}\varDelta)$ and $({}_{\overline{d}^{\mathrm{a}}}\varDelta, {}_{d^{\mathrm{a}}}\varDelta)$ are G.c. between $(K_1{}^E, \le)$ and $(K_2{}^E, \le)$;

for any $e^{\mathrm{a}} \in \mathrm{E}^{\mathrm{a}}(K_2, K_1{}^{W_1})$, $({}_{\underline{e}^{\mathrm{a}}}E, {}_{e^{\mathrm{a}}}E)$, $(E_{e^{\mathrm{a}}}, {}_{e^{\mathrm{a}}}E)$ and $(E_{e^{\mathrm{a}}}, E_{\underline{e}^{\mathrm{a}}})$ are G.c. between $(K_1{}^E, \ge)$ and $(K_2{}^E, \ge)$;

for any $e^{\mathrm{a}} \in \mathrm{E}^{\mathrm{a}}(K_1, K_2{}^{W_2})$, $({}_{e^{\mathrm{a}}}E, {}_{e^{\mathrm{a}}}E)$, $({}_{e^{\mathrm{a}}}E, E_{e^{\mathrm{a}}})$ and $(E_{\underline{e}^{\mathrm{a}}}, E_{e^{\mathrm{a}}})$ are G.c. between $(K_1{}^E, \ge)$ and $(K_2{}^E, \ge)$.

□

Proof – Let us prove the first two statements. For any $e \in \mathrm{E}(K_2, K_1^{W_1})$ and $d \in \mathrm{D}(K_1, K_2^{W_2})$,

$$d = \underline{e} \Leftrightarrow e \text{ and } d \text{ are companions} \qquad \text{(equivalent companion definition)}$$

$$\Rightarrow (\mathit{\Delta}_d, {}_eE) \text{ is a G.c. between } (K_1^E, \leq) \text{ and } (K_2^E, \geq). \qquad \text{(Proposition 1.27)}$$

That is, $(\mathit{\Delta}_{\underline{e}}, {}_eE)$ is a G.c. between $(K_1^E, \leq)$ and $(K_2^E, \geq)$. By Proposition 1.26, $({}_e\mathit{\Delta}, {}_eE)$ and $({}_e\mathit{\Delta}, E_{\underline{e}})$ are also G.c. between $(K_1^E, \leq)$ and $(K_2^E, \geq)$. For any $e \in \mathrm{E}(K_2, K_1^{W_1})$ and $d \in \mathrm{D}(K_1, K_2^{W_2})$,

$$e = \overline{d} \Leftrightarrow e \text{ and } d \text{ are companions} \qquad \text{(equivalent companion definition)}$$

$$\Rightarrow (\mathit{\Delta}_d, {}_eE) \text{ is a G.c. between } (K_1^E, \leq) \text{ and } (K_2^E, \geq). \qquad \text{(Proposition 1.27)}$$

That is, $(\mathit{\Delta}_d, {}_{\overline{d}}E)$ is a G.c. between $(K_1^E, \leq)$ and $(K_2^E, \geq)$. By Proposition 1.26, $(\mathit{\Delta}_d, E_d)$ and $({}_{\overline{d}}\mathit{\Delta}, E_d)$ are also G.c. between $(K_1^E, \leq)$ and $(K_2^E, \geq)$.

The other statements follow from the first two by duality. □

3.4 – CHARACTERIZATION BY DECOMPOSITION

Let us introduce our third way to characterize an elementary operator, the so–called characterization by decomposition. In this characterization, the t.i. window elementary operators (with window W) from K_1^E to K_2^E will depend on a family of elementary operators from K_1 to K_2 that we call elementary transformation tables or *Elementary Look Up Tables* (ELUTs). The characterization will be derived from the characterization by selection.

We define the following four pairs of useful expressions. In these expressions $\circ$ stands for the mapping composition.

For any dilation $\mathit{\Delta} \in \Delta_W(K_1^E, K_2^E)$ and any family of dilations $(d_v) \in \mathrm{D}(K_1, K_2)^{W^t}$, let

$$(d_{\mathit{\Delta}})_v(s) \triangleq \mathit{\Delta}(f_{o,s})(v) \quad (s \in K_1, v \in W^t) \tag{96}$$

$$\boxed{\mathit{\Delta}_{(d)}(f) \triangleq \bigvee_{v \in W^t} d_v \circ (f + v) \quad (f \in K_1^E),} \tag{97}$$

$((d_{\mathit{\Delta}})_v)$ is called the *family of ELUTs of the dilation* $\mathit{\Delta}$, and $\mathit{\Delta}_{(d)}(f)$ is called the *dilation of* f *w.r.t. the family of ELUTs* (d_v).

For any erosion $E \in \mathrm{E}_W(K_1^E, K_2^E)$ and any family of erosions $(e_v) \in \mathrm{E}(K_1, K_2)^{W^t}$, let

$$({}_Ee)_v(s) \triangleq E(f^{o,s})(v) \quad (s \in K_1, v \in W^t) \tag{98}$$

$$\boxed{{}_{(e)}E(f) \triangleq \bigwedge_{v \in W^t} e_v \circ (f + v) \quad (f \in K_1^E),} \tag{99}$$

$(({}_Ee)_v)$ is called the *family of ELUTs of the erosion* E, and ${}_{(e)}E(f)$ is called the *erosion of* f *w.r.t. the family of ELUTs* (e_v).

For any anti–dilation $\mathit{\Delta}^a \in \Delta^a_W(K_1^E, K_2^E)$ and any family of anti–dilations $(d^a_v) \in \mathrm{D}^a(K_1, K_2)^{W^t}$, let

$$(d^{a}_{\varDelta^{a}})_{v}(s) \triangleq \varDelta^{a}(f_{o,s})(v) \quad (s \in K_1, v \in W^{t}) \tag{100}$$

$$\boxed{\varDelta^{a}_{(d^{a})}(f) \triangleq \bigwedge_{v \in W^{t}} d^{a}_{v} \circ (f + v) \quad (f \in K_1^{E}),} \tag{101}$$

$((d^{a}_{\varDelta^{a}})_{v})$ is called the *family of ELUTs of the anti–dilation* $\varDelta^{a}$, and $\varDelta^{a}_{(d^{a})}(f)$ is called the *anti–dilation of* f *w.r.t. the family of ELUTs* (d^{a}_{v}).

