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Formal introduction to digital image processing

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Preface

The objects like digital image, scanner, display, look–up–table, filter that we deal with in digital image processing can be defined in a precise way by using various algebraic concepts such as set, Cartesian product, binary relation, mapping, composition, operation, operator and so on.

The useful operations on digital images can be defined in terms of mathematical properties like commutativity, associativity or distributivity, leading to well known algebraic structures like monoid, vector space or lattice.

Furthermore, the useful transformations that we need to process the images can be defined in terms of mappings which preserve these algebraic structures. In this sense, they are called morphisms.

The main objective of this book is to give all the basic details about the algebraic approach of digital image processing and to cover in a unified way the linear and morphological aspects.

With all the early definitions at hand, apparently difficult issues become accessible.

Our feeling is that such a formal approach can help to build a unified theory of image processing which can benefit the specification task of image processing systems.

The ultimate goal would be a precise characterization of any research contribution in this area.

This book is the result of many years of works and lectures in signal processing and more specifically in digital image processing and mathematical morphology. Within this process, the years we have spent at the Brazilian Institute for Space Research (INPE) have been decisive.

This second edition contains many small improvements which are the result of our teaching experience at INPE during the years of 1998, 1999 and 2000. The major of them are the intoduction of poset graphs which give a better foundation to the Hasse diagram in Chaper 1 and the introduction of two types of window operators in Chapter 4.

São José dos Campos, July 2000.

Gerald Jean Francis Banon

Content

Figure list

Chapter 1

Digital image

The digital images are the first objects to be dealt with in image processing. In this chapter, we restrict ourselves to the case of the **gray–level** images. This case includes the binary and black–and–white images.

The next section provides the precise definition of pixel (picture element). As will be seen, this definition precede the definition of image.

1.1 Image definition

We begin introducing two different sets of elements which will enter in the gray–level image definition.

The first set, that we denote by E, is a set of adjacent squares arranged along a certain number of rows and columns, and forming a rectangular surface (here we will not try to formalize the concepts of square, row and column, we just accept them as geometrical features). The squares are distinguished by their position.

Figure 1.1 shows a set of 110 squares arranged along 10 rows and 11 columns. A small $\times$ has been drawn in each square to indicate that it is **not** white colored.

The second set, that we denote by K, is a set of gray–levels. We call it *gray–scale*. Figure 1.2 shows a gray–scale with 4 gray–levels.

Definition 1.1 (pixel) – A gray square or *picture–element* or *pixel* for short, *located in* E and *with a gray–level in* K, is an element of the Cartesian product (see Definition 1.14) of E and K. □

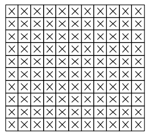
$E =$ 

Fig. 1.1 – A set of squares.

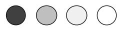
$K =$ 

Fig. 1.2 – A gray–scale.

Figure 1.3 shows the Cartesian product $E \times K$ where E is the set of squares shown in Figure 1.1 and K is the gray–scale shown in Figure 1.2.

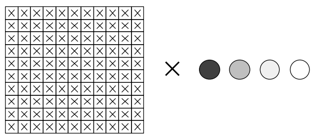
$E \times K =$ 

Fig. 1.3 – Cartesian product of a set of squares and a gray–scale.

By definition of the Cartesian product, a gray square or pixel located in E and with a gray–level in K is an ordered pair (x, s) whose elements are a (non colored) square x of E and a gray–level s of K. Sometimes, we say that x is the *pixel location* in E.

Figure 1.4 shows two graphical representations of a certain pixel (x, s) of the Cartesian product $E \times K$ shown in Figure 1.3.

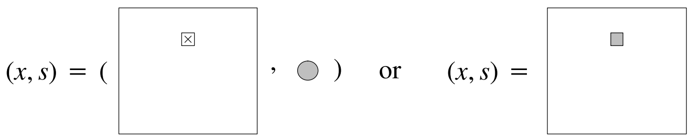


Fig. 1.4 – A pixel.

Here, we must distinguish between a square and a pixel: a square does not have any color (even black or white) as opposed to a pixel which is a gray colored square.

Definition 1.2 (digital gray–level image) – A *digital gray–level image* or simply a *gray–level image* or *image with domain E* and *gray–scale K*, is a **graph–of–a–mapping** on E to K (see Definition 1.16). □

In other words (see Definition 1.16), an image I with domain E and gray–scale K is a subset of pixels located in E and with gray–levels in K such that **each** square of E is colored with one and **only** one gray–level of K. Using the mathematical notation, we write $I \in \mathcal{G}(E \times K)$.

An element p of $E \times K$ is a *pixel of I* iff (if and only if) if $p \in I$.

Figure 1.5 shows an image I having as domain E the set of squares shown in Figure 1.1 and as gray–scale K the set of gray–levels shown in Figure 1.2.

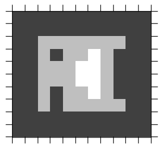
$I =$ 

Fig. 1.5 – An image.

If the number of rows of E is m and the number of columns is n, then the image and its domain are said to be of size m by n, or simply is m by n. Sometimes, the expression m by n is written $m \times n$.

The image of Figure 1.5 is of size 10 by 11.

Two images with the same domain and the same gray–scale are said to be *equal* if they have the same pixels.

As a direct consequence of the image definition, we can make the following observations.

On one hand, an image I on E to K is necessarily a subset of $E \times K$. Figure 1.6 illustrates this point for the image of Figure 1.5. In other words, I is a binary relation on E to K (see Definition 1.15). As a relation, I is read "*is colored with*".

On the other hand, not all subset of $E \times K$ is an image on E to K. A necessary condition for a subset X of $E \times K$ to be an image is that the number of pixels of X must be equal to the number of elements of E, but this is not a sufficient condition.

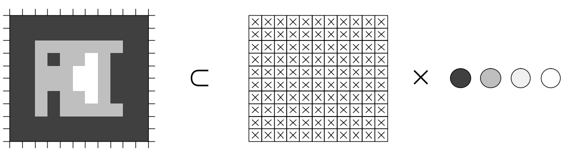


Fig. 1.6 – An image as a subset of a Cartesian product.

Figure 1.7 shows a subset of the Cartesian product of a set of 9 squares and a gray–scale with 4 gray–levels which **is** an image since this subset is a graph–of–a–mapping.

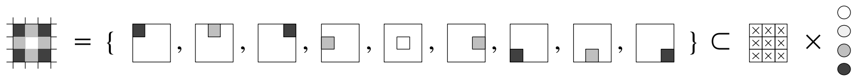


Fig. 1.7 – A Cartesian product subset which is an image.

Exercise 1.3 (image) – Does the subset of Figure 1.8 form an image on the set of 9 squares to the gray–scale with 4 gray–levels? Explain your answer. If so, draw the image. □

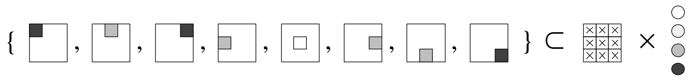


Fig. 1.8 – A Cartesian product subset.

Exercise 1.4 (image) – Does the subset of Figure 1.9 form an image on the set of 9 squares to the gray–scale with 4 gray–levels? Explain your answer. If so, draw the image. □

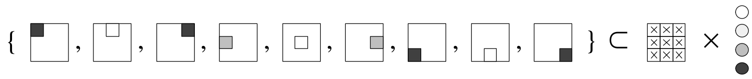


Fig. 1.9 – A Cartesian product subset.

Exercise 1.5 (image) – Does the subset of Figure 1.10 form an image on the set of 9 squares to the gray–scale with 4 gray–levels? Explain your answer. If so, draw the image. □

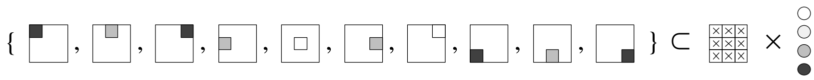


Fig. 1.10 – A Cartesian product subset.

An image is a graph–of–a–mapping, but not every graph–of–a–mapping is an image. When we refer to an image, we must consider the properties of E and K. Some graph–of–a–mappings have the same nature of the images.

Before ending this section, we want to point out the analogy between the definition of gray–level image and the definition of rectangular array of natural numbers or simply numerical array.

We denote by K' an interval of natural numbers and we consider the set K' in place of the gray–scale K.

We call numbered square or *array–element* located in E and with a number in K', an element of the Cartesian product of E and K'. In other words, an array–element located in E and with a number in K' is an ordered pair (x, s) whose elements are a (non numbered) square x of E and a number s of K'.

A *rectangular array of natural numbers* or *numerical array with domain E* with m rows and n columns, and *with an interval* K' (of natural numbers), is a **graph–of–a–mapping** on E to K'.

In other words, a numerical array with domain E and interval K' is a subset of array–elements located in E and with numbers in K' such that **each** square of E is numbered with one and **only** one number of K'. Using the mathematical notation, we write $A \in \mathcal{G}(E \times K')$.

Figure 1.11 shows a numerical array A having as domain E the set of squares shown in Figure 1.1 and as interval of natural numbers the set $\{0, 1, 2, 3\}$. In this figure the square edges have been omitted instead two square braces have been used to group the array elements.

$$A = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&1&1&1&1&1&0&0 \\ 0&0&1&0&1&1&3&1&0&0&0 \\ 0&0&1&1&1&3&3&1&0&0&0 \\ 0&0&1&1&1&3&3&1&0&0&0 \\ 0&0&1&0&1&1&3&1&0&0&0 \\ 0&0&1&0&1&1&1&1&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

Fig. 1.11 – A numerical array.

In the next section we will show the relationship between images and rectangular arrays of natural numbers.

1.2 Image characterization

In this section we introduce several image characterizations, the more important one being the characterization in terms of matrix. This characterization will lead to the notions of scanner device and display device. First, we introduce the characterization in terms of mapping.

By the graph–of–a–mapping characterization (see Proposition 1.32), we can assign, in a bijective way, to each image I on E to K, the mapping f_I from E to K (see Expression (1.1) in Section 1.3) defined by, for any x of E and s of K,

$$f_I(x) \triangleq s \text{ iff } x\, I\, s.$$

Figure 1.12 illustrates, by mean of block diagrams, the image characterization in terms of mapping.

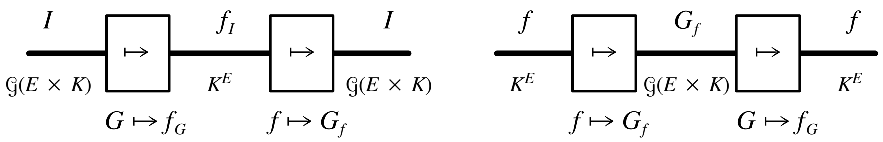


Fig. 1.12 – Image characterization in terms of mapping.

Since $I \mapsto f_I$ is a bijection, very often, no distinction will be made between an image I and its representation by the mapping f_I. Abuse of notation will also occur in that a mapping will often be called an image, and conversely, an image will often be called a mapping and denoted by f.

The elements of G_f (see Expression (1.2) in Section 1.3) are of the form $(x, f(x))$. Hence, by abuse of notation, we say that they are *pixels of f*.

From the characterization in terms of mapping, we can introduce the characterization in terms of matrix. Actually, to do that, we need to consider some algebraic properties of the image domain and gray–scale.

With respect to an image, it is possible to identify two binary relations, one defined on the set E of adjacent squares and an another one defined on the gray–scale K.

On the set E of squares, we can define the binary relation: "*is ((on the same row as) or (on a row above the row of)) and ((on the same column as) or (on a column to the left of the column of))*".

This binary relation is a partial ordering (see Definition 1.36). When no confusion is likely to arise, we will denote it by $\leq$.

Figure 1.13 shows four squares, called a, b, c and d, in a set of squares with four rows and four columns. Following the above definition for $\leq$, the expression $a \leq b$ is true, but both expressions $c \leq d$ and $d \leq c$ are false. We say that a and b can be compared but c and d not.

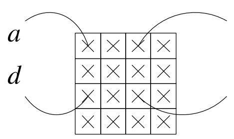


Fig. 1.13 – Four squares in a set of squares.

We observe that on the set E of squares, the binary relations: "*is (on the same row as) or (on a row above the row of)*" and "*is (on the same column as) or (a column to the left of the column of)*" are reflexive and transitive but not anti–symmetric.

Exercise 1.6 (partial ordering on the squares) – Prove that the binary relation "*is ((on the same row as) or (on a row above the row of)) and ((on the same column as) or (on a column to the left of the column of))*" defined on the set E of squares is a partial ordering. □

Solution – First, two identical squares are on the same row and column, that is, the reflexive condition is satisfied; second, if a square a "*is ((on the same row as) or (on a row above the row of)) and ((on the same column as) or (on a column to the left of the column of))*" a square b and conversely, then the square a must be on the same row and column as the square b, in other words, they are identical, that is, the antisymmetry condition is satisfied; third, the relations between the squares: "*is (on the same row as) or (on a row above the row of)*" and "*is (on the same column as) or (a column to the left of the column of)*" are transitive, consequently their intersection (set intersection) is also transitive, that is, the transitivity condition is satisfied. □

Hence, the set E of adjacent squares arranged along rows and columns with its binary relation is a poset (see Definition 1.38).

On the set K of gray levels we can define the binary relation: "*is darker or equal than*".

This binary relation is a partial ordering. When no confusion is possible, we will denote it by $\leq$. Hence, the gray–scale K with this binary relation is another poset.

Unlike the set E, the gray–scale K with its binary relation is a **chain** (see Definition 1.39).

Between the rows and columns of squares of E, we can identify, respectively, the binary relations "*is (identical to) or (above)*" and "*is (identical to) or (on the left of)*".

With these binary relations, the set of rows and the set of columns form two chains.

From Proposition 1.52, each of these two chains are poset–isomorphic to some ordinal number, so we can refer to the "first", the "second" ... row and column of the image domain.

This leads up to the following algebraic property of an image domain.

Proposition 1.7 (image domain property) – A set E of adjacent squares arranged along m rows and n columns, with the partial ordering "*is ((on the same row as) or (on a row above the row of)) and ((on the same column as) or (on a column to the left of the column of))*" is poset–isomorphic to the direct product $\mathbf{m} \times \mathbf{n}$ of the ordinal number $\mathbf{m}$ with the ordinal number $\mathbf{n}$. □

Proof – We have to show the existence of a bijection between the set E of the squares and the direct product $\mathbf{m} \times \mathbf{n}$, which is order–preserving. Let us consider the mapping that assigns to the square at the i^{th} row and the j^{th} column the pair (i, j). By construction, this mapping is a bijection from E to $\mathbf{m} \times \mathbf{n}$, in other words, $E \cong (\mathbf{m} \times \mathbf{n})$. Furthermore, if, under this mapping, a square a maps to (i, j) and a square a' maps to (i', j') then the square a "*is ((on the same row as) or (on a row above the row of)) and ((on the same column as) or (on a column to the left of the column of))*" the square a' if and only if $(i, j) \leq (i', j')$. □

Since there is only one poset–isomorphism from $\mathbf{m} \times \mathbf{n}$ to $(E, \leq)$ we can give it a name. We will denote it by b_1. Figure 1.14 shows the poset–isomorphism b_1 for a 2 by 2 image domain. We observe that this mapping preserve the ordering.