For any anti–erosion $E^{a} \in \mathrm{E}^{a}_{W}(K_1^{E}, K_2^{E})$ and any family of anti–erosions $(e^{a}_{v}) \in \mathrm{E}^{a}(K_1, K_2)^{Wt}$, let

$$({}_{E^{a}}e^{a})_{v}(s) \triangleq E^{a}(f^{o,s})(v) \quad (s \in K_1, v \in W^{t}) \tag{102}$$

$$\boxed{{}_{(e^{a})}E^{a}(f) \triangleq \bigvee_{v \in W^{t}} e^{a}_{v} \circ (f + v) \quad (f \in K_1^{E}),} \tag{103}$$

$(({}_{E^{a}}e^{a})_{v})$ is called the *family of ELUTs of the anti–erosion* E^{a}, and ${}_{(e^{a})}E^{a}(f)$ is called the *anti–dilation of* f *w.r.t. the family of ELUTs* (e^{a}_{v}).

In the next proposition, we state a result relative to the class of dilations, similar results could be stated for the other classes of elementary operators.

Proposition 1.29 (composition properties) – Let $(E, +)$ be an Abelian group, let W be a nonempty subset of E, and let K_1 and K_2 be two bounded chains. We have the following statements:

$\varDelta \mapsto ((d_{\varDelta})_{v})$ (Exp. (96)) is the composition of $\varDelta \mapsto d_{\varDelta}$ (Exp. (80)) with $p \mapsto (p_{v})$ (Exp. (52))

$(d_{v}) \mapsto \varDelta_{(d)}$ (Exp. (97)) is the composition of $(p_{v}) \mapsto p$ (Exp. (53)) with $d \mapsto \varDelta_{d}$ (Exp. (81)).

□

Proof – Let us prove that the mapping $\varDelta \mapsto ((d_{\varDelta})_{v})$ (Exp. (96)) from $\Delta_{W}(K_1^{E}, K_2^{E})$ to $\mathrm{D}(K_1, K_2)^{Wt}$ is the composition of the mapping $\varDelta \mapsto d_{\varDelta}$ (Exp. (80)) from $\Delta_{W}(K_1^{E}, K_2^{E})$ to $\mathrm{D}(K_1, K_2^{W^{t}})$, with the mapping $p \mapsto (p_{v})$ (Exp. (52)) from $\mathrm{D}(K_1, K_2^{W^{t}})$ to $\mathrm{D}(K_1, K_2)^{Wt}$. For any $\varDelta \in \Delta_{W}(K_1^{E}, K_2^{E})$, $v \in W^{t}$ and $s \in K_1$,

$$
\begin{aligned}
(d_{\varDelta})_{v}(s) &= d_{\varDelta}(s)(v) && \text{(Expression (52))}\\
&= (\varDelta(f_{o,s})/W^{t})(v) && \text{(Expression (80))}\\
&= \varDelta(f_{o,s})(v) && \text{(Expression (63))}\\
&= (d_{\varDelta})_{v}(s). && \text{(Expression (96))}
\end{aligned}
$$

Let us prove that the mapping $(d_{v}) \mapsto \varDelta_{(d)}$ (Exp. (97)) from $\mathrm{D}(K_1, K_2)^{Wt}$ to $\Delta_{W}(K_1^{E}, K_2^{E})$ is the composition of $(p_{v}) \mapsto p$ (Exp. (53)) from $\mathrm{D}(K_1, K_2)^{Wt}$ to $\mathrm{D}(K_1, K_2^{W^{t}})$, with $d \mapsto \varDelta_{d}$ (Exp. (81)) from $\mathrm{D}(K_1, K_2^{W^{t}})$ to $\Delta_{W}(K_1^{E}, K_2^{E})$. For any $(d_{v}) \in \mathrm{D}(K_1, K_2)^{Wt}$, $f \in K_1^{W}$ and $y \in E$, let d be the mapping given by Expression (53), then we have

$$
\begin{aligned}
\varDelta_{d}(f)(y) &= \bigvee_{u \in W} d^{t}((f - y)(u))(u) && \text{(Expression (81))}\\
&= \bigvee_{u \in W} d((f - y)(u))(-u) && \text{(Expression (79))}
\end{aligned}
$$

$$= \bigvee_{u \in W} d((f - u)(x))(-u) \qquad \text{(Expression (62) and } x = y\text{)}$$

$$= \bigvee_{v \in W^t} d((f + v)(x))(v) \qquad \text{(Exp. (77), } v = -u \text{ and } u = -v\text{)}$$

$$= \bigvee_{v \in W^t} d_v((f + v)(x)) \qquad \text{(Expression (53))}$$

$$= \bigvee_{v \in W^t} (d_v \circ (f + v))(y) \qquad \text{(definition of the composition and } y = x\text{)}$$

$$= \varDelta_{(d)}(f)(y). \qquad \text{(Expression (97))}$$

□

Proposition 1.30 (characterization by decomposition of the t.i. elementary operators) – Let $(E, +)$ be an Abelian group, let W be a nonempty subset of E, and let K_1 and K_2 be two bounded chains. We have the following statements:

$\varDelta \mapsto ((d_\varDelta)_v)$ (Exp. (96)) from $\Delta_W(K_1{}^E, K_2{}^E)$ to $\mathrm{D}(K_1, K_2)^{W^t}$ is a bijection, its inverse is $(d_v) \mapsto \varDelta_{(d)}$ (Exp. (97));

$E \mapsto ((_E e)_v)$ (Exp. (98)) from $\mathrm{E}_W(K_1{}^E, K_2{}^E)$ to $\mathrm{E}(K_1, K_2)^{W^t}$ is a bijection, its inverse is $(e_v) \mapsto {}_{(e)}E$ (Exp. (99));

$\varDelta^a \mapsto ((d^a{}_{\varDelta^a})_v)$ (Exp. (100)) from $\Delta^a{}_W(K_1{}^E, K_2{}^E)$ to $\mathrm{D}^a(K_1, K_2)^{W^t}$ is a bijection, its inverse is $(d^a{}_v) \mapsto \varDelta^a{}_{(d^a)}$ (Exp. (101));

$E^a \mapsto ((_{E^a} e^a)_v)$ (Exp. (102)) from $\mathrm{E}^a{}_W(K_1{}^E, K_2{}^E)$ to $\mathrm{E}^a(K_1, K_2)^{W^t}$ is a bijection, its inverse is $(e^a{}_v) \mapsto {}_{(e^a)}E^a$ (Exp. (103)).

□

Proof – Let us prove the first statement. By Proposition 1.29, $\varDelta \mapsto ((d_\varDelta)_v)$ (Exp. (96)) is the composition of $\varDelta \mapsto d_\varDelta$ (Exp. (80)) with $p \mapsto (p_v)$ (Exp. (52)). By Propositions 1.24 and 1.11 these two mappings are bijections, therefore their composition $\varDelta \mapsto ((d_\varDelta)_v)$ is also a bijection. Furthermore, the mapping $(d_v) \mapsto \varDelta_{(d)}$ (Exp. (97)) is its inverse since it is, by Proposition 1.29, the composition of $(p_v) \mapsto p$ (Exp. (53)) with $d \mapsto \varDelta_d$ (Exp. (81)) which are by Propositions 1.11 and 1.24, the inverses of the above two bijections. The three other statements follow from the first one by duality. □

From the above characterization we can state the following two propositions.

Proposition 1.31 (number of t.i. elementary operators) – Let $(E, +)$ be an Abelian group, let W be a nonempty subset of E, let K_1 and K_2 be two bounded chains, and let $k_1 = (\#K_1) - 1$, $k_2 = (\#K_2) - 1$ and $w = \#W$. For any $k_1 \geq 1$ and $k_2 \geq 1$, the number of operators in a given class of t.i. window elementary operators from K_1 to K_2 with window W is $\left(\dfrac{(k_1 + k_2)!}{k_1! k_2!}\right)^w$. □

Proof – The result derives from Propositions 1.6 and 1.30. □

Figure 12 shows an example of number of t.i. elementary operators (e.g. erosions) with $k_1 = k_2 = 255$ and $w = 9$.

Proposition 1.32 (construction/decomposition of elementary operators) – Let $(E, +)$ be an Abelian group, let W be a nonempty subset of E, and let K_1 and K_2 be two bounded chains. Let $(\varDelta_v)_{v \in W^t}$, $(E_v)_{v \in W^t}$, $(\varDelta^a{}_v)_{v \in W^t}$ and $(E^a{}_v)_{v \in W^t}$ be the families of operators from $K_1{}^E$ to $K_2{}^E$ given by, respectively,

```
(510!/(255! 255!))^9
4562037540935177602017762245962398091159262900078411894941681 20\
  5373097509950657481522055492860742601795118933504459086883897\
  4953454359295434971703706939840852758135000304591578537624069\
  5596056096755517925868475417853293056422385049325569327243096\
  7992225071008877537075007279134119715796008381107349853562979\
  4428324672596730527199299223348538707090891066046391785020488\
  2570625900486964609586849760094937419518708945620648079148128\
  0289218048085815329276549831593895640784026233676946482040894\
  6314699869570934583552362817346187821682287541733602877866164\
  7728570340604772382153061679348625114197325694552113070543334\
  7036757177219439953885069491571275217759314583669854727275884\
  1315293404314521764590929646651232269189613102132683016295657\
  5228621781573348022491013037903287409623958125948392499346558\
  0860024102632920412316802880506020267702153176377746846603629\
  6393323069832763516545260141269461308505559839207227579085159\
  7071987386780823676350126232219004431373337281534963135796148\
  5965625101447153524185108314744534813802976119492915669350088\
  4208291628892667947101366390356997884333460732309043279437643\
  4964438695063531735576862041537448405714876532923024786796263\
  2113823650692331948829830729225062784989100210199842025828796\
  5567321437336291518475121613111121525624254552702295996388914\
  5829890113229824264870556885026260116064084999421139950616701\
   692882172548108187 2842752
```

Fig. 12 – Example of number of erosions.

$$\Delta_v = (f \mapsto d_v \circ f) \circ \tau_v$$

$$E_v = (f \mapsto e_v \circ f) \circ \tau_v$$

$$\Delta^a{}_v = (f \mapsto d^a{}_v \circ f) \circ \tau_v$$

$$E^a{}_v = (f \mapsto e^a{}_v \circ f) \circ \tau_v,$$

where, $(d_v)_{v \in W^t}$, $(e_v)_{v \in W^t}$, $(d^a{}_v)_{v \in W^t}$ and $(e^a{}_v)_{v \in W^t}$ are, respectively, families of dilations, erosions, anti–dilations and anti–erosions from K_1 to K_2. For any $v \in W^t$, the operators Δ_v, E_v, $\Delta^a{}_v$ and $E^a{}_v$ from $K_1{}^E$ to $K_2{}^E$ are, respectively, a dilation, an erosion, an anti–dilation and an anti–erosion. The operators Δ, E, Δ^a and E^a from $K_1{}^E$ to $K_2{}^E$ given by, respectively,

$$\Delta = \bigvee_{v \in W^t} \Delta_v$$

$$E = \bigwedge_{v \in W^t} E_v$$

$$\Delta^a = \bigwedge_{v \in W^t} \Delta^a{}_v$$

$$E^{a} = \bigvee_{v \in W^{t}} E^{a}{}_{v}.$$

are, respectively, a dilation, an erosion, an anti–dilation and an anti–erosion. Conversely, any operator Δ, E, Δ^{a} and E^{a} from $K_1{}^{E}$ to $K_2{}^{E}$ which is, respectively, a dilation, an erosion, an anti–dilation and an anti–erosion has this form. Furthermore, $(d_v)_{v \in W^t}$, $(e_v)_{v \in W^t}$, $(d^{a}{}_{v})_{v \in W^t}$ and $(e^{a}{}_{v})_{v \in W^t}$ are given by, respectively, for any $v \in W^{t}$,

$$d_v(s) = \Delta(f_{o,s})(v) \quad (s \in K_1)$$

$$e_v(s) = E(f^{o,s})(v) \quad (s \in K_1)$$

$$d^{a}{}_{v}(s) = \Delta^{a}(f_{o,s})(v) \quad (s \in K_1)$$

$$e^{a}{}_{v}(s) = E^{a}(f^{o,s})(v) \quad (s \in K_1).$$

□

Proof – Let us prove the result for the class of dilations. By construction, for any $v \in W^{t}$, Δ_v is the composition of τ_v with the mapping $f \mapsto d_v \circ f$; since both are dilations, Δ_v is also a dilation. The rest of the result is Proposition 1.30. The results for the other classes of elementary operators follow by duality. □

Figure 13 shows an elementary operator ${}_{(e)}E$ which is a t.i. window erosion (with window W) from $K_1{}^{E}$ to $K_2{}^{E}$, where $E = [-5, 5] \subset \mathbf{Z}$ (the set of integers), $W = [-2, 2] \subset E$, $K_1 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $K_2 = \{0, 1, 2, 3\}$ and E is equipped with the addition $+$ modulus 11. In its characterization by decomposition, this erosion depends on a family (e_x) of 5 erosions from K_1 to K_2 which are derived from the erosion e given in Figure 10 by applying Expression (52). In this way the operator ${}_{(e)}E$ is identical to ${}_{e}E$ shown in Figure 10. In Figure 13, a particular function f is shown; the transformation of f through ${}_{(e)}E$ at 3 of E is 1. This value is obtained through Expression (99). Figure 13 shows a *decomposition* of ${}_{(e)}E$. Each branch is a particular erosion as stated in Proposition 1.32.