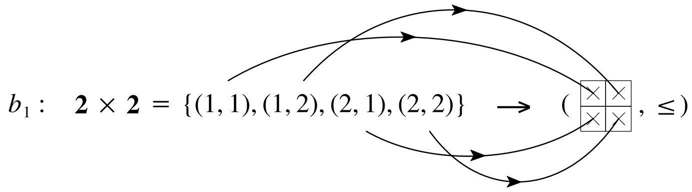


Fig. 1.14 – The poset–isomorphism for an image domain.

In relation to a gray–scale, we can identify a similar algebraic property. Let K_p be the interval $[0, p - 1]$ of the first p natural numbers.

Proposition 1.8 (gray–scale property) – A gray–scale K with p gray levels, with the partial ordering "*is darker or equal than*" is poset–isomorphic to the interval K_p of the first p natural numbers with the usual partial ordering. □

Proof – The interval K_p of the first n natural numbers with the usual partial ordering, and a gray–scale K with p gray levels, with the partial ordering "*is darker or equal than*" are both poset–isomorphic to the ordinal number $\mathbf{p}$ (see Proposition 1.52). Therefore, the proposition is a consequence of the transitivity of the binary relation "*is poset–isomorphic to*". □

Since there is only one poset–isomorphism from $(K, \leq)$ to $(K_p, \leq)$, we can give it a name. We will denote it by b_2. Figure 1.15 shows the poset–isomorphism b_2 for a gray–scale with 4 gray levels. We observe that this mapping preserve the ordering.

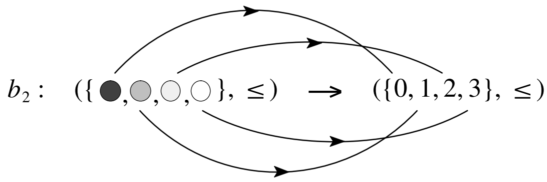


Fig. 1.15 – The poset–isomorphism for a gray–scale.

We are now ready to establish the image characterization in terms of matrix.

Let f be a mapping from E to K (we recall that such f may characterize an image on E to K), we denote by g_f the expression defined by

$$g_f \triangleq b_2 \circ f \circ b_1.$$

Let g be a mapping from $\mathbf{m} \times \mathbf{n}$ to K_p (g can be seen as a matrix), we denote by f_g the expression defined by

$$f_g \triangleq {b_2}^{-1} \circ g \circ {b_1}^{-1}.$$

Figure 1.16 is an illustration of both expressions.

$$
\begin{tikzcd}
E \arrow[d, "f"'] & \mathbf{m} \times \mathbf{n} \arrow[l, "b_1"'] \arrow[d, "g_f"] \\
K \arrow[r, "b_2"'] & K_p
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "b_1^{-1}"] \arrow[d, "f_g"'] & \mathbf{m} \times \mathbf{n} \arrow[d, "g"] \\
K & K_p \arrow[l, "b_2^{-1}"]
\end{tikzcd}
$$

Fig. 1.16 – Two composites used in the image characterization.

We observe that g_f is a mapping from $\mathbf{m} \times \mathbf{n}$ to K_p and that f_g is a mapping from E to K.

In the next proposition, we write simply g_{f_I} instead of $g_{(f_I)}$ and G_{f_g} instead of $G_{(f_g)}$.

Proposition 1.9 (image characterization in terms of matrix) – Let E be the set of adjacent squares arranged along m rows and n columns and let K be the gray–scale with p gray–levels. The mapping $I \mapsto g_{f_I}$ from $\mathbb{G}(E \times K)$ to ${K_p}^{\mathbf{m}\times\mathbf{n}}$ is a bijection, its inverse is the mapping $g \mapsto G_{f_g}$. $\square$

Proof – We observe that by construction g_{f_I} is an element of ${K_p}^{\mathbf{m}\times\mathbf{n}}$ and that G_{f_g} is an element of $\mathbb{G}(E \times K)$. The mapping $I \mapsto g_{f_I}$ from $\mathbb{G}(E \times K)$ to ${K_p}^{\mathbf{m}\times\mathbf{n}}$ is the composite of the bijection $I \mapsto f_I$ from $\mathbb{G}(E \times K)$ to K^E and of the bijection $f \mapsto g_f$ from K^E to ${K_p}^{\mathbf{m}\times\mathbf{n}}$ (see Exercise 1.29), consequently it is a bijection. Furthermore, $g \mapsto f_g$ is the inverse of $f \mapsto g_f$ (see Exercise 1.29), and $f \mapsto G_f$ is the inverse of $I \mapsto f_I$ (see Proposition 1.32). Thus $g \mapsto G_{f_g}$ being the composite of $g \mapsto f_g$ and $f \mapsto G_f$, it is the inverse of $I \mapsto g_{f_I}$ by Proposition 1.31. $\square$

Figure 1.17 illustrates Proposition 1.9.

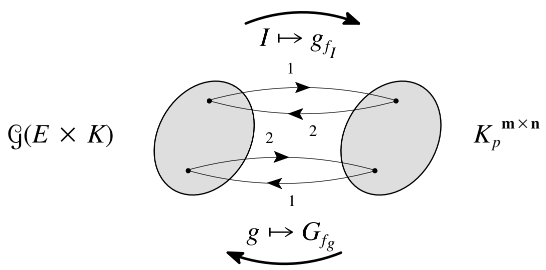


Fig. 1.17 – Image characterization in terms of matrix.

The mapping $I \mapsto g_{f_I}$ is an elementary mathematical model for a *scanner device* which transforms a digital image into a matrix. We will denote this first mapping by α. Its inverse is an elementary mathematical model for a *display device* which transforms a matrix into a digital image. We will denote this second mapping by β.

Figure 1.18 shows a scanner device α and a display device β in the case $m = n = 3$ and $p = 4$.

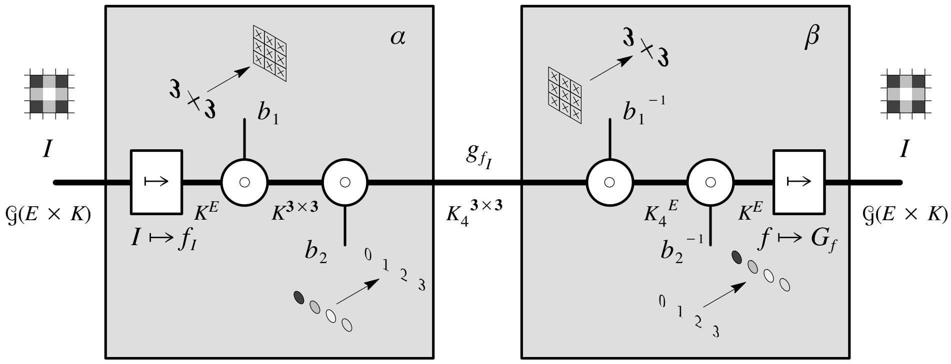


Fig. 1.18 – A scanner and a display device.

Before ending this section, we want to explicit the relationship between images and rectangular arrays of natural numbers of the previous section.

We first define two more mappings.

Let λ be a mapping from $\mathcal{G}(E \times K_p)$ to $K_p{}^{\mathbf{m} \times \mathbf{n}}$ defined by, for any array A of $\mathcal{G}(E \times K_p)$,

$$\lambda(A) \triangleq f_A \circ b_1,$$

where f_A is the mapping from E to K_p defined by, for any x of E and k of K_p,

$$f_A(x) \triangleq k \ \text{ iff } \ x \, A \, k.$$

The mapping λ transforms an array into a matrix.

Let μ be a mapping from $K_p{}^{\mathbf{m} \times \mathbf{n}}$ to $\mathcal{G}(E \times K_p)$ defined by, for any matrix g of $K_p{}^{\mathbf{m} \times \mathbf{n}}$,

$$\mu(g) \triangleq G_{g \circ b_1{}^{-1}},$$

where G_g is the binary relation, between $\mathbf{m} \times \mathbf{n}$ and K_p, defined by

$$G_g \triangleq \{((i,j),k) \in (\mathbf{m} \times \mathbf{n}) \times K_k : \ k = g(i,j)\}.$$

The mapping μ transforms a matrix into an array.

Proposition 1.10 (array characterization in terms of matrix) – Let E be the set of adjacent squares arranged along m rows and n columns and let K_p be the interval of the first p natural numbers. The mapping λ from $\mathcal{G}(E \times K_p)$ to ${K_p}^{\mathbf{m\times n}}$ is a bijection, its inverse is the mapping μ. □

Proof – The idea is to decompose λ into two bijections. Let us divide the proof into four main steps.

1. For any A in $\mathcal{G}(E \times K_p)$, $f_A \in {K_p}^E$ by definition of f_A, and for any f in ${K_p}^E$, $f \circ b_1 \in {K_p}^{\mathbf{m\times n}}$ by definition of b_1 and composition. In other words, the composite of $A \mapsto f_A$ and $f \mapsto f \circ b_1$ is a mapping from $\mathcal{G}(E \times K_p)$ to ${K_p}^{\mathbf{m\times n}}$.

2. Let us prove that this composite is λ. For any A in $\mathcal{G}(E \times K_p)$,

$$\begin{aligned}
(f \mapsto f \circ b_1) \circ (A \mapsto f_A)(A) &= (f \mapsto f \circ b_1)((A \mapsto f_A)(A)) && \text{(composition definition)}\\
&= (f \mapsto f \circ b_1)(f_A) && \text{(mapping definition)}\\
&= f_A \circ b_1 && \text{(mapping definition)}\\
&= \lambda(A). && (\lambda \text{ definition})
\end{aligned}$$

3. By Proposition 1.32 $A \mapsto f_A$ is a bijection and $f \mapsto G_f$ is its inverse. Let us prove that $f \mapsto f \circ b_1$ is also a bijection and $g \mapsto g \circ {b_1}^{-1}$ is its inverse. For any f in ${K_p}^E$,

$$\begin{aligned}
(g \mapsto g \circ {b_1}^{-1}) \circ (f \mapsto f \circ b_1)(f) &= (g \mapsto g \circ {b_1}^{-1})((f \mapsto f \circ b_1)(f)) && \text{(comp. definition)}\\
&= (g \mapsto g \circ {b_1}^{-1})(f \circ b_1) && \text{(mapping definition)}\\
&= (f \circ b_1) \circ {b_1}^{-1} && \text{(mapping definition)}\\
&= f \circ (b_1 \circ {b_1}^{-1}) && \text{(composition associativity)}\\
&= f, && (b_1 \text{ is a bijection})
\end{aligned}$$

that is, $g \mapsto g \circ {b_1}^{-1}$ is a left inverse of $f \mapsto f \circ b_1$.

For any g in ${K_p}^{\mathbf{m\times n}}$,

$$\begin{aligned}
(f \mapsto f \circ b_1) \circ (g \mapsto g \circ {b_1}^{-1})(g) &= (f \mapsto f \circ b_1)((g \mapsto g \circ {b_1}^{-1})(g)) && \text{(comp. definition)}\\
&= (f \mapsto f \circ b_1)(g \circ {b_1}^{-1}) && \text{(mapping definition)}\\
&= (g \circ {b_1}^{-1}) \circ b_1 && \text{(mapping definition)}\\
&= g \circ ({b_1}^{-1} \circ b_1) && \text{(composition associativity)}\\
&= g, && (b_1 \text{ is a bijection})
\end{aligned}$$

that is, $g \mapsto g \circ {b_1}^{-1}$ is a right inverse of $f \mapsto f \circ b_1$.

Therefore, by Proposition 1.28, $f \mapsto f \circ b_1$ is a bijection and its inverse is $g \mapsto g \circ {b_1}^{-1}$.

4. Hence, by Proposition 1.31, λ is a bijection as composite of two bijections and its inverse is $(g \mapsto g \circ {b_1}^{-1}) \circ (g \mapsto G_g)$. We just have to prove that this mapping is μ. For any g in $K_p^{\mathbf{m}\times\mathbf{n}}$,

$$
\begin{aligned}
(g \mapsto g \circ {b_1}^{-1}) \circ (g \mapsto G_g)(g) &= (g \mapsto g \circ {b_1}^{-1})((g \mapsto G_g)(g)) && \text{(comp. definition)}\\
&= (g \mapsto g \circ {b_1}^{-1})(G_g) && \text{(mapping definition)}\\
&= G_{g \circ {b_1}^{-1}} && \text{(mapping definition)}\\
&= \mu(g). && \text{(μ definition)}
\end{aligned}
$$

□

The composites $\mu \circ \alpha$ and $\beta \circ \lambda$ are mutually inverse and establish the relationship between images an arrays as illustrated in Figure 1.19 where K is the gray–scale of Figure 1.2.

The composite $\mu \circ \alpha$ transforms an image into an array of natural numbers, and conversely, the composite $\beta \circ \lambda$ transforms an array into an image.

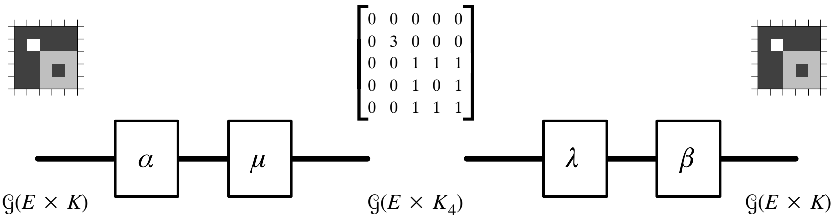


Fig. 1.19 –Relationship between an image and an array of natural numbers.

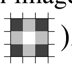
The pair consisting of the image I of Figure 1.5 and the array A of Figure 1.11 is an example of a pair satisfying the equalities: $A = (\mu \circ \alpha)(I)$ and $I = (\beta \circ \lambda)(A)$.

Since $\mu \circ \alpha$ is a bijection, very often, no distinction will be made between an image I and its representation by the array $A = (\mu \circ \alpha)(I)$. Abuse of notation will also occur in that an array will often be called an image, and conversely, an image will often be called an array.