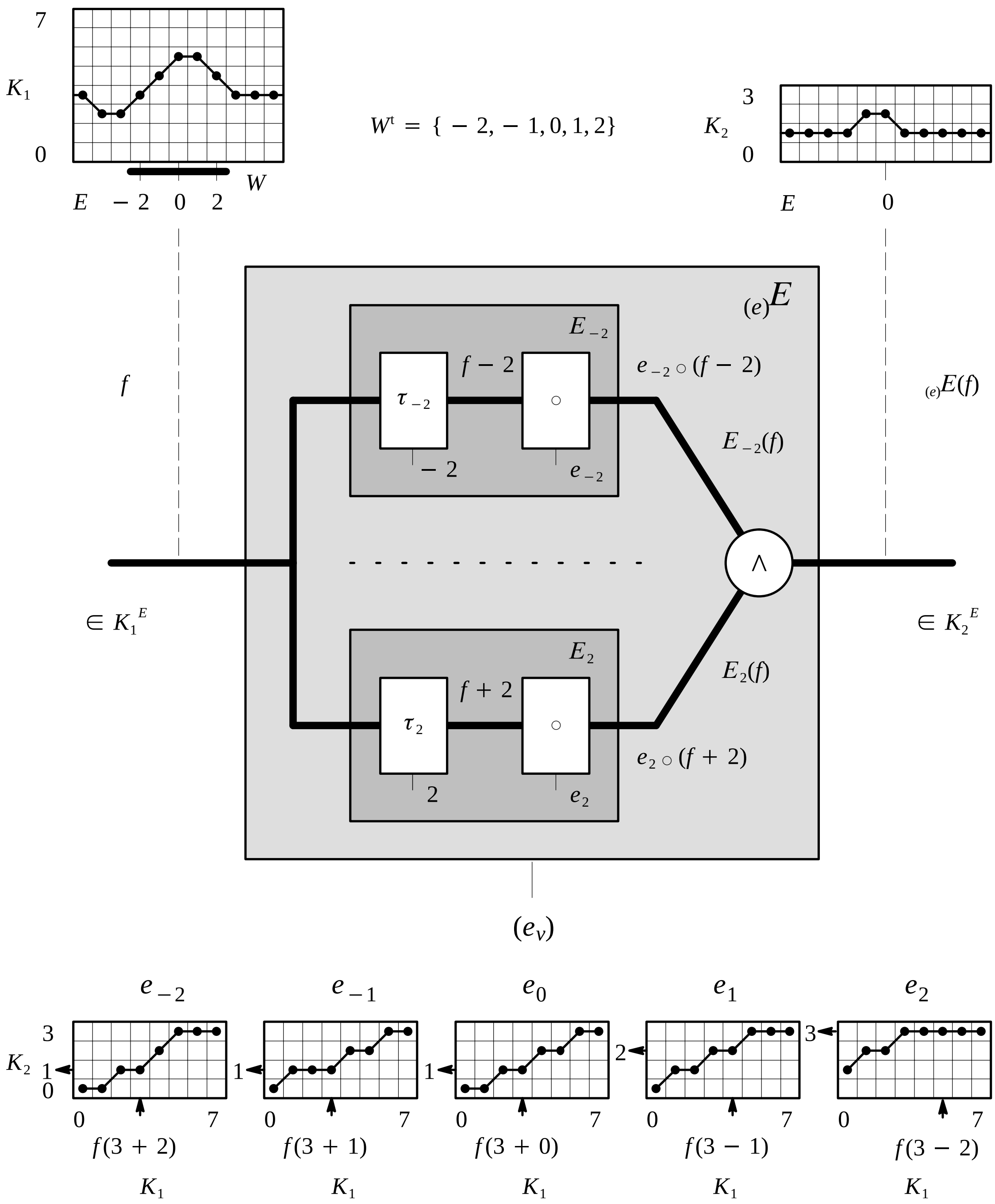

Fig. 13 – Erosion characterization by decomposition.

CHAPTER 4

EXAMPLES

4.1 – NEURAL NETWORKS

In the previous chapter we saw that the elementary t.i. operators for gray–level images can be characterized alternatively in terms of three types of mapping: the structuring elements, the impulse responses and the Elementary Look Up Tables. We will call these mappings *characteristic functions*. In this section and in the following ones, we show that the classes of dilations and erosions recalled in the first chapter are part of our general characterization.

In this chapter, we assume that $K_1 = K_2 = K = [0, k] \subset \mathbf{Z}$.

We first consider the case of the elementary parametric neural networks. We define the following useful expressions.

For any increasing function a from K to K such that $a(0) = 0$ and $a(k) = k$, any nonempty subset B of E, any function b from B to $K = [0, k] \subset \mathbf{Z}$, and any families of binary operations $(+_u)_{u \in B}$ and $(-_u)_{u \in B}$ in, respectively, $\mathfrak{D}^+$ and $\mathfrak{D}^-$, such that $+_u$ and $-_u$ are companions for any $u \in B$, let

$$d_{a,b,+}(t)(u) \triangleq a(t +_u b(u)) \quad (t \in K,\ u \in B) \tag{104}$$

$$e_{a,b,-}(s)(v) \triangleq \overline{a}(s) -_{-v}\ b(-v) \quad (s \in K,\ v \in B^t) \tag{105}$$

$$d_{b,a,+}(t)(u) \triangleq \underline{a}(t) +_u b(u) \quad (t \in K,\ u \in B) \tag{106}$$

$$e_{b,a,-}(s)(v) \triangleq a(s -_{-v}\ b(-v)) \quad (s \in K,\ v \in B^t). \tag{107}$$

We observe that he underlying operators $d_{a,b,+}$ from K to K^B and $e_{a,b,-}$ from K to K^{B^t} are, respectively, a dilation and an erosion. By using Theorem 2.7–(v) of Heijmans and Ronse (6) we can deduce that $d_{a,b,+}$ and $e_{a,b,-}$ are companions. The same is true for $d_{b,a,+}$ and $e_{b,a,-}$.