From the definitions of α, β, λ and μ, we have for any image I and any array A:

$$(\beta \circ \lambda)(A) = G_{{b_2}^{-1} \circ f_A} \quad \text{and} \quad (\mu \circ \alpha)(I) = G_{b_2 \circ f_I}\,.$$

Exercise 1.11 (relationship between images and arrays) – Using the mapping of Figure 1.15 draw the image $(\beta \circ \lambda)(\begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 0 & 2 & 1 \\ 2 & 0 & 3 & 0 & 2 \\ 1 & 2 & 0 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix})$. □

Exercise 1.12 (relationship between images and arrays) – Using the mapping of Figure 1.15, write down the array $(\mu \circ \alpha)($$)$. □

1.3 Some mathematical definitions and properties

To help the understanding of the definitions given in the previous sections, we recall some important mathematical definitions such as set, ordered pair, Cartesian product, binary relation, graph–of–a–mapping, mapping, bijection, composition, equivalence relation, partial ordering, poset, chain, ordinal number, matrix, Hasse diagram, poset–isomorphism, and we recall some of their properties.

Set

Intuitively, a *set* is any collection of objects called *elements*. It is denoted by writing its elements between brackets, the order being meaningless. For instance, the set having the two elements a and b is denoted by $\{a, b\}$. This intuitive definition of set will be sufficient for the moment because all we will need are sets that can be seen as elements of other sets. For an axiomatic definition of set, see Dugundji (1965, p. 17) or Mac Lane and Birkhoff (1967, p. 506).

Two elements a and b of a set are *equal* iff (if and only if) they are the same, we write $a = b$. If two elements are not equal, we say that they are distinct.

We can extend the equality between elements to the equality between sets. Two sets A and B are *equal* iff they have the same elements; we write $A = B$. For example the set {a, b} and {a, b, b} are equal. To be equal, two (finite) sets must have the same number of distinct elements, that is we can associate to each element of a set a different element of the other set and conversely.

The set A is said to be *included in* B, or A is a *subset of* B iff the elements of A are elements of B. We denote by $\mathcal{P}(B)$ the set of all subsets of a set B.

From the notion of set, we can define the notion of ordered pair.

Ordered pair and Cartesian product

Definition 1.13 (ordered pair) – The set $\{\{a\}, \{a, b\}\}$ is called the *ordered pair a and b*, in this order; it is denoted by (a, b). □

Following this definition, the ordered pair (b, a) is the collection $\{\{b\}, \{a, b\}\}$. Hence, $(a, b) \neq (b, a)$ iff $a \neq b$.

Furthermore, two ordered pairs (a, b) and (c, d) are equal iff $a = c$ and $b = d$.

Definition 1.14 (Cartesian product) – Let A and B be two non–empty sets, the *Cartesian product of A and B* is the set of the **all** ordered pairs (a, b) with a in A and b in B. The Cartesian product of A and B is denoted by $A \times B$. □

From the notion of Cartesian product, we can define the notion of binary relation.

Binary relation and Graph–of–a–mapping

Definition 1.15 (binary relation) – Let A and B be two non–empty sets, a *binary relation R on A to B* (or between A and B) is a subset of $A \times B$. The expression "(a, b) *belongs to R*" is written as $a\ R\ b$. □

For example, the set of all ordered pairs (X_1, X_2) in $\mathcal{P}(A) \times \mathcal{P}(A)$ such that X_1 is a subset of X_2 is a binary relation. It is denoted $\subset$ and it is called *inclusion*.

We can represent a binary relation using a two–entries table. Each line of the table corresponds to an element of A and each column to an element of B. In this table, we place a $\times$ in the line a and the column b whenever $a\ R\ b$. In this way all the elements of R are represented by all the $\times$ in the table. Figure 1.20 shows two binary relations.

R_1	b_1	b_2	b_3
a_1	$\times$		
a_2		$\times$	
a_3		$\times$	

R_2	a_1	a_2	a_3
b_1	$\times$		
b_2		$\times$	$\times$
b_3			

Fig. 1.20 – Two binary relations.

Two binary relations, R_1 between A and B, and R_2 between B and A, are said *mutually converse* iff

$$b\ R_2\ a \text{ iff } a\ R_1\ b \quad (a \in A \text{ and } b \in B).$$

The binary relations R_1 and R_2 of Figure 1.20 form an example of mutually converse relations.

We now consider a particular class of binary relation. Actually, this class is introduced here for the first time. This class is useful to define precisely what is a digital image.

Definition 1.16 (graph–of–a–mapping) – Let A and B be two non–empty sets. The binary relation G on A to B is a *graph–of–a–mapping* on A to B, iff for **each** a in A, there exists one b in B and **only one**, such that $a\ G\ b$. □

The binary relation R_1 of Figure 1.20 is a graph–of–a–mapping but R_2 is not.

We denote by $\mathcal{G}(A \times B)$ the set of graph–of–a–mappings on A to B.

Related to the graph–of–a–mapping we have the notion of mapping, may be one of the most basic in algebra. From it we establish relationships among sets.

Mapping

Definition 1.17 (mapping) – Let A and B be two non–empty sets. A *mapping* (or *function*) f with *domain* A and *range* (or codomain) B assigns to **each** element a of A a **unique** element b of B. The element b is denoted by $f(a)$ and called the *value of f at a*. A mapping f from A to B is denoted by $f:\ A \to B$. □

Let b_a denote an expression depending on a. If b_a belongs to B whenever a belongs to A, then the correspondence a to b_a defines a mapping from A to B that is denoted by $a \mapsto b_a$.

Figure 1.21 shows two graphical representations of a mapping f from A to B.

As a consequence of Definition 1.17, if $a_1 = a_2$ then $f(a_1) = f(a_2)$. That is, by using the notion of mapping, we can deduce an equality from another one.

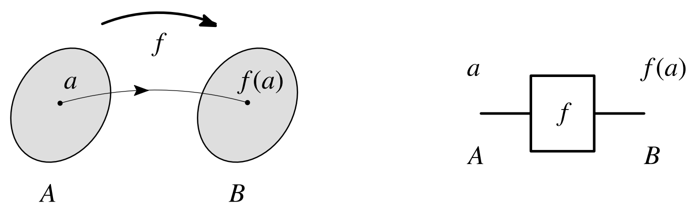


Fig. 1.21 – A mapping.

We call the set of mappings from A to B a *mapping* (or *function*) *set*, and we denote it by B^A.

Two mappings f and g are equal iff they have the same domain, range and values, i.e., $f(a) = g(a)$ for every a in the domain.

For any non–empty set A, the *identity mapping on A*, denoted by 1_A, is the mapping from A to A which assigns each element of A to itself.

Let X be a subset of A. The *image of X through a mapping f from A to B*, denoted by $f(X)$, is a subset of B such that b belongs to it iff for some element a in X, $f(a) = b$, i.e.,

$$f(X) \triangleq \{b \in B : \exists a \in X, f(a) = b\}.$$

Figure 1.22 shows the image $f(X)$ of a subset X through a mapping f.

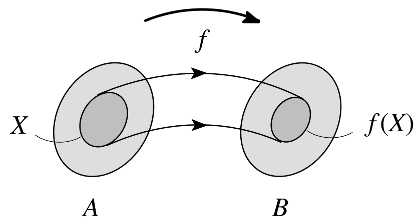


Fig. 1.22 – Image of a set through a mapping.

Here, the notion of image of a subset must not be mistaken with that of digital image of Section 1.1.

Exercise 1.18 (properties of the image of a mapping) – Let f be a mapping from A to B. Prove that for any subsets X_1 and X_2 of a set A,

$$X_1 \subset X_2 \Rightarrow f(X_1) \subset f(X_2) \quad \text{and} \quad f(X_1 \cap X_2) \subset f(X_1) \cap f(X_2),$$

where $\cap$ denotes the intersection between subsets. □

Proof – Let us prove the first inclusion. For any b in B,

$$\begin{aligned} b \in f(X_1) &\Leftrightarrow \exists a \in X_1,\ f(a) = b && \text{(image definition)} \\ &\Rightarrow \exists a \in X_2,\ f(a) = b && \text{(hypothesis)} \\ &\Leftrightarrow b \in f(X_2). && \text{(image definition)} \end{aligned}$$

Let us prove the second inclusion. We observe that since $X_1 \cap X_2$ is a subset of X_1 and X_2 we can apply the above result twice, hence, for any b in B,

$$\begin{aligned} b \in f(X_1 \cap X_2) &\Rightarrow b \in f(X_1) \text{ and } b \in f(X_2) && \text{(above observation)} \\ &\Leftrightarrow b \in f(X_1) \cap f(X_2). && \text{(intersection definition)} \end{aligned}$$

□

Let f be a mapping from A to B, the *inverse image* of a subset Y of B through f is the subset of A denoted by $f^{-1}(Y)$ and defined by

$$f^{-1}(Y) \triangleq \{a \in A : f(a) \in Y\}.$$

Figure 1.23 shows the inverse image $f^{-1}(Y)$ of a subset Y through a mapping f.

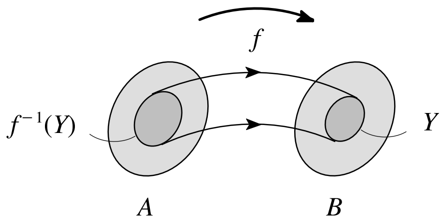


Fig. 1.23 – Image inverse of a set through a mapping.

Exercise 1.19 (inverse image property) – Let f be a mapping from A to B and let b be an element of B. Prove that if $a \in f^{-1}(\{b\})$ then $f(a) = b$. □

Some special mappings are called bijections.

Bijection and composition

A mapping f from A to B is *injective* or an *injection* iff for all $a_1 \in A$ and $a_2 \in A$, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$.

A mapping f from A to B is *surjective* or a *surjection* iff the image of A through f is B, i.e., $f(A) = B$.

A mapping f is *bijective* or a *bijection* iff it is injective **and** surjective.

The identity mapping is an example of bijection.

Injective and surjective mappings are dual concepts. This can be seen by using the concept of mapping composition.

Definition 1.20 (composite of two mappings) – Let f be a mapping from A to B, and let g be a mapping from B to C. The *composite of f and g* is the mapping from A to C, denoted by $g \circ f$ or gf, and given by, for any a of A,

$$(g \circ f)(a) \triangleq g(f(a)).$$ □

Figure 1.24 shows two graphical representations of the composite of two mappings.

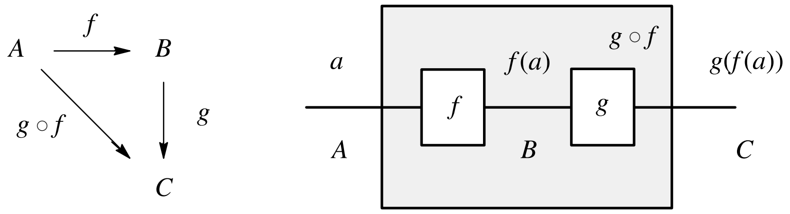


Fig. 1.24 – Composite of two mappings.

From the definition of composite of two mappings, we can defined an operation between appropriate mappings, that we call the *composition of mappings*. Figure 1.25 shows a graphical representation of the composition of mappings.

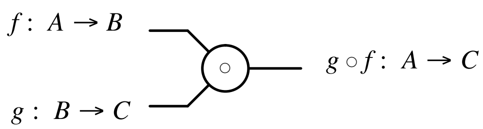


Fig. 1.25 – Composition of mappings.

The composition of mappings obeys the following laws. For any mappings f from A to B, g from B to C, and h from C to D,

$$h \circ (g \circ f) = (h \circ g) \circ f \qquad \text{(associativity)}$$

$$f \circ 1_A = f = 1_B \circ f. \qquad \text{(identity law)}$$

Exercise 1.21 (composition properties) – Prove the associativity and identity laws of the composition. □

Solution – Let us prove the associativity law. For any a in A,

$$\begin{aligned}
(h \circ (g \circ f)(a)) &= h((g \circ f)(a)) && \text{(composition definition)}\\
&= h(g(f(a))) && \text{(composition definition)}\\
&= (h \circ g)(f(a)) && \text{(composition definition)}\\
&= ((h \circ g) \circ f)(a). && \text{(composition definition)}
\end{aligned}$$

Let us prove the identity law. For any a in A,

$$\begin{aligned}
(f \circ 1_A)(a) &= f(1_A(a)) && \text{(composition definition)}\\
&= f(a) && \text{(identity mapping definition)}\\
&= 1_A(f(a)) && \text{(identity mapping definition)}\\
&= (1_A \circ f)(a). && \text{(composition definition)}
\end{aligned}$$

□

Figure 1.26 shows three graphical representations of the composition associativity.

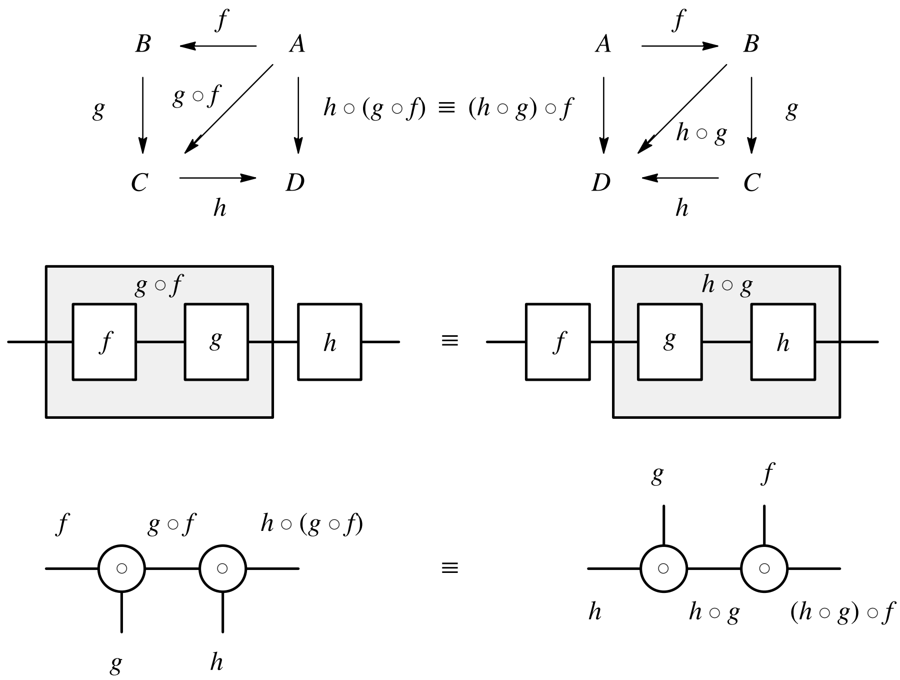


Fig. 1.26 – Composition associativity.

The mapping composition is useful to define the dual concepts of left and right inverse mapping.

Definition 1.22 (left and right inverse mapping) – Let f be a mapping from A to B. A mapping g from B to A is a *left inverse of* f iff $g \circ f$ is the identity mapping on A, that is, $g \circ f = 1_A$. It is a *right inverse of* f iff $f \circ g$ is the identity mapping on B, that is, $f \circ g = 1_B$. □

The left side of Figure 1.27 shows a mapping g which is the left inverse of a mapping f, the right side of the figure shows a mapping g which is the right inverse of a mapping f.

The left and right inverse mapping concepts can be used to characterize the injections and surjections.

Proposition 1.23 (first characterization of the injections and surjections) – A mapping is an injection iff it has a left inverse, and a surjection iff it has a right inverse. □

For the proof, see Mac Lane & Birkhoff (1967, Theorem 1, p.9).

Alternatively, the injections and surjections can be characterized by using the concepts of image and inverse image through a mapping.

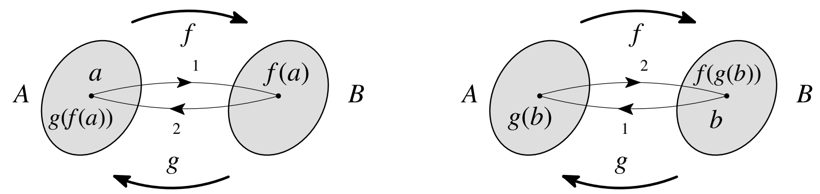


Fig. 1.27 – Left and right inverses of a mapping.

Proposition 1.24 (second characterization of the injections and surjections) – A mapping f from A to B is an injection iff, for any subset X of A, $f^{-1}(f(X)) = X$, and a surjection iff, for any subset Y of B, $f(f^{-1}(Y)) = Y$. □

Proposition 1.24 is another way to see that the injections and the surjections are dual concepts.

Exercise 1.25 (second characterization of the injections and surjections) – By using Proposition 1.23, prove Proposition 1.24. □

Exercise 1.26 (surjection property) – Let f be a surjection from A to B, and let X be a subset of A and Y a subset of B. Prove that, if $f(X) \subset Y$ and $f^{-1}(Y) \subset X$ then $f(X) = Y$.□

We are now ready to introduce and characterize the notion of inverse mapping.

Definition 1.27 (inverse mapping) – Let f be a mapping from A to B. The mapping g from B to A is a *two sided–inverse*, or simply an *inverse*, *of* f iff g is the left **and** right inverse of f. □

Proposition 1.28 (characterization of the bijections) – Let f be a mapping from A to B. The following statements are equivalent:

1. f is a bijection;

2. f has a left and a right inverse;

3. f has an inverse (a two–sided inverse).

When this is the case, any two inverses (left, right or two–sided) of f are equal. This unique inverse of f, denoted by f^{-1}, is a bijection from B to A and

$$(f^{-1})^{-1} = f.$$ □

For the proof, see Mac Lane & Birkhoff (1967, Corollary, p. 10).

The last statement says that if g is the inverse of f, then f is the inverse of g. In this case, we say that f and g are *mutually inverse*.

Exercise 1.29 (bijection example) – Let b_1 be a bijection from A to B, let f be a mapping from B to C and let b_2 be a bijection from C to D. Prove that the mapping F from C^B to D^A defined by, for any f of C^B,

$$F(f) \triangleq b_2 \circ f \circ b_1,$$

is a bijection and that its inverse G is defined by, for any g of D^A,

$$G(g) \triangleq {b_2}^{-1} \circ g \circ {b_1}^{-1}. \qquad \square$$

Exercise 1.30 (bijection property) – Let f be a bijection from A to B, and let X_1 and X_2 be two subsets of A. Prove that $f(X_1 \cap X_2) = f(X_1) \cap f(X_2)$. □

If f is a bijection from A to B, then the image of any subset Y of B through its inverse f^{-1} is equal to the inverse image of Y through f. This justifies the unique notation $f^{-1}(X)$ for both concepts.

Proposition 1.31 (composition of bijections) – Let f be a bijection from A to B and let g be a bijection from B to C, then $g \circ f$ is a bijection from A to C and its inverse is the mapping $f^{-1} \circ g^{-1}$ from C to A. □

For the proof, see Mac Lane & Birkhoff (1967, p. 10).

At this point, we can introduce the graph–of–a–mapping characterization in terms of mappings.

Let G be a graph–of–a–mapping on A to B. For any a of A and b of B, we denote by $f_G(a)$ the expression defined by

$$f_G(a) \triangleq b \text{ iff } a\, G\, b. \tag{1.1}$$

Let f be a mapping from A to B. We denote by G_f the binary relation, between A and B, defined by

$$G_f \triangleq \{(a,b) \in A \times B : \; b = f(a)\}. \tag{1.2}$$

Proposition 1.32 (graph–of–a–mapping characterization in terms of mappings) – The mapping $G \mapsto f_G$ from $\mathbb{G}(A \times B)$ to B^A is a bijection, its inverse is the mapping $f \mapsto G_f$. □

Proof – Let us divide the proof into four main steps.

1. f_G is a mapping from A to B since, G being a graph–of–a–mapping, for each a we have one and only one b such that $a\,G\,b$, or equivalently, such that $f_G(a) = b$.

2. f being a mapping from A to B, for each a of A, there is one and only one b such that $f(a) = b$, or equivalently, such that (a, b) belongs to G_f, therefore G_f is a graph–of–a–mapping on A to B.

3. Let G be a graph–of–a–mapping on A to B,

$$\begin{aligned} G_{f_G} &= \{(a,b) \in A \times B : \; b = f_G(a)\} && \text{(definition of } G_f\text{)} \\ &= \{(a,b) \in A \times B : \; a\,G\,b\} && \text{(definition of } f_G\text{)} \\ &= G && \text{(binary relation definition)} \end{aligned}$$

that is $f \mapsto G_f$ is a left inverse of $G \mapsto f_G$. Therefore, by Proposition 1.23, $G \mapsto f_G$ is an injection.

4. Let f be a mapping from A to B, for any a in A,

$$\begin{aligned} f_{G_f}(a) &= b \quad \text{iff} \quad a\, G_f\, b && \text{(definition of } f_G) \\ &= b \quad \text{iff} \quad b = f(a) && \text{(definition of } G_f) \\ &= f(a) \end{aligned}$$

that is, by mapping equality, $f_{G_f} = f$. Consequently $f \mapsto G_f$ is a right inverse of $G \mapsto f_G$. Therefore, by Proposition 1.23, $G \mapsto f_G$ is a surjection.

Hence, by definition, $G \mapsto f_G$ is a bijection and by Proposition 1.28 its inverse is $f \mapsto G_f$. □

G_f is called the *graph of the mapping f* and can be read "is assigned by f to".

For example, if $b = f(a)$, then a "is assigned by f to" b.

Figure 1.28 illustrates Proposition 1.32.

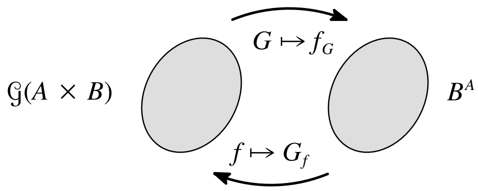


Fig. 1.28 – Graph–of–a–mapping characterization.

Returning to the binary relations, we can defined two more classes.

Equivalence and partial ordering

When a binary relation is on A to itself, we say that the binary relation is *on A*.

A binary relation R on A is

reflexive iff $a\,R\,a$ for any a in A,

antireflexive iff $a\,R\,a$ for no a in A,

symmetric iff $a\,R\,b$ implies that $b\,R\,a$ for any a and b in A,

antisymmetric iff $a\,R\,b$ and $b\,R\,a$ implies that $a = b$ for any a and b in A,

strictly antisymmetric iff $a\,R\,b$ and $b\,R\,a$ for no a and b in A,

transitive iff $a\,R\,b$ and $b\,R\,c$ implies that $a\,R\,c$ for any a, b and c in A.

antitransitive iff $a\,R\,c$ implies that $a\,R\,b$ and $b\,R\,c$ for no b in A, and any a and c in A.

For the antireflexivity definition see also Birkhoff, 1967, p. 20. The strict antisymmetry and the antitransitivity are original definitions.

Let A be the points of a plan in which we have chosen an axe which we interpret as showing the upward direction. We respect to this direction we can define a binary relation on A called "*is below*". We verify that this relation is strictly antisymmetric and transitive. If a binary relation is strictly antisymmetric and transitive we says that it is of the type "*is below*".

Exercise 1.33 (antireflexivity) – Prove that if a relation is strictly antisymmetric then it is antireflexive. □

Exercise 1.34 (strict antisymmetry) – Prove that if a relation is antireflexive and transitive then it is strictly antisymmetric. □

Solution – Let R be a binary relation on A. For any a and b in A, by transitivity of R, if the assertion $a\ R\ a$ is false then the assertion $a\ R\ b$ and $b\ R\ a$ is false. By antireflexivity of R, the assertion $a\ R\ a$ is always false, consequently for such R, the assertion $a\ R\ b$ and $b\ R\ a$ is always false, in other words, the assertion $a\ R\ b$ and $b\ R\ a$ is true for no a and b in A. □

Based on some of these properties, let us define two classes of binary relation. The first class is that of equivalence relation.

Definition 1.35 (equivalence relation) – Let A be a non–empty set. A binary relation on A which is reflexive, symmetric and transitive is called an *equivalence relation*. □

The equality between elements of a set is an equivalence relation. The binary relation $\cong$ on the "set" whose elements are sets, defined by $a \cong b$ iff "there exists a bijection from a to b" is an equivalence relation.

The second class of binary relation is that of partial ordering or inclusion. Later on, this binary relation will be used to introduce the Mathematical Morphology.

Definition 1.36 (partial ordering relation or inclusion) – Let A be a non–empty set. A binary relation on A which is reflexive, antisymmetric and transitive is called a *partial ordering relation* or *inclusion relation*. □

A partial ordering is usually denoted by $\leq$ and it is read "*is less than or equal to*" or "*is contained in*". Hence if $a \leq b$ we say that a "*is less than or equal to*" b or a "*is contained in*" b.

Exercise 1.37 (mutually converse partial orderings) – Let R_1 and R_2 be two mutually converse relations. Prove that R_1 is a partial ordering iff R_2 is a partial ordering. □

The partial ordering concept leads up to the algebraic structure of partially ordered set and chain.

Poset and chain

Definition 1.38 (partially ordered set) – A *partially ordered set*, or *poset* for short, is a non–empty set on which a binary relation is defined which is a partial ordering. If A is the set and $\leq$ the partial ordering, then the poset is denoted by $(A, \leq)$. □

The set $\mathbf{N}$ of natural numbers with its usual partial ordering $\leq$ is a poset.

Definition 1.39 (chain) – A poset $(A, \leq)$ is a *chain* iff for all a and b belonging to A, either $a \leq b$ or $b \leq a$. □

The poset $(\mathbf{N}, \leq)$ of natural numbers is a chain.

Any *subset X of a poset* is itself a poset under the same ordering relation (restricted to X) (Birkhoff, 1967, Theorem 1, p.2). Furthermore, any subset of a chain is a chain.

For example, the subset $\{1, 2\}$ of $\mathbf{N}$ under the usual partial ordering $\leq$ is a chain.

A subset X of a poset A is a (*closed*) *interval* iff there exist two elements a and b in A such that x belongs to X iff $a \leq x \leq b$. The elements a and b are elements of X, are unique by the antisymmetry of $\leq$, and are called, respectively, the *left* and *right extremities* of X. The subset X is denoted by $[a, b]$.

From two posets we can create another poset as shown below.

The *direct product of two posets* $(A, \leq)$ and $(B, \leq)$ is the Cartesian product $A \times B$ with the binary relation $\underset{\times}{\leq}$ defined by, for any a and a' of A, and b and b' of B:

$$(a, b) \underset{\times}{\leq} (a', b') \quad \text{iff} \quad a \leq a' \text{ and } b \leq b'.$$

The direct product of two posets is a poset. But the direct product of two chain is not necessarily a chain.

For example, the direct product of $(\{1, 2\}, \leq)$ by itself is a poset but not a chain for we have neither $(1, 2) \underset{\times}{\leq} (2, 1)$ nor $(2, 1) \underset{\times}{\leq} (1, 2)$.

Important examples of chain are the ordinal numbers.

Ordinal number and matrix

Let n be a non zero natural number (Mac Lane & Birkhoff 1967, p.35), the interval $\mathbf{n} \triangleq [1, n]$ of natural numbers with the usual partial ordering $\leq$ is a chain called *ordinal number* (Birkhoff 1967, in Theorem 5, p.5). We shall often use $\mathbf{n}$ instead of $(\mathbf{n}, \leq)$ if there is no danger of misunderstanding. That is, in this case, $\mathbf{n}$ is an ordinal number.

A non–empty set A is *finite* if for some natural number n there exists a bijection from $\mathbf{n}$ to A, that is, $A \cong \mathbf{n}$. The natural number n is the number of elements of A, we denote it by $\#A$.

The direct product of two ordinal numbers $\mathbf{m}$ and $\mathbf{n}$, that we shall denote by $\mathbf{m} \times \mathbf{n}$ if there is no danger of misunderstanding, is an example of poset.

Let m and n be two non zero natural numbers, and let K be any non–empty set. A mapping from $\mathbf{m} \times \mathbf{n}$ to K is called a *$m \times n$ matrix with entries in K*. If A is a matrix, its entry at (i, j) is denoted by a_{ij}. Conversely, if a_{ij} denotes the entry at (i, j) of a matrix, then the matrix is denoted by $[a_{ij}]$.

When the entries are in the set $\mathbf{R}$ of real numbers, the matrix is called *real*. A $1 \times n$ matrix is called a *row matrix*, and a $n \times 1$ *column matrix*.

Finite poset are conveniently graphically represented by its Hasse diagram.

Hasse diagram

In order to “visualize” a finite poset $(A, \leq)$, we draw its graph (Szász, 1971, p. 8) or *Hasse diagram* (Heijmans, 1994). To define such diagram it is worthful to introduce two new binary relations on A which are one to one related with the partial orderings. We call them strict inclusion and covering.

Definition 1.40 (strict inclusion) – Let A be a non–empty set. A binary relation on A which is antireflexive and transitive is called a *strict inclusion relation*. □

A strict inclusion is usually denoted by $<$ and it is read "*is less than*" or "*is properly contained in*". Hence if $a < b$ we say that a "*is less than*" b or a "*is properly contained in*" b.

Definition 1.41 (covering) – Let A be a non–empty set. A binary relation on A which is strictly antisymmetric and antitransitive is called a *covering relation*. □

A covering is usually denoted by $\prec$ and it is read "*covered by*", its converse is to be read "*covers*". Hence if $a \prec b$ we say that a "*is covered by*" b or b "*covers*" a (Szász, 1971, p. 7).