For any increasing function a from K to K such that $a(0) = 0$ and $a(k) = k$, any nonempty subset B of E, any function b from B to $K = [0, k] \subset \mathbf{Z}$, and any families of binary operations $(+_u)_{u \in B}$ and $(-_u)_{u \in B}$ in, respectively, $\mathfrak{D}^+$ and $\mathfrak{D}^-$, such that $+_u$ and $-_u$ are companions for any $u \in B$, let

$$((d_{a,b,+})_u(t) \triangleq a(t +_u b(u)) \quad (t \in K)) \quad (u \in B) \tag{108}$$

$$((e_{a,b,-})_v(s) \triangleq \overline{a}(s) -_{-v}\ b(-v) \quad (s \in K)) \quad (v \in B^t) \tag{109}$$

$$((d_{b,a,+})_u(t) \triangleq \underline{a}(t) +_u b(u) \quad (t \in K)) \quad (u \in B) \tag{110}$$

$$((e_{b,a,-})_v(s) \triangleq a(s -_{-v}\ b(-v)) \quad (s \in K)) \quad (v \in B^t). \tag{111}$$

We observe that the underlying families $((d_{a,b,+})_u)$ and $((e_{a,b,-})_v)$ are, respectively, families of dilations and erosions from K to K. The same is true for $((d_{b,a,+})_u)$ and $((e_{b,a,-})_v)$.

Proposition 1.33 (elementary parametric neural networks) – Let $(E, +)$ be an Abelian group, and let K be a bounded chain. Let $\Delta_{a,b}$, $E_{a,b}$, $\Delta_{b,a}$ and $E_{b,a}$ be the elementary operators from K^E to K^E defined, respectively, by Expressions (19) to (22). For any increasing function a from K to K such that $a(0) = 0$ and $a(k) = k$, any nonempty subset B of E, any function b from B to $K = [0, k] \subset \mathbf{Z}$, and any families of binary operations $(+_u)_{u \in B}$ and $(-_u)_{u \in B}$ in, respectively, $\mathfrak{D}^+$ and $\mathfrak{D}^-$, such that $+_u$ and $-_u$ are companions for any $u \in B$, we have the following statements:

$$_{e_{a,b,-}}\Delta = \Delta_{a,b,+} \quad \text{and} \quad E_{d_{a,b,+}} = E_{a,b,-} \qquad \text{(for confrontation)}$$

$$\Delta_{d_{a,b,+}} = \Delta_{a,b,+} \quad \text{and} \quad _{e_{a,b,-}}E = E_{a,b,-} \qquad \text{(for selection)}$$

$$\Delta_{(d_{a,b,+})} = \Delta_{a,b,+} \quad \text{and} \quad _{(e_{a,b,-})}E = E_{a,b,-} \qquad \text{(for decomposition)}$$

$$_{e_{b,a,-}}\Delta = \Delta_{b,a,+} \quad \text{and} \quad E_{d_{b,a,+}} = E_{b,a,-} \qquad \text{(for confrontation)}$$

$$\Delta_{d_{b,a,+}} = \Delta_{b,a,+} \quad \text{and} \quad _{e_{b,a,-}}E = E_{b,a,-} \qquad \text{(for selection)}$$

$$\Delta_{(d_{b,a,+})} = \Delta_{b,a,+} \quad \text{and} \quad _{(e_{b,a,-})}E = E_{b,a,-}. \qquad \text{(for decomposition)}$$

□

Proof – Let us prove the second statement in the case of the dilations. For any $g \in K^E$ and any $x \in E$,

$$\begin{aligned}
\Delta_{d_{a,b,+}}(g)(x) &= \bigvee_{v \in B^t} d_{a,b,+}{}^t((g - x)(v))(v) && \text{(Expression (81))}\\
&= \bigvee_{v \in B^t} d_{a,b,+}((g - x)(v))(-v) && \text{(Expression (79))}\\
&= \bigvee_{v \in B^t} a((g - x)(v) +_{-v} b(-v)) && \text{(Expression (104))}\\
&= a(\bigvee_{v \in B^t} (g - x)(v) +_{-v} b(-v)) && \text{(a is a dilation (Prop. 1.4))}\\
&= a(\bigvee_{v \in B^t} g(v + x) +_{-v} b(-v)) && \text{(Expression (62))}\\
&= a(\bigvee_{-v \in B} g(v + x) +_{-v} b(-v)) && \text{(Expression (77))}\\
&= a(\bigvee_{u \in B} g(x - u) +_u b(u)) && (u = -v)\\
&= \Delta_{a,b,+}(g)(x). && \text{(Expression (19))}
\end{aligned}$$

The case of the erosions can be derived from the case of the dilations by duality. The first statement derives from the second one by observing that $d_{a,b,+}$ and $e_{a,b,-}$ satisfy $d_{a,b,+} = \underline{e_{a,b,-}}$ and $e_{a,b,-} = \overline{d_{a,b,+}}$ ($d_{a,b,+}$ and $e_{a,b,-}$ are companions) and by applying Proposition 1.26. The third statement derives from the second one by observing that $(((d_{a,b,+})_u), d_{a,b,+})$ and $(((e_{a,b,-})_v), e_{a,b,-})$ belong to the graph of $(p_v) \mapsto p$ (Exp. (53)) and by applying Proposition 1.29. The last three statements can be proved in the same way as the first three ones.

□

Figures 14 and 15 show two examples of characteristic functions $d_{a,b,+}$, $e_{a,b,-}$, $((d_{a,b,+})_u)$ and $((e_{a,b,-})_v)$ for a given increasing function a from $K = \{0, 1, 2, 3, 4, 5, 6, 7\}$ to K such that $a(0) = 0$ and $a(k) = k$, a given function b defined on B, where $B = [-2, 2] \subset E \subset \mathbf{Z}$ (E is equipped with the addition $+$ modulus $\#E$), and two given families of binary operations $(+_u)_{u \in B}$ and $(-_u)_{u \in B}$ in, respectively, $\mathfrak{D}^+$ and $\mathfrak{D}^-$, such that $+_u$ and $-_u$ are companions for any $u \in B$. Actually, Figure 14 corresponds to the case of the Heijmans' dilation and erosion with $b(-2) = b(2) = -1$, $b(-1) = b(1) = 0$ and $b(0) = 1$.