Exercise 1.42 (strict inclusion property) – Prove that a strict inclusion is strictly antisymmetric. □

Solution – The property is a consequence of Exercise 1.34. □

Exercise 1.43 (strict inclusion property) – Let $<$ be a strict inclusion on a finite set A. Prove that, for any a and b in A,

$$\exists x : a < x < b \;\Leftrightarrow\; \begin{array}{c} \exists (x_i) : a < x_1 < \ldots < x_n < b \\ \text{and} \\ a < y_1 < x_1 < \ldots < x_n < y_{n+1} < b \;\text{ for no } (y_i) \end{array}.$$ □

Solution – Let x be one of the x_i, then by transitivity of $<$, the second statement implies the first one. To prove that the first statement implies the second one, we give below an algorithm to compute the x_i from x.

$$x_1 \leftarrow \text{FIRST}(a, x)$$

$$\text{NEXT}(x_1, 1)$$

where the procedures FIRST and NEXT are defined below.

$$\begin{array}{c} \text{FIRST}(x, y) \\ \exists ? s : x < s < y \\ \text{yes : return FIRST}(x, s) \\ \text{no : return } y \end{array} \qquad \begin{array}{c} \text{NEXT}(x, i) \\ y \leftarrow \text{FIRST}(x, b) \\ y ? = x \\ \text{no : } x_i \leftarrow y; \text{NEXT}(y, i+1) \\ \text{yes : return} \end{array}$$

The procedure FIRST(x, y) returns the element z, if it exists, such that $x < z < y$ and there is no s such that $x < s < z$, otherwise it returns y. The procedure NEXT$(x, 1)$ finds the elements $x_2, \ldots x_n$, if they exist, such that $x < x_2 < \ldots < x_n < b$. □

We now state the relationships between partial ordering, strict inclusion and covering.

Let us consider first the relationship between partial ordering and strict inclusion (Birkhoff, 1967, Lemma 1).

Let $\leq$ be a partial ordering on A. For any a and b in A, we denote by $a <_{\leq} b$ the expression given by

$$a <_{\leq} b \overset{\Delta}{\Leftrightarrow} a \neq b \text{ and } a \leq b.$$

Let $<$ be a strict inclusion on A. For any a and b in A, we denote by $a \leq_< b$ the expression given by

$$a \leq_< b \overset{\Delta}{\Leftrightarrow} a = b \text{ or } a < b.$$

Proposition 1.44 (characterization of the partial orderings in terms of strict inclusions) – The mapping $\leq \mapsto <_\leq$ from the set of partial orderings to the set of strict inclusions is a bijection and its inverse is $< \mapsto \leq_<$. □

Proof – Let us divide the proof into four main steps.

1. Let $\leq$ be a partial ordering on A, then $<_\leq$ is a binary relation on A which is a strict inclusion. The antireflexivity comes from the antireflexivity of the relation $\neq$. The transitivity comes from the transitivity of $\leq$.

2. Let $<$ be a strict inclusion on A, then $\leq_<$ is a binary relation on A which is a partial ordering. The reflexivity comes from the reflexivity of the relation $=$ and the transitivity comes from the transitivity of $=$ and $<$. The antisymmetry comes from the symmetry of $=$ and the strict antisymmetry of $<$. This last assertion, for example, can be proved in the following way. For any a and b in A,

$$\begin{array}{ccll}
\begin{array}{c} a \leq_< b \\ \text{and} \\ b \leq_< a \end{array} & \Leftrightarrow & \begin{array}{c} a = b \text{ or } a < b \\ \text{and} \\ b = a \text{ or } b < a \end{array} & \text{(definition of } \leq_<) \\[2ex]
 & \Leftrightarrow & \begin{array}{c} a = b \text{ and } b = a \\ \text{or} \\ a = b \text{ and } b < a \\ \text{or} \\ a < b \text{ and } b = a \\ \text{or} \\ a < b \text{ and } b < a \end{array} & \text{(distributivity of “and” over “or”)} \\[2ex]
 & \Leftrightarrow & a = b \text{ and } b = a & \text{(strict antisymmetry of } <) \\
 & \Leftrightarrow & a = b. & \text{(symmetry of } =)
\end{array}$$

3. Let $\leq$ be a partial ordering on A. For any a and b in A,

$$\begin{array}{ccll}
a \leq_{<_\leq} b & \Leftrightarrow & a = b \text{ or } a <_\leq b & \text{(definition of } \leq_<) \\
 & \Leftrightarrow & a = b \text{ or } (a \neq b \text{ and } a \leq b) & \text{(definition of } <_\leq) \\
 & \Leftrightarrow & \begin{array}{c} a = b \text{ or } a \neq b \\ \text{and} \\ a = b \text{ or } a \leq b \end{array} & \text{(distributivity of “or” over “and”)} \\
 & \Leftrightarrow & a = b \text{ or } a \leq b & (a = b \text{ or } a \neq b \text{ is always true}) \\
 & \Leftrightarrow & a \leq b, & \text{(reflexivity of } \leq)
\end{array}$$

that is, $< \mapsto \leq_<$ is a left inverse of $\leq \mapsto <_\leq$. Therefore, by Proposition 1.23 $\leq \mapsto <_\leq$ is injective.

4. Let $<$ be a strict inclusion on A. For any a and b in A,

$$\begin{aligned}
a <_{\leq_<} b &\Leftrightarrow a \neq b \text{ and } a \leq_< b && \text{(definition of } <_{\leq}\text{)}\\
&\Leftrightarrow a \neq b \text{ and } (a = b \text{ or } a < b) && \text{(definition of } \leq_<\text{)}\\
&\Leftrightarrow \begin{array}{c} a \neq b \text{ and } a = b \\ \text{or} \\ a \neq b \text{ and } a < b \end{array} && \text{(distributivity of “and” over “or”)}\\
&\Leftrightarrow a \neq b \text{ and } a < b && (a = b \text{ and } a \neq b \text{ is always false})\\
&\Leftrightarrow a < b, && \text{(antireflexivity of } <\text{)}
\end{aligned}$$

that is, $< \mapsto \leq_<$ is a right inverse of $\leq \mapsto <_\leq$. Therefore, by Proposition 1.23 $\leq \mapsto <_\leq$ is surjective.

Hence by definition, $< \mapsto \leq_<$ is a bijection and by Proposition 1.28 its inverse is $< \mapsto \leq_<$. □

Let us consider now the relationship between strict inclusion and covering.

Let $<$ be a strict inclusion on A. For any a and b in A, we denote by $a \prec_< b$ the expression given by

$$a \prec_< b \overset{\Delta}{\Leftrightarrow} a < b \text{ and } a < x < b \text{ for no } x.$$

Let $\prec$ be a covering on a finite set A. For any a and b in A, we denote by $a <_\prec b$ the expression given by

$$a <_\prec b \overset{\Delta}{\Leftrightarrow} a \prec b \text{ or } \exists (x_i) : a \prec x_1 \prec \ldots \prec x_n \prec b.$$

Proposition 1.45 (characterization of the strict inclusions in terms of the coverings) – Let A be a finite set. The mapping $< \mapsto \prec_<$ from the set of strict inclusions on A to the set of coverings on A is a bijection and its inverse is $\prec \mapsto <_\prec$. □

Proof – Let us divide the proof into four main steps.

1. Let $<$ be a strict inclusion on A, then $\prec_<$ is a binary relation on A which is a covering. The strict antisymmetry comes from the strict antisymmetry of $<$ (see Exercise 1.42). The antitransitivity comes from the transitivity of $\leq$. This last assertion can be proved in the following way. For any a and b in A,

$$\begin{aligned}
a \prec_< b &\Leftrightarrow a < b \text{ and } a < x < b \text{ for no } x && \text{(definition of } \prec_<\text{)}\\
&\Rightarrow a < x < b \text{ for no } x && \text{(transitivity of } <\text{)}\\
&\Rightarrow a < y < x < z < b \text{ for no } x, y \text{ and } z && \text{(a fortiori)}\\
&\Leftrightarrow a \prec_< x \prec_< b \text{ for no } x. && \text{(definition of } \prec_<\text{)}
\end{aligned}$$

2. Let $\prec$ be a covering on A, then $<_\prec$ is a binary relation on A which is a strict inclusion. The antireflexivity comes from the antireflexivity of the relation $\prec$ (see Exercise 1.33). The transitivity comes from the fact that if $a <_\prec b$ and $b <_\prec c$ then there always exists a family of element of A (this family contains at least the element b) in between a and c (in the sense of $\prec$), that is $a <_\prec c$.

3. Let $<$ be a strict inclusion on A. For any a and b in A,

$$a <_{\prec_<} b \Leftrightarrow a \prec_< \text{ or } \exists(x_i) : a \prec_< x_1 \prec_< \ldots \prec_< x_n \prec_< b \qquad \text{(definition of } <_{\prec}\text{)}$$

$$\Leftrightarrow \begin{array}{c} a < b \text{ and } a < x < b \text{ for no } x \\ \text{or} \\ \exists(x_i) : a < x_1 < \ldots < x_n < b \\ \text{and} \\ a < y_1 < x_1 < \ldots < x_n < y_{n+1} < b \text{ for no } (y_i) \end{array} \qquad \text{(definition of } \prec_<\text{)}$$

$$\Leftrightarrow \begin{array}{c} a < b \text{ or } \exists(x_i)\ldots \\ \text{and} \\ a < x < b \text{ for no } x \text{ or } \exists(x_i)\ldots \end{array} \qquad \text{(distributivity of “or” over “and”)}$$

$$\Leftrightarrow \begin{array}{c} a < b \text{ or } \exists x : a < x < b \\ \text{and} \\ a < x < b \text{ for no } x \text{ or } \exists x : a < x < b \end{array} \qquad \text{(Exercise 1.43)}$$

$$\Leftrightarrow (a < b \text{ or } \exists x : a < x < b) \text{ and TRUE} \qquad \text{(incompatible statements)}$$

$$\Leftrightarrow a < b \text{ or } \exists x : a < x < b \qquad \text{(TRUE is the unit element of “and”)}$$

$$\Leftrightarrow a < b, \qquad \text{(the second term implies the first one)}$$

that is, $\prec \mapsto <_{\prec}$ is a left inverse of $< \mapsto \prec_<$. Therefore, by Proposition 1.23 $< \mapsto \prec_<$ is injective.

4. Let $\prec$ be a covering on A. For any a and b in A,

$$a \prec_{<_\prec} b \Leftrightarrow a <_\prec b \text{ and } a <_\prec x <_\prec b \text{ for no } x \qquad \text{(definition of } \prec_<\text{)}$$

$$\Leftrightarrow \begin{array}{c} a \prec b \text{ or } \exists(x_i) : a \prec x_1 \prec \ldots \prec x_l \prec b \\ \text{and} \\ (a \prec x \text{ or } \exists(y_j) : a \prec y_1 \prec \ldots \prec y_m \prec x \\ \text{and} \\ x \prec b \text{ or } \exists(z_k) : x \prec z_1 \prec \ldots \prec z_n \prec b) \text{ for no } x \end{array} \qquad \text{(definition of } <_\prec\text{)}$$

$$\Leftrightarrow \begin{array}{c} a \prec b \text{ or } \exists(x_i) : a \prec x_1 \prec \ldots \prec x_l \prec b \\ \text{and} \\ (a \prec x \prec b \\ \text{or} \\ \exists(y_j) : a \prec y_1 \prec \ldots \prec y_m \prec x \prec b \\ \text{or} \\ \exists(z_k) : a \prec x \prec z_1 \prec \ldots \prec z_n \prec b \\ \text{or} \\ \exists(y_j)(z_k) : a \prec y_1 \prec \ldots \prec x \prec z_1 \prec \ldots \prec z_n \prec b) \text{ for no } x \end{array} \qquad \text{(distribut.)}$$

$$\Leftrightarrow \begin{array}{c} a \prec b \text{ and } (a \prec x \ \ldots) \text{ for no } x \\ \text{or} \\ \exists(x_i) : a \prec x_1 \prec \ldots \prec x_l \prec b \text{ and } (a \prec x \ \ldots) \text{ for no } x \end{array} \qquad \text{(distribut.)}$$

$$\Leftrightarrow \begin{array}{c} a \prec b \\ \text{or} \\ \exists(x_i) : a \prec x_1 \prec \ldots \prec x_l \prec b \text{ and } (a \prec x \ \ldots) \text{ for no } x \end{array} \qquad \text{(antitransitiv.)}$$

$$\Leftrightarrow a \prec b \text{ or FALSE}, \qquad \text{(incompatible statements)}$$

$$\Leftrightarrow a \prec b, \qquad \text{(FALSE is the unit element of “or”)}$$

that is, $\prec \mapsto <_\prec$ is a right inverse of $< \mapsto \prec_<$. Therefore, by Proposition 1.23 $< \mapsto \prec_<$ is surjective.

Hence by definition, $< \;\mapsto\; \prec_{<}$ is a bijection and by Proposition 1.28 its inverse is $\prec \;\mapsto\; <_{\prec}$. □

By combining the above both characterizations we can state a new characterization of the partial orderings in terms of the coverings.

Let $\leq$ be a partial ordering on A. For any a and b in A, we denote by $a \prec_{\leq} b$ the expression given by

$$a \prec_{\leq} b \overset{\Delta}{\Leftrightarrow} a \neq b \text{ and } a \leq b \text{ and } a \leq x \leq b \text{ for no } x \neq a \text{ or } b.$$

Let $\prec$ be a covering on a finite set A. For any a and b in A, we denote by $a \leq_{\prec} b$ the expression given by

$$a \leq_{\prec} b \overset{\Delta}{\Leftrightarrow} a = b \text{ or } a \prec b \text{ or } \exists(x_i) : a \prec x_1 \prec \ldots \prec x_n \prec b.$$

Proposition 1.46 (characterization of the partial orderings in terms of the coverings) – Let A be a finite set. The mapping $\leq \;\mapsto\; \prec_{\leq}$ from the set of strict inclusions on A to the set of coverings on A is a bijection and its inverse is $\prec \;\mapsto\; \leq_{\prec}$. □

Exercise 1.47 (characterization of the partial orderings in terms of the coverings) – Prove Proposition 1.46. Hint: prove that $\leq \;\mapsto\; \prec_{\leq}$ is the composition of $\leq \mapsto <_{\leq}$ and $< \mapsto \prec_{<}$, and apply Propositions 1.44, 1.45 and 1.31. □

Figure 1.29 illustrates the characterizations given in Propositions 1.44 and 1.46.

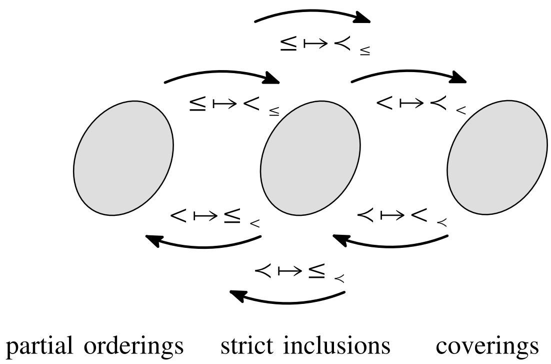


Fig. 1.29 – Partial ordering characterizations.