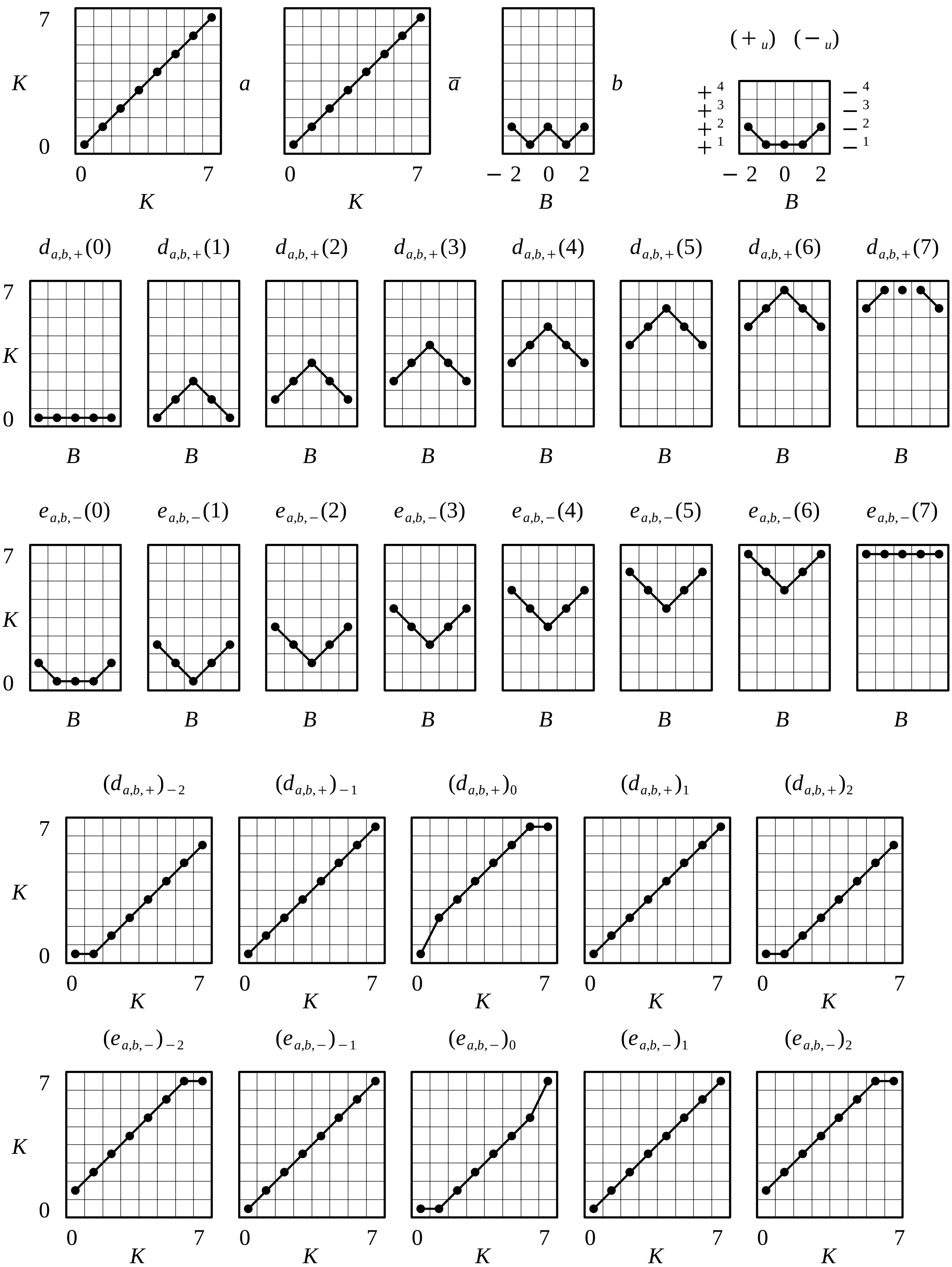

Fig. 14 – Example of characteristic functions for the Heijmans' elementary operators.