If $\leq$, $<$ and $\prec$ are, respectively, a partial ordering, a strict inclusion and a covering on the same set A and are in a mutual one–to–one relationship, then we have,

$$a \prec b \;\Rightarrow\; a < b \;\Rightarrow\; a \leq b,$$

in other words,

$$\prec \subset < \subset \leq.$$

From Exercise 1.34, we observe that a strict inclusion is a binary relation of the type "*is below*".

If $(A, \leq)$ is a finite poset then we call $(A, \prec_{\leq})$ its *graph*.

Since $<_{\leq}$ is of the type "*is below*", the graph of a finite poset $(A, \leq)$ can be represented graphically by drawing its elements as small circles, placing a below b whenever $a <_{\leq} b$, and drawing a line segment connecting them, whenever $a \prec_{\leq} b$. The resulting figure is called the *Hasse diagram of* $(A, \leq)$. Figure 1.30 shows the Hasse diagrams of four posets.

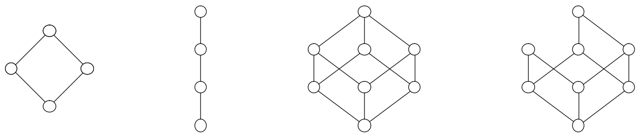


Fig. 1.30 – Hasse diagrams of four posets.

For example, the first Hasse diagram in this figure is the graph of the direct product of $(\{1, 2\}, \leq)$ by itself if we named the small circles as in Figure 1.31.

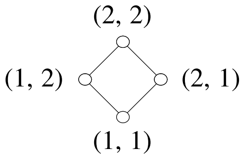


Fig. 1.31 – Hasse diagram of the direct product of a two–elements chain.

Exercise 1.48 (partial order of a poset and covering of the graph of a poset) – Draw the table given the partial order of the poset which graph is shown in Figure 1.31. Draw the table given the covering of its graph. □

Solution – The left table in Figure 1.32 represents the partial order of the poset which graph is shown in Figure 1.31. The right table in Figure 1.32 represents the covering of its graph.

	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
$(1,1)$	×	×	×	×
$(1,2)$		×		×
$(2,1)$			×	×
$(2,2)$				×

	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
$(1,1)$		×	×	
$(1,2)$				×
$(2,1)$				×
$(2,2)$				

Fig. 1.32 – Partial order and covering.

□

Exercise 1.49 (partial order of a poset and covering of the graph of a poset) – Draw the table given the partial order of the poset which graph is the third one (from left to right) shown in Figure 1.30. Draw the table given the covering of its graph. Hint: give a letter to each of the small circles. □

Returning to the bijection and poset notions, we recall the notion of poset–isomorphism.

Poset–isomorphism

We now give the definitions of increasing and decreasing mappings, and of poset–isomorphism and of poset–dual–isomorphism.

Let $(A, \le)$ and $(B, \le)$ be two posets. A mapping f is *increasing* (or *isotone*, or a *poset–morphism*) iff for any a and a' in A,

$$a \le a' \Rightarrow f(a) \le f(a');$$

it is *decreasing* (or *antitone*) iff for any a and b in A_1,

$$a \le a' \Rightarrow f(a') \le f(a).$$

Definition 1.50 (poset–isomorphism and poset–dual–isomorphism) – Let $(A, \le)$ and $(B, \le)$ be two posets. A *poset–isomorphism f from A to B* is a bijection which is order–preserving, that is, f and its inverse are increasing. A *poset–dual–isomorphism f from A to B* is a bijection which is order–reversing, that is, f and its inverse are decreasing. □

Two posets are *poset–isomorphic* iff there exists a poset–isomorphism between them.

The binary relation "is poset–isomorphic to" defined on the set whose elements are posets is an equivalence relation.

Figure 1.33 shows two bijections. On the left hand side the bijection is a poset–isomorphism. On the right hand side the bijection is **not** a poset–isomorphism.

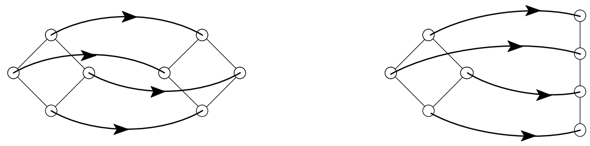


Fig. 1.33 – Poset–isomorphism and none poset–isomorphism.

Exercise 1.51 (equivalent definition of a poset–isomorphism) – Let $(A, \le)$ and $(B, \le)$ be two posets. Prove that a bijection f from A to B is a poset isomorphism iff for any a and a' in A,

$$a \le a' \Leftrightarrow f(a) \le f(a').$$ □

Proposition 1.52 (finite chain) – Any finite chain with n elements is poset–isomorphic to the ordinal number **n**. □

For the proof see Birkhoff (1967, Theorem 5, p.5); Mac Lane & Birkhoff (1967, Proposition 2, p.483).

There is only one way to build a poset–isomorphism between two isomorphic finite chains. In other words, this isomorphism is unique.

Chapter 2

Image moments

We will define the moments of an image or, equivalently (see Chapter 1), of a matrix by the moments of its associated random variable. This will lead to the concept of mean and variance of an image, of covariance matrix of a matrix of images, and finally to the principal component analysis.

First we will show how an image can be associated with a real function and consequently be seen as a vector in a linear vector space.

2.1 Image as a vector

We begin by introducing the abstract concept of real image. Here, the word **real** comes from the mathematical concept of real numbers and as we will see through its definition, a real image may not exist from the point of view of the real world.

Let f be an image from E to K, where K is an interval of the natural numbers, for example the interval $[0, p - 1]$ considered in Chapter 1. Since K is a subset of the set $\mathbf{R}$ of real numbers, we can associate to f the mapping g_f from E to $\mathbf{R}$ defined by $g_f(x) \triangleq f(x)$, for any x in E. The mapping g_f is an extension of f (see Section 2.4). We call g_f the *real extension of the image f.*

Figure 2.1 illustrates the real extension of an image.

We call a mapping from E to $\mathbf{R}$ a *real image* on E, or simply an image when no misunderstanding is likely to arise.

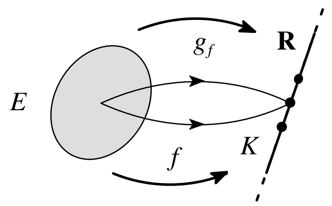


Fig. 2.1 – Real extension of an image.

A real image g on E is the real extension of an image from E to K iff $g(E)$ is included in K. In practice, "few" real images will satisfy this condition. This is why we say that a real image may not exist from the real world point of view.

We will now extend some of the properties of the real numbers to the real images and we will show that they may be viewed as vectors in a linear vector space.

By using the same notation as we did in Chapter 1 for a mapping set, we denote by $\mathbf{R}^E$ the set of real images with domain in E. By introducing appropriate operations on the real images, we can structure $\mathbf{R}^E$ as a linear vector space.

Let us introduce the following three operations on $\mathbf{R}^E$.

The *image addition*, denoted by $+$, is the extension to $\mathbf{R}^E$ of the addition $+$ on $\mathbf{R}$. If f and g are two images in $\mathbf{R}^E$, then $f + g$ is called their *sum*.

The *image multiplication by a scalar*, denoted by $\cdot$, is the external extension to $\mathbf{R}^E$ of the multiplication on $\mathbf{R}$. If α is a real number and f is an image in $\mathbf{R}^E$, then $\alpha . f$ is called *product of the image f by the scalar α*.

It is important to observe that the above operations do not guarantee, when applied to real images which are the extensions of images from E to K, producing another real image which is the extension of an image from E to K. That is, in general, the operation result, let say h, will **not** satisfy the condition $h(E)$ being included in K.

The *black image selection*, denoted by o, is the extension to $\mathbf{R}^E$ of the nullary operation on $\mathbf{R}$ which select the real value 0. This operation selects the *black image*, that is, the constant image, $x \mapsto 0$.

Figure 2.2 shows through block diagrams the above three operations on real images of size 2 by 2. In this figure we do not make distinction between the matrix from $\mathbf{2} \times \mathbf{2}$ to K and the numerical array of size 2 by 2.

We give without proof the following proposition. Actually, the proof uses the results of Propositions 2.46 and 2.47.

Proposition 2.1 (linear vector space of images) – The set $\mathbf{R}^E$ equipped with the internal binary operation of image addition $+$, the external binary operation of image multiplication by a scalar $\cdot$ and the nullary operation o (which select the black image), denoted by $(\mathbf{R}^E, +, \cdot, o)$, is linear vector space over the field $\mathbf{R}$. □

Because of Proposition 2.1, a real image is called a vector.

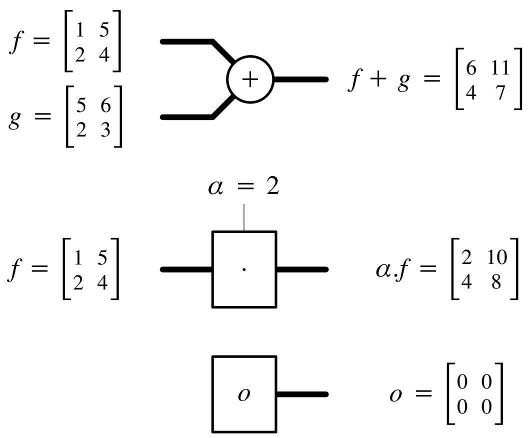


Fig. 2.2 – Operations on images.

The negative of f in $\mathbf{R}^E$, denoted by $-f$, is called the *negative image of f*.

Exercise 2.2 (negative image) – For any f in $\mathbf{R}^E$, prove that $-f$ is the real image $x \mapsto -f(x)$. Hint: show that $(-f)(x) = -f(x)$ for any x in E. □

For any f and g in $\mathbf{R}^E$, the *image difference of f* and g, denoted by $f - g$, is the image in $\mathbf{R}^E$ given by

$$f - g \triangleq f + (-g).$$

In order to introduce an inner product on the real images, we need to define one more operation on them as well as the concept of sum of an image.

The *image multiplication*, denoted by $\cdot$, is the extension to $\mathbf{R}^E$ of the multiplication on $\mathbf{R}$. If f and g are two images in $\mathbf{R}^E$, then $f \cdot g$ is called their *product*. Figure 2.3 shows through a block diagram the product of two images of size 2 by 2. Even though the images can be seen as matrices, the above definition of product should not be confused with the traditional matrix product.

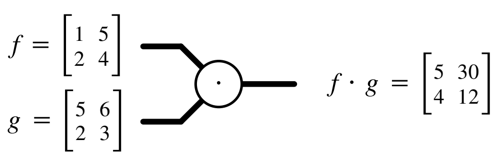


Fig. 2.3 – Multiplication on images.

For any f in $\mathbf{R}^E$ (E being a finite set), the *sum of the image* f, denoted by $\sum f$, is the element of $\mathbf{R}$ given by

$$\sum f \triangleq \sum_{x \in E} f(x).$$

In Section 4.4, the mapping $f \mapsto \sum f$ from $\mathbf{R}^E$ to $\mathbf{R}$ is called a reduction.

By combining the above two notions we can define the following expression involving two images. For any f and g in $\mathbf{R}^E$, let $< f, g >$ be the element of $\mathbf{R}$ given by

$$< f, g > \triangleq (\textstyle\sum f \cdot g)/\#E. \tag{2.1}$$

We give without proof the following proposition.

Proposition 2.3 (image inner product) – For any f and g in $\mathbf{R}^E$, $< f, g >$ is an inner product of f and g. □

The scalar $< f, g >$ is called the *inner product of the images f and* g, and $\| f \|$ the *norm of f.*

The linear vector space $\mathbf{R}^E$ equipped with the operation $(f, g) \mapsto < f, g >$ is the Euclidian vector space (see Section 2.4) that we will work with.

Exercise 2.4 (image inner product) – Show that, for the images f and g given in Figure 2.2, we have $< f, g > = 51/4$. □

The images f and g are said *orthogonal* iff $< f, g > = 0$. We observe that any image f is orthogonal to the black image o.

Exercise 2.5 (image norm) – Show that, for the images f and g given in Figure 2.2, we have $\| f \| = \sqrt{23/2}$ and $\| g \| = \sqrt{37/2}$. □

Let denote by i the constant image $x \mapsto 1$. We have $\| i \| = 1$.

2.2 Image as a random variable

A real image f in $\mathbf{R}^E$ can be seen as a simple random variable with respect to a random experiment that consists of randomly picking, with the constant probability $1/\#E$, any pixel of f. From this point of view, we can define the first moments of an image (Banon, 1992a). These moments will be expressed in terms of the image inner product defined by expression (2.1).

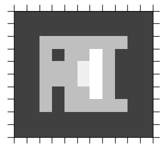
Definition 2.6 (mean of an image) – For any f in $\mathbf{R}^E$, the *mean of the image* f, denoted by $\mathrm{m}(f)$, is the element of $\mathbf{R}$ given by

$$\mathrm{m}(f) \triangleq < f, i > . \qquad \square$$

The mean (or average) of an image is a measure of its brightness. The greater the mean the brighter the image. Figure 2.4 shows two images with different means, the right-hand image as a greater mean and appears brighter compare to the left-hand one.

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&1&1&1&1&1&1&0&0\\ 0&0&1&0&1&1&3&1&0&0&0\\ 0&0&1&1&1&2&3&1&0&0&0\\ 0&0&1&1&1&2&3&1&0&0&0\\ 0&0&1&0&1&1&3&1&0&0&0\\ 0&0&1&0&1&1&1&1&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} \qquad \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1\\ 1&1&2&2&2&2&2&2&2&1&1\\ 1&1&2&1&2&2&3&2&1&1&1\\ 1&1&2&2&2&3&3&2&1&1&1\\ 1&1&2&2&2&3&3&2&1&1&1\\ 1&1&2&1&2&2&3&2&1&1&1\\ 1&1&2&1&2&2&2&2&2&1&1\\ 1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1 \end{bmatrix}$$

f $\qquad$ g



$\mathrm{m}(f) = 0.41$

$\mathrm{m}(g) = 1.41$

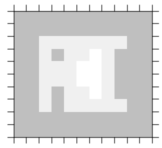


Fig. 2.4 – Two images with different means.

Exercise 2.7 (mean of an image) – Show that, for the images f and g given in Figure 2.2, we have $\mathrm{m}(f) = 3$ and $\mathrm{m}(g) = 4$. □

We give without proof the following mean properties.

Proposition 2.8 (mean properties) – For any f and g in $\mathbf{R}^E$ and $\alpha \in \mathbf{R}$, we have

$$\mathrm{m}(i) = 1,$$
$$\mathrm{m}(f + g) = \mathrm{m}(f) + \mathrm{m}(g),$$
$$\mathrm{m}(\alpha . f) = \alpha \mathrm{m}(f). \qquad \square$$

The mean is the first moment. To define the centralized second moment, we need to introduce the following definition.