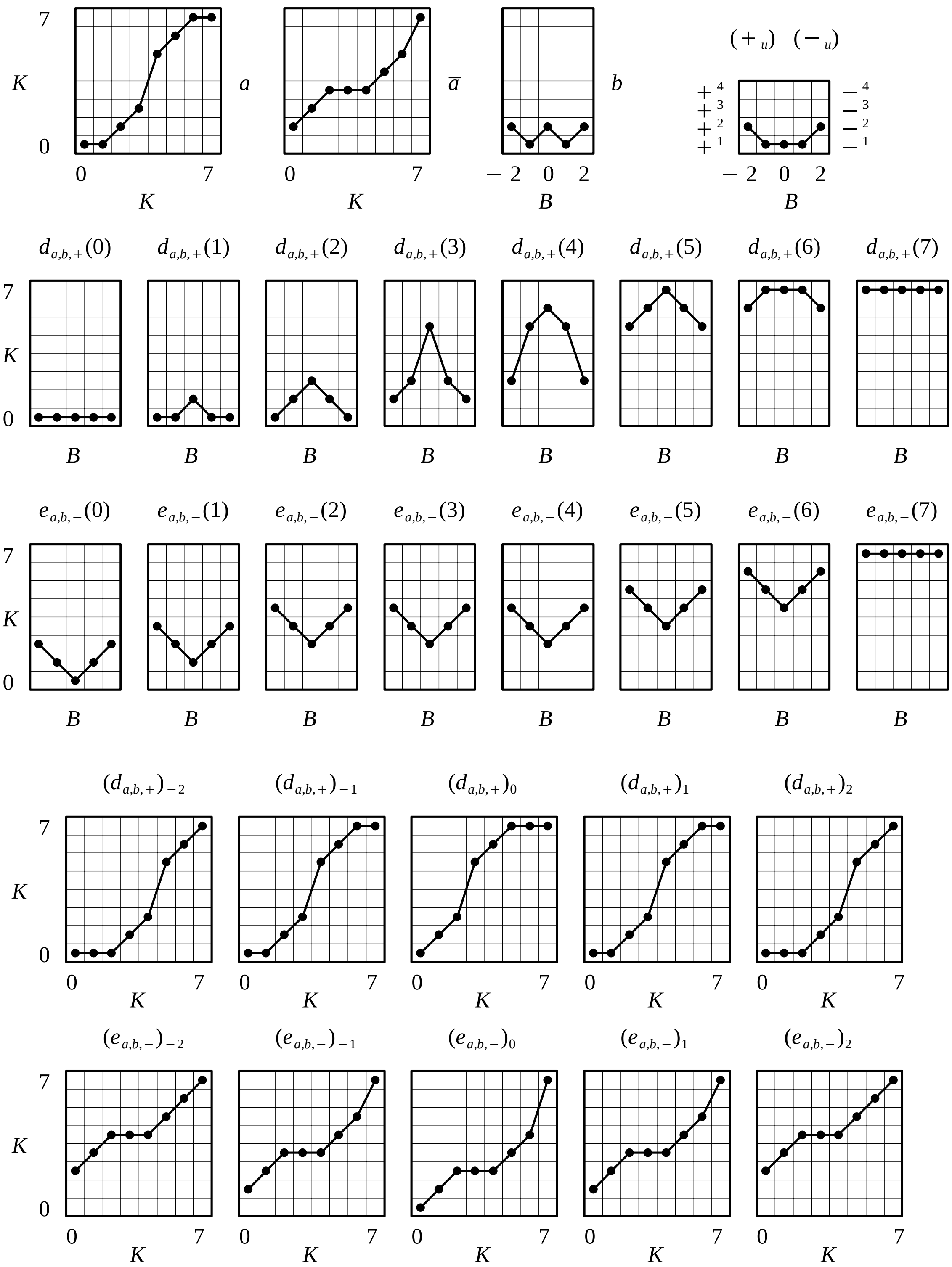

Fig. 15 – Example of characteristic functions for the elementary neural networks.

4.2 – FLAT OPERATORS

Let us consider the case of the flat elementary operators (9). We define the following useful expressions.

For any nonempty subset B of E, let

$$d_B(t)(u) \stackrel{\Delta}{=} t \quad (t \in K,\ u \in B) \tag{112}$$

$$e_B(s)(v) \stackrel{\Delta}{=} s \quad (s \in K,\ v \in B^t). \tag{113}$$

As special cases of Expressions (104) and (107), we observe that the underlying operators d_B from K to K^B and e_B from K to K^{B^t} are both a dilation and an erosion, and they are companions.

For any $u \in B$, let

$$(d_B)_u(t) \stackrel{\Delta}{=} t \quad (t \in K). \tag{114}$$

For any $v \in B^t$, let

$$(e_B)_v(s) \stackrel{\Delta}{=} s \quad (s \in K). \tag{115}$$

As special cases of Expressions (108) and (111), we observe that the underlying families $((d_B)_u)$ and $((e_B)_v)$ are both families of dilations and erosions from K to K.

Proposition 1.34 (flat elementary operators) – Let $(E, +)$ be an Abelian group, and let K be a bounded chain. Let $\varDelta_B$ and E_B be the elementary operators from K^E to K^E defined, respectively, by Expressions (11) and (12). For any nonempty subset B of E, we have the following statements:

$${}_{e_B}\varDelta = \varDelta_B \quad \text{and} \quad E_{d_B} = E_B \qquad \text{(for confrontation)}$$

$$\varDelta_{d_B} = \varDelta_B \quad \text{and} \quad {}_{e_B}E = E_B \qquad \text{(for selection)}$$

$$\varDelta_{(d_B)} = \varDelta_B \quad \text{and} \quad {}_{(e_B)}E = E_B \qquad \text{(for decomposition).}$$

□

Proof – Let us prove the second statement in the case of the dilations. Let id denote the identity mapping from K to K, and let $b_B(u) \stackrel{\Delta}{=} 0$ for any $u \in B$. We have

$$\begin{aligned} \varDelta_{d_B} &= \varDelta_{d_{\mathrm{id}, b_B, +}} && (d_{\mathrm{id}, b_B, +} = d_B) \\ &= \varDelta_{b_B} && \text{(Proposition 1.33)} \\ &= \varDelta_B. && \text{(definition of id, } b_B\text{, } \mathcal{C}^+ \text{ and Expression (13))} \end{aligned}$$

The case of the erosions can be derived from the case of the dilations by duality. The first statement derives from the second one by observing that d_B and e_B satisfy $d_B = \underline{e_B}$ and $e_B = \overline{d_B}$ and by applying Proposition 1.26. The third statement derives from the second one by observing that $(((d_B)_u), d_B)$ and $(((e_B)_v), e_B)$ belong to the graph of $(p_v) \mapsto p$ (Exp. (53)) and by applying Proposition 1.29. □

Figure 16 shows an example of characteristic functions p_B and $((p_B)_u)$ derived from a subset B, where $B = [-2, 2] \subset E \subset \mathbf{Z}$, $K = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and E is equipped with the addition $+$ modulus $\#E$. In this case, B is symmetrical with respect to the origin 0 (i.e., $B = B^t$), d_B and e_B are identical to a mapping from K to K^B that we will denote by p_B, and for any $u \in B$, $(d_B)_u$ and $(e_B)_u$ are identical to a mapping from K to K that we will denote by $(p_B)_u$.

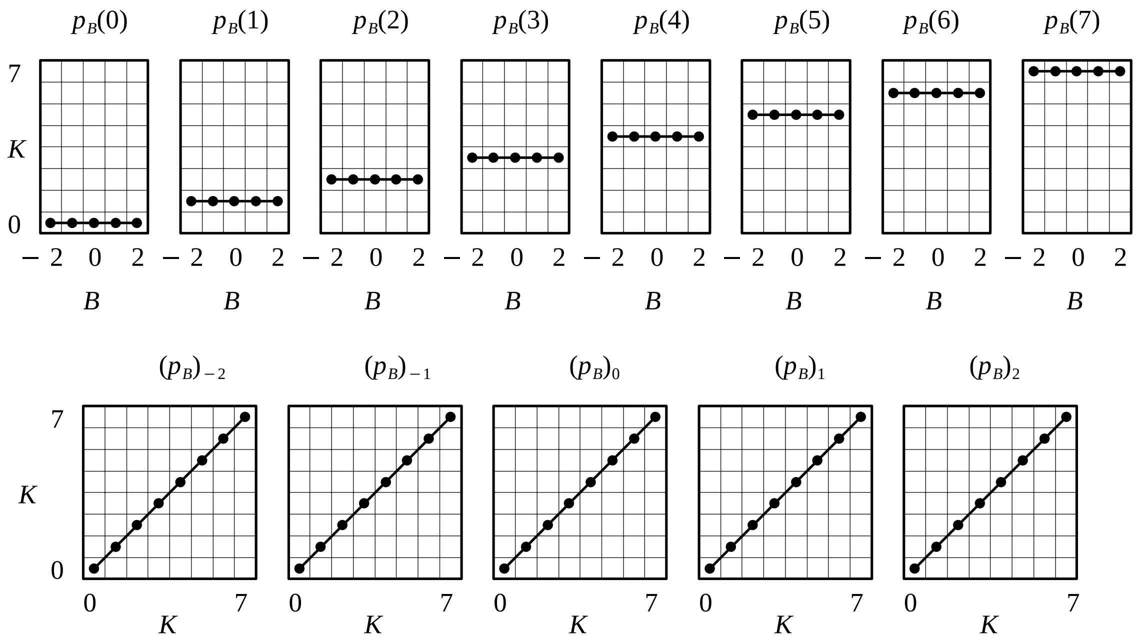

Fig. 16 – Example of characteristic functions for the flat elementary operators.