Definition 2.9 (centralized version of an image) – For any f in $\mathbf{R}^E$, the *centralized version of the image* f, denoted by $\tilde{f}$, is the image in $\mathbf{R}^E$ given by

$$\tilde{f} \triangleq f - \mathrm{m}(f) . i. \qquad \square$$

We give without proof the following properties of the centralized version of an image.

Proposition 2.10 (centralized version of an image) – For any f and g in $\mathbf{R}^E$ and $\alpha \in \mathbf{R}$, we have

$$\mathrm{m}(\tilde{f}) = 0,$$
$$\tilde{i} = o,$$
$$\tilde{\tilde{f}} = \tilde{f},$$
$$\widetilde{f + g} = \tilde{f} + \tilde{g},$$
$$\widetilde{\alpha . f} = \alpha . \tilde{f}. \qquad \square$$

Exercise 2.11 (centralized version of an image) – Show that, for the images f and g given in Figure 2.2, we have $\tilde{f} = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}$ and $\tilde{g} = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$. □

From the definition of the mean and the first statement of Proposition 2.10 we observe that $\tilde{f}$ and i are orthogonal. Furthermore, from the definition of the centralized version, we have $f = \tilde{f} + \mathrm{m}(f) \cdot i$. In other words, any image f can be decomposed in terms of the sum of two orthogonal images: $\tilde{f}$ and $\mathrm{m}(f) \cdot i$. This observation is illustrated in Figure 2.5. In this figure orthogonal images are represented by orthogonal vectors.

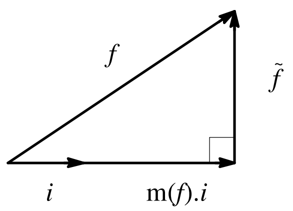


Fig. 2.5 – The decomposition of an image.

The images f and g are said *uncorrelated* iff $\tilde{f}$ and $\tilde{g}$ are orthogonal, otherwise, they are said *correlated*. In particular, from the second statement of Proposition 2.10, any image f is uncorrelated with i, and more generally to any constant images. We observe that the unique images that are uncorrelated with themselves are the constant images, the image i is an example.

As we can see from the above observation, the mathematical concept of correlation is not intuitive for the case of constant images. Actually, it is meaningful for the non constant images.

We can now define the notion of covariance of two images and that of variance of an image.

Definition 2.12 (covariance of two images) – For any f and g in $\mathbf{R}^E$, the *covariance of the images f and g*, denoted by $\mathrm{cov}(f, g)$, is the element of $\mathbf{R}$ given by

$$\mathrm{cov}(f, g) \triangleq \, < \tilde{f}, \tilde{g} > . \qquad \square$$

From the above definition, we see that the images f and g are uncorrelated iff their covariance is zero, that is, $\mathrm{cov}(f, g) = 0$.

We observe that $\mathrm{cov}(f, g)$ is also an inner product of f and g. With respect to it, f and g are uncorrelated iff f and g are orthogonal (and no more $\tilde{f}$ and $\tilde{g}$). Nevertheless, we will continue to use $< f, g >$ as inner product of f and g.

Proposition 2.13 (equivalent definition of the covariance) – For any f and g in $\mathbf{R}^E$, we have

$$\mathrm{cov}(f, g) = \, < f, g > - \, \mathrm{m}(f)\mathrm{m}(g). \qquad \square$$

Exercise 2.14 (covariance of two images) – Show that, for the images f and g given in Figure 2.2, we have $\mathrm{cov}(f, g) = 3/4$. Use both, Definition 2.12 and the equivalent definition given in Proposition 2.13. □

Definition 2.15 (variance of an image) – For any f in $\mathbf{R}^E$, the *variance of the image* f, denoted by $\mathrm{var}(f)$, is the element of $\mathbf{R}$ given by

$$\mathrm{var}(f) \triangleq \|\tilde{f}\|^2. \qquad \square$$

We observe that $\mathrm{var}(f)$ is $\mathrm{cov}(f, f)$. For any f in $\mathbf{R}^E$, the *standard deviation of the image* f, denoted by $\mathrm{sd}(f)$, is the element of $\mathbf{R}$ given by

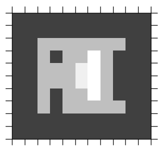
$$\mathrm{sd}(f) \triangleq \|\tilde{f}\|,$$

that is the standard deviation of f is the square root of the variance of f.

The variance of an image is a measure of its contrast. The greater the variance the higher the image contrast. Figure 2.6 shows two images with different variances, the right-hand image has a smaller variance and appears with less contrast compared to the left-hand one.

$$\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&1&1&1&1&0&0\\
0&0&1&0&1&1&3&1&0&0&0\\
0&0&1&1&1&2&3&1&0&0&0\\
0&0&1&1&1&2&3&1&0&0&0\\
0&0&1&0&1&1&3&1&0&0&0\\
0&0&1&0&1&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
\qquad
\begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&2&1&1&1&1\\
1&1&1&1&1&2&2&1&1&1&1\\
1&1&1&1&1&2&2&1&1&1&1\\
1&1&1&1&1&1&2&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1
\end{bmatrix}$$

f $\qquad$ g



$\mathrm{var}(f) = 0.50$
$\mathrm{sd}(f) = 0.71$

$\mathrm{var}(g) = 0.05$
$\mathrm{sd}(g) = 0.23$

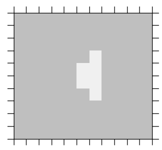


Fig. 2.6 – Two images with different variances.

Nevertheless, the above comment about contrast should be applied carefully. Let $\alpha > 1$, then $\alpha.f$ should have a better contrast than f. But this does not mean that we may increase arbitrarily α to get a better contrast. We must recall that $\alpha.f$ is a real image only, and it may not exist from the point of view of the real world.

We give without proof the following variance properties.

Proposition 2.16 (variance properties) – For any f and g in $\mathbf{R}^E$ and $\alpha \in \mathbf{R}$, we have

$$\begin{aligned} \mathrm{var}(f) &= \mathrm{var}(-f), \\ \mathrm{var}(f) &= \|f\|^2 - \mathrm{m}(f)^2, \qquad \text{(equivalent definition)} \\ \mathrm{var}(i) &= 0, \\ \mathrm{var}(f+g) &= \mathrm{var}(f) + 2\mathrm{cov}(f,g) + \mathrm{var}(g), \\ \mathrm{var}(\alpha{\cdot}f) &= \alpha^2\,\mathrm{var}(f). \end{aligned}$$ □

We observe from the fourth statement of Proposition 2.16 that if f and g are uncorrelated then the variance of the sum is the sum of the variances, that is,

$$\mathrm{var}(f+g) = \mathrm{var}(f) + \mathrm{var}(g).$$

Exercise 2.17 (variance property) – Using the result of Exercise 2.42 and Proposition 2.10, prove that, for any f and g in $\mathbf{R}^E$, and α and β in $\mathbf{R}$,

$$\mathrm{var}(\alpha{\cdot}f + \beta{\cdot}g) = \alpha^2\,\mathrm{var}(f) + 2\alpha\beta\mathrm{cov}(f,g) + \beta^2\,\mathrm{var}(g).$$ □

Exercise 2.18 (variance of an image) – Show that, for the images f and g given in Figure 2.2, we have $\mathrm{var}(f) = \mathrm{var}(g) = 5/2$. Use both, Definition 2.15 and the equivalent definition given in Proposition 2.16. □

Instead of indicating simply that two images are uncorrelated or not, we can define a correlation coefficient.

Definition 2.19 (correlation coefficient of two images) – For any f and g in $\mathbf{R}^E$ such that $\mathrm{var}(f)$ and $\mathrm{var}(g)$ are not zero, the *correlation coefficient of the images f and g*, denoted by $\rho_{f,g}$, is the element of $\mathbf{R}$ given by

$$\rho_{f,g} \triangleq \frac{\mathrm{cov}(f,g)}{\sqrt{\mathrm{var}(f)\mathrm{var}(g)}}.$$ □

From the Cauchy–Schwarz inequality (cf. Proposition 2.41), we observe that

$$-1 < \rho_{f,g} < 1.$$

Furthermore, $\rho_{f,g}$ is zero iff f and g are uncorrelated.

Exercise 2.20 (correlation property) – Let f be in $\mathbf{R}^E$ and let i be the constant image $x \mapsto 1$ in $\mathbf{R}^E$. Prove that $\rho_{f,f+i} = 1$. □

Exercise 2.21 (sufficient condition of linear independence) – Show that a set of two images f and g, such that $\mathrm{var}(f)$ and $\mathrm{var}(g)$ are not zero, is linearly independent if their correlation coefficient is not ± 1. (Hint: develop the expressions $\mathrm{cov}(\alpha{\cdot}f + \beta{\cdot}g, f)$ and $\mathrm{cov}(\alpha{\cdot}f + \beta{\cdot}g, g)$, and make use of the property that the black image is uncorrelated with any image, i.e, $\mathrm{cov}(o,f) = 0$ for any image f. □

Exercise 2.22 (correlation of two images) – Show that, for the images f and g given in Figure 2.2, we have $\rho_{f,g} = 3/10$. Show that the set of the images f and g is linearly independent. □

When we deal with a family of images, it is useful to use the notion of covariance matrix. We now denote by F a $1 \times n$ matrix with entry in $\mathbf{R}^E$, that is F is a row matrix whose elements are real images.

Definition 2.23 (covariance matrix) – Let F be a $1 \times n$ matrix with entry in $\mathbf{R}^E$, the *covariance matrix of the row matrix of images F*, denoted by Σ_F, is the $n \times n$ real valued matrix $\mathbf{R}$ given by

$$\Sigma_F \triangleq [\mathrm{cov}(f_i, f_j)]. \qquad \square$$

We observe that any covariance matrix is symmetric and the elements of the principal diagonal are the variances of the images in the family, and the matrix is diagonal iff the images in the family are two by two uncorrelated. From a family of images having a given covariance matrix, one can derive a new family having a diagonal covariance matrix by applying the so–called Principal Component Analysis (see Section 2.3).

Exercise 2.24 (covariance matrix of two images) – Show that, for the images f and g given in Figure 2.2, we have $\Sigma_{[f\,g]} = \frac{1}{4}\begin{bmatrix} 10 & 3 \\ 3 & 10 \end{bmatrix}$. $\square$

The moments of an image can be alternatively derived from the histogram of an image.

Definition 2.25 (histogram of an image) – For any f in K^E, the *histogram of the image f*, denoted by $\mathrm{H}(f)$, is the mapping from K to $\{0, 1/\#E, 2/\#E, ..., 1\}$ given by

$$\mathrm{H}(f)(i) \triangleq \frac{\#\{x \in E : f(x) = i\}}{\#E} \quad (i \in K). \qquad \square$$

We can rewrite the expression of the histogram of f as

$$\mathrm{H}(f)(i) = \frac{1}{\#E} \sum 1_i \circ f, \tag{2.2}$$

where, for any $i \in K$, 1_i is the mapping from K to K given by

$$1_i(s) \triangleq \begin{cases} 1 & \text{if } \; i = s \\ 0 & \text{otherwise} \end{cases} \quad (s \in K). \tag{2.3}$$

Figure 2.7 shows an image f and the binary component $1_i \circ f$ when i equal 2.

$$\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&1&1&1&1&0&0\\
0&0&1&0&1&1&3&1&0&0&0\\
0&0&1&1&1&2&3&1&0&0&0\\
0&0&1&1&1&2&3&1&0&0&0\\
0&0&1&0&1&1&3&1&0&0&0\\
0&0&1&0&1&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
\qquad
\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

f $\qquad$ $1_2 \circ f$

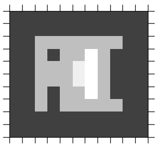
Fig. 2.7 – An image and one of its binary component.

The value $\mathrm{H}(f)(i)$ can be seen as the probability of picking a pixel from f with value i in the random experiment mentioned above.

Figure 2.8 shows the graph of the histogram of an image.

$$\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&1&1&1&1&0&0\\
0&0&1&0&1&1&3&1&0&0&0\\
0&0&1&1&1&2&3&1&0&0&0\\
0&0&1&1&1&2&3&1&0&0&0\\
0&0&1&0&1&1&3&1&0&0&0\\
0&0&1&0&1&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

f



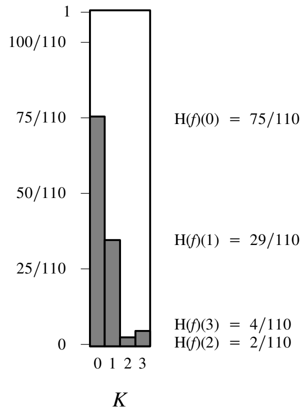


Fig. 2.8 – The histogram of an image.

Exercise 2.26 (histogram of an image) – Assuming that $K = [0, 7] \subset \mathbf{Z}$ is the gray–scale of the images f and g given in Figure 2.2, find the histograms of f and g, and draw their graphs. □

Proposition 2.27 (histogram properties) – For any f in K^E, we have

$$\sum_{i \in K} \mathrm{H}(f)(i) = 1,$$

$$\mathrm{m}(f) = \sum_{i \in K} i\mathrm{H}(f)(i),$$

$$\mathrm{var}(f) = \sum_{i \in K} (i - \mathrm{m}(f))^2 \mathrm{H}(f)(i). \quad \square$$

Proof – Let us prove the statement relative to the mean. For any f in K^E

$$\begin{aligned}
\sum_{i \in K} i\mathrm{H}(f)(i) &= \sum_{i \in K} i \frac{1}{\#E} \sum 1_i \circ f, && \text{(histogram, Exp. 2.2)}\\
&= \sum_{i \in K} i \frac{1}{\#E} \sum_{x \in E} 1_i(f(x)), && \text{(composition def.)}\\
&= \frac{1}{\#E} \sum_{x \in E} \sum_{i \in K} i 1_i(f(x)), && \text{(sum properties)}\\
&= \frac{1}{\#E} \sum_{x \in E} \sum_{i \in K} f(x) 1_i(f(x)), && \text{(Exp. 2.3)}\\
&= \frac{1}{\#E} \sum_{x \in E} f(x) \sum_{i \in K} 1_i(f(x)), && \text{(sum properties)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\#E}\sum_{x\in E} f(x), && \text{(mapping definition)}\\
&= \frac{1}{\#E}\sum_{x\in E} f(x)i(x), && \text{(definition of } i\text{)}\\
&= \frac{1}{\#E}\sum f\cdot i, && \text{(product definition)}\\
&= < f, i >. && \text{(Expression (2.1))}\\
&= \mathrm{m}(f). && \text{(Definition 2.6)}
\end{aligned}
$$

The other statements can be proved under similar arguments. □

Exercise 2.28 (histogram properties) – Using Proposition 2.27 and the histograms of Exercise 2.26, show that, for the images f and g given in Figure 2.2, we have $\mathrm{m}(f) = 3$ and $\mathrm{m}(g) = 4$. □

2.3 Principal component analysis

From a matrix of images defined on the same domain, we can derive a new matrix having interesting statistical properties by applying the so–called Principal Component Analysis (Anderson, 1966; Gnadesikan, 1977; Banon, 1992b).

Let F be a $1 \times p$ matrices with entries in $\mathbf{R}^E$, i.e., F is a row matrix of images ($F \in (\mathbf{R}^E)^{\mathbf{1}\times\mathbf{p}}$). We denote, by f_j its entry at $(1,j)$. When the f_j's are images of a same scene captured along different spectral bands, F is called a *multispectral image* and the f_j's are its *bands*.

We assume that the covariance matrix of F is not diagonal. That is, there exist at least two f_j's which are correlated.

An image g in $\mathbf{R}^E$ is a *linear combination of the* f_i's iff there exist p real numbers α_i's such that

$$g = \sum_{i\in\mathbf{p}} \alpha_i \cdot f_i.$$

Forming such linear combinations, we can construct new interesting images and matrices of images.

For example, we may want to find a linear combination which displays the best possible contrast. More precisely, we may look for α_i's such that $\mathrm{var}(\sum_{i\in\mathbf{p}} \alpha_i \cdot f_i)$ is maximum under the restriction $\sum_{i\in\mathbf{p}} \alpha_i^2 = 1$. A such linear combination is called a *first principal component of the row matrix F.* We denote it by g_1. We say **a** first principal component because $\mathrm{var}(g) = \mathrm{var}(-g)$ implies that the first principal components come in pairs.

Let us apply this idea to the row matrix $[f\,g]$ where f and g are the image of Figure 2.2. We will denote by $\alpha{\cdot}f + \beta{\cdot}g$ a linear combination of f and g. From Exercise 2.17 and recalling that the variances of f and g are equal, the variance of $\alpha{\cdot}f + \beta{\cdot}g$ is proportional to $\alpha^2 + \rho_{f,g}\alpha\beta + \beta^2$. Hence, the values of α and β which maximize the variance under the restriction $\alpha^2 + \beta^2 = 1$ are those which maximize the product $\alpha\beta$ under that restriction (we recall that $\rho_{f,g}$ is positive, see Exercise 2.22). A solution is $\alpha = \beta = 1/\sqrt{2}$. In other words, one of the best contrasted linear combinations of the images f and g, or one of the first principal components of $[f\,g]$ is $(1/\sqrt{2})(f + g)$. Figure 2.9 illustrates geometrically this solution. The rectangle of side α and β inscribed in the circle of radius 1 which has the greatest area is the square of side $1/\sqrt{2}$.

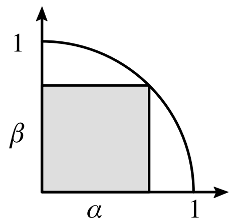


Fig. 2.9 – Optimal solution for the best contrast problem.

Continuing along the same reasoning, we may want to find a linear combination which displays the best possible contrast and which is uncorrelated with the above best contrasted linear combination g_1. More precisely, we may look for α_i's such that $\mathrm{var}(\sum_{i \in \mathbf{p}} \alpha_i{\cdot}f_i)$ is maximum under the restriction $\sum_{i \in \mathbf{p}} \alpha_i^2 = 1$, and $\sum_{i \in \mathbf{p}} \alpha_i{\cdot}f_i$ is uncorrelated with first principal component g_1. A such linear combination is called a *second principal component of the row matrix F.* We denote it by g_2.

If the f_i's form a linearly independent set of p vectors, then we can repeat the above construction to find the next principal components until the last one, that is a p^{th} principal component of the matrix F.

The images f and g of Figure 2.2 forming a linearly independent set of vectors (see Exercise 2.22), we can find a second (and last) principal component. Since this is the last one, we do not have to worry about finding the best contrast, we just have to find a linear combination $\alpha{\cdot}f + \beta{\cdot}g$ which is uncorrelated with the first component $(1/\sqrt{2})(f + g)$ and such that $\alpha^2 + \beta^2 = 1$.

Writing that the covariance between the first and the second component must be zero, we get that $\alpha + \beta = 0$. Since (α,β) must be a point of the circle of radius 1, a solution is $\alpha = -1/\sqrt{2}$ and $\beta = 1/\sqrt{2}$. In other words, one of the second principal components of $[f\,g]$ is $(1/\sqrt{2})(g - f)$. We observe that, in this example, we have never used explicitly the value of $\rho_{f,g}$.

Figure 2.10 shows the geometrical construction of a pair of principal components of $[f\,g]$ where f and g are two images with equal variances (as in Figure 2.2). In this figure uncorrelated images are represented by orthogonal vectors.

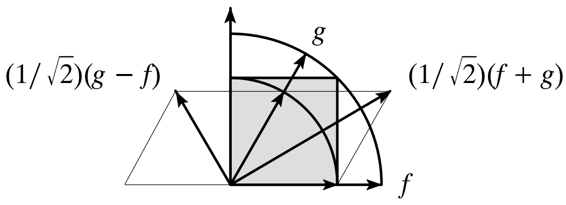


Fig. 2.10 – Principal component construction.

Let denote by $\sum_{i \in \mathbf{p}} \alpha_{ij} \cdot f_i$ a j^{th} principal component g_j and by G the row matrix of p principal components g_j of the row matrix f. The row matrix G is called the *Karhunen–Loève* or *Hotelling transform of the row matrix F.*

The problem of finding a set of p principal components, i.e., the problem of finding the p^2 α_{ij}'s has a very nice solution in Matrix Theory.

Let F and G be two $1 \times p$ matrices with entries in $\mathbf{R}^E$. In order to establish the next propositions we define, the following expression,

$$F^{t} \otimes G \triangleq [\operatorname{cov}(f_i, g_j)].$$

By construction, $F^{t} \otimes G$ is a $p \times p$ real matrix.

We observe that for any row matrix F, we have $F^{t} \otimes F = \Sigma_F$.

Proposition 2.29 (mixt associativity) – Let F and G be two $1 \times p$ matrices with entry in $\mathbf{R}^E$, and let A be a $p \times p$ real matrix, then

$$G^{t} \otimes (FA) = (G^{t} \otimes F)A \quad \text{and} \quad (A^{t}F^{t}) \otimes G = A^{t}(F^{t} \otimes G),$$

where FA is the usual matrix product of F by A. □

Proof – For any matrices $F = [f_k]$, $G = [g_i]$ and any $A = [\alpha_{kj}]$,

$$\begin{aligned} G^{t} \otimes (FA) &= [\operatorname{cov}(g_i, \textstyle\sum_k \alpha_{kj} \cdot f_k)] && \text{(product definitions)} \\ &= [\textstyle\sum_k \alpha_{kj} \operatorname{cov}(g_i, f_k)] && \text{(covariance properties)} \\ &= (G^{t} \otimes F)A. && \text{(product definitions)} \end{aligned}$$

The second equality can be proved in a similar manner. □

Proposition 2.30 (covariance matrix of a row matrix of linear combinations of images) – Let F be a matrix of p images and let A be a $p \times p$ real matrix, then

$$\Sigma_{FA} = A^{t}(\Sigma_F A)$$ □

Proof – For any matrices F and A,

$$\begin{aligned}\Sigma_{FA} &= (A^t F^t) \otimes (FA) && (\Sigma \text{ and } \otimes \text{ definitions})\\ &= A^t(F^t \otimes (FA)) && (\text{Proposition 2.29})\\ &= A^t((F^t \otimes F)A) && (\text{Proposition 2.29})\\ &= A^t(\Sigma_F A). && (\Sigma \text{ and } \otimes \text{ definitions})\end{aligned}$$

□

Exercise 2.31 (covariance matrix of a row matrix of linear combinations of images) – Let f and g be the images given in Figure 2.2, and let $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Using Exercise 2.24 and Proposition 2.30, show that we have $\Sigma_{[f\,g]A} = \frac{1}{4}\begin{bmatrix} 13 & 0 \\ 0 & 7 \end{bmatrix}$. □

Proposition 2.32 (principal component construction) – Let F be a row matrix of images. If A is an orthogonal matrix which diagonalizes Σ_F, then FA is a row matrix of principal components of F. □

Proof – Let F be any row matrix of p images. If A is an orthogonal matrix which diagonalizes Σ_F (we know that such matrix exists since Σ_F is symmetric, see Section 2.4), then, by Proposition 2.30, Σ_{FA} is diagonal. Hence, the images in FA are two by two uncorrelated. Furthermore, since A is orthogonal, $\sum_k \alpha_{kj}^2 = 1$ for $j = 1, 2, ..., p$. That is FA is a matrix of principal components of F. □

Since the diagonal elements of Σ_{FA} are the eigenvalues of Σ_F and the column vectors of A (in a basis $\mathcal{V}$) are the corresponding eigenvectors, the above proposition lead to the construction of a set of principal components. A **first** principal component of F is $F[\boldsymbol{a}]_{\mathcal{V}}^t$ where $\boldsymbol{a}$ is the eigenvector of Σ_F (i.e., the column vector of A in the basis $\mathcal{V}$) corresponding to the **greatest** eigenvalue of Σ_F. A **second** principal component is obtained by selecting the second greatest eigenvalue and so on.

Exercise 2.33 (principal component construction) – Let f and g be the images given in Figure 2.2. Verify that the matrix $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ diagonalizes the covariance matrix of $[f\,g]$. Applying Proposition 2.32, show that a matrix of principal components of $[f\,g]$ is $\left[\frac{1}{\sqrt{2}}\begin{bmatrix} 6 & 11 \\ 4 & 7 \end{bmatrix} \;\; \frac{1}{\sqrt{2}}\begin{bmatrix} 4 & 1 \\ 0 & -1 \end{bmatrix}\right]$. Which one is a first principal component of $[f\,g]$? Explain your answer. □

Some authors (Anderson, 1966, Introduction of Chapter 11) called also principal components the column vectors of A.

We call *energy of a row matrix of images*, $F = [f_k]$, the trace of Σ_F, that is the sum of the image variances, $\sum_k \mathrm{var}(f_k)$. Since the trace of Σ_F and the trace of Σ_{FA} are the equal, we can say that the Karhunen–Loève transform preserves the energy.

For example, the trace of the covariance matrix of $[f\, g]$, where f and g are the images of Figure 2.2, is $5/2 + 5/2 = 5$ (see Exercise 2.24); the trace of the covariance matrix of principal components of $[f\, g]$ is $13/4 + 7/4 = 5$ (see Exercises 2.33 and 2.31). We observe the energy conservation between both matrices. Whereas the energy in $[f\, g]$ is uniformly distributed over f and g, the same energy in the matrix of principal components of $[f\, g]$ in concentrated over the first principal component (65% of the total energy).

One of the possible application of the principal component analysis is compression. When the matrix of images is formed with highly correlated images, we can reach a good compression rate by keeping just the principal components with greatest variance.

Before ending this section, we would like to give a geometrical interpretation of the principal component analysis.

The row matrices of images that we have considered up to now are elements of the set $(\mathbf{R}^E)^{\mathbf{1}\times\mathbf{p}}$. We now consider the set $(\mathbf{R}^p)^E$ of the mappings from E to $\mathbf{R}^p$. Actually, we have $(\mathbf{R}^E)^{\mathbf{1}\times\mathbf{p}} \cong (\mathbf{R}^p)^E$. Let $f_1, ..., f_p$ be p images in $\mathbf{R}^E$ and let $F = [f_1\, f_2\, ...\, f_p]$, a bijection of interest from $(\mathbf{R}^E)^{\mathbf{1}\times\mathbf{p}}$ to $(\mathbf{R}^p)^E$ is $F \mapsto \boldsymbol{f}_F$, where $\boldsymbol{f}_F$ is defined by, for any x in E,

$$\boldsymbol{f}_F(x) \triangleq (f_1(x), f_2(x), ..., f_p(x)),$$

its inverse $\boldsymbol{f} \mapsto F_{\boldsymbol{f}}$ being defined by

$$F_{\boldsymbol{f}} \triangleq [f_1\, f_2\, ...\, f_p],$$

where the f_j's are defined by, for any x in E,

$$f_j(x) \triangleq (\boldsymbol{f}(x))_j^{\mathbb{E}} \quad \text{for any} \quad j = 1, 2, ..., p,$$

$(\boldsymbol{f}(x))_j^{\mathbb{E}}$ being the j^{th} coordinate of $\boldsymbol{f}(x)$ in the canonical basis $\mathbb{E}$ of $\mathbf{R}^p$.

For example, if f and g are the two images of Figure 2.2, then the above bijection assigns to the row matrix $[f\, g]$, the image $\boldsymbol{f}_{[f\,g]} = \begin{bmatrix} (1,5) & (5,6) \\ (2,2) & (4,3) \end{bmatrix}$.

The pixel values of an image in $(\mathbf{R}^p)^E$ are vectors in $\mathbf{R}^p$. When p is 2, these vectors can be represented graphically as points in the plane by choosing a pair of axes. Figure 2.11 shows in the plane the four values assumed by the pixels of $\boldsymbol{f}_{[f\,g]}$, f and g being the images of Figure 2.2.

For any $\boldsymbol{f}$ in $(\mathbf{R}^p)^E$ and any $p \times p$ real matrix A, we would like to compute the mapping that transforms the image $\boldsymbol{f}$ into $\boldsymbol{f}_{F_{\boldsymbol{f}}A}$. This mapping is illustrated in Figure 2.12.

The next proposition gives the solution for such computation.

Proposition 2.34 (pixel values transformation) – Let F be in $(\mathbf{R}^E)^{\mathbf{1}\times\mathbf{p}}$ and let A be a $p \times p$ real matrix, then, for any x in E,

$$[\boldsymbol{f}_{FA}(x)]_{\mathbb{E}} = [\boldsymbol{f}_F(x)]_{\mathbb{E}}\, A,$$

where $\mathbb{E}$ is the canonical basis of $\mathbf{R}^p$. □

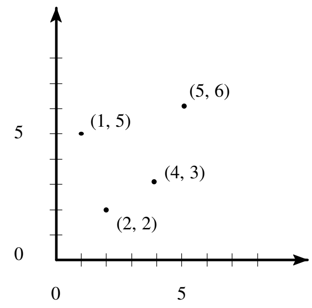


Fig. 2.11 – Pixel values of an image.

$$\begin{tikzcd}
(\mathbf{R}^p)^E \arrow[r, "\boldsymbol{f} \mapsto F_{\boldsymbol{f}}"] \arrow[d, "\boldsymbol{f} \mapsto \boldsymbol{f}_{F_{\boldsymbol{f}}A}"'] & (\mathbf{R}^E)^{\mathbf{1} \times \mathbf{p}} \arrow[d, "F \mapsto FA"] \\
(\mathbf{R}^p)^E & (