4.3 – SET OPERATORS

From the above propositon, we can derive some relationship between the set elementary operators and our general characterization. Let 1_X be the characteristic function of a subset X of E and let (E) be the collection of all subsets of E. We assume that $K = [0, 1] \subset \mathbf{Z}$.

Proposition 1.35 (set elementary operators) – Let $(E, +)$ be an Abelian group. Let Δ_B and E_B be the elementary operators from (E) to (E) defined, respectively, by Expressions (1) and (2). For any nonempty subset B of E, and any X and Y in (E), we have the following statements:

$${}_{e_B}\Delta(1_Y) = 1_{\Delta_B(Y)} \quad \text{and} \quad E_{d_B}(1_X) = 1_{E_B(X)} \qquad \text{(for confrontation)}$$

$$\Delta_{d_B}(1_Y) = 1_{\Delta_B(Y)} \quad \text{and} \quad {}_{e_B}E(1_X) = 1_{E_B(X)} \qquad \text{(for selection)}$$

$$\Delta_{(d_B)}(1_Y) = 1_{\Delta_B(Y)} \quad \text{and} \quad {}_{(e_B)}E(1_X) = 1_{E_B(X)} \qquad \text{(for decomposition).}$$

□

Proof – The result derives from Proposition 1.34 recalling that the mapping $X \mapsto 1_X$ is a bijection (7). □

REFERENCES

1. Banon, G.J.F.; Barrera J. Minimal representations for translation–invariant set mappings by mathematical morphology. **SIAM J. Appl. Math.**, **51**(6):1782–1798, Dec. 1991.

2. Banon, G.J.F.; Barrera J. Decomposition of mappings between complete lattices by mathematical morphology. Part I: general lattices. **Signal Process.**, **30**:299–327, 1993.

3. Matheron, G. **Eléments pour une théorie des milieux poreux.** Paris, Masson, 1967.

4. Minkowski, H. Volumem und oberfläche. **Math. Annalen**, **57**:447–495, 1903.

5. Hadwiger, H. Minkowskische addition und subtraktion beleibiger punktmengen und die theoreme von Erhard Schmidt. **Math. Zeitschrift.**, **53**(3):210–218, 1950.

6. Heijmans, H.J.A.M.; Ronse, C. The algebraic basis of mathematical morphology. Part I: dilations and erosions. **Comput. Vision Graph. Image Process.**, **50**(3):245–295, 1990.

7. Banon, G.J.F.; Barrera J. **Bases da morfologia matemática para a análise de imagens binárias.** IX Escola de Computação, Recife, July 1994. 248 p..

8. Banon, G.J.F.; Barrera J. Set operator decomposition and conditionally translation–invariant elementary operators. In: Serra, J.S. and Soille, P., eds. **Mathematical morphology and its applications to image processing**. Dordrecht, Kluwer Academic Publishers, 1994. p. 5–12.

9. Heijmans, H.J.A.M. **Morphological image operators.** Advances in eletronics and electron physics. Supplement 24. Boston, Academic Press, Inc., 1994.

10. Heijmans, H.J.A.M. Theoretical aspects of gray–level morphology. **IEEE Trans. on PAMI**, **PAMI–13(**6):568–582, June 1991.

11. Bloch, I; Maitre, H **Ensembles flous et morphologie mathématique.** Paris, Telecom Paris, 1992.

12. Baets, B. de; Kerre, E.; Gupta, M.M. The fundamentals of fuzzy mathematical morphology. Part 1: basic concepts. **Int. J. General Syst.**, **23**(2):155–171, 1994.

13. Sinha, D.; Dougherty, E.R. An intrinsically fuzzy approach to mathematical morphology. In: SPIE conference, November, Boston, 1991. **Proceedings of the SPIE.** v. 1607.

14. Davidson, J.L.; Hummer, F. Morphology neural networks: an introduction with applications. **Circuits Syst. Signal Process.**, **12**(2):177–210, 1993.

15. Gupta, M.M.; Rao, D.H. On the principle of fuzzy neural networks. **Fuzzy Set Syst.**, **61**:1–18, 1994.

16. Dougherty, E.R.; Sinha, D. **Computational mathematical morphology**. Morphological Laboratory, Center for Imaging Science, Rochester Institute of Technology, December 1992. (Technical report MIL–92–17.)

17. Birkhoff, G. **Lattice theory.** 3.ed. Providence, Rhode Island, American Mathematical Society, 1967.

18. Serra, J. **Image analysis and mathematical morphology.** London, Academic Press, 1988. v. 2.

19. Serra, J. **Thickenings, thinnings.** Centre de Morphologie Mathématique, École Nationale Supérieure des Mines de Paris, Fontainebleau, June 1987. (Technical report N–39.)

20. Achache, A. Galois Connexion of fuzzy subset. **Fuzzy Sets Syst.**, **8**:215–218, 1982.

21. Sugeno, M. **Theory of fuzzy integrals and its applications.** (Ph.D. thesis) – Tokyo, Tokyo Institute of Technology, 1974.

22. Banon, G.J.F. Distinction between several subsets of fuzzy measures. **Fuzzy Set Syst.**, **5**:291–305, 1981.

23. Banon, G.J.F. Constructive decomposition of fuzzy measures in terms of possibility or necessity measures. In: **International Fuzzy Systems Association World Congress, 6th.** São Paulo, July, Proceedings, 1995, p. 217–220.

24. Zadeh, L.A. Fuzzy sets as a basis for a theory of possibility. **Fuzzy Set Syst.**, **1**(1):3–28, 1978.

25. Barrera, J. **Uma abordagem unificada para problemas de visão computacional: a morphologia matemática.** (PhD thesis) – São Paulo, USP, 1992. 228p. (in Portuguese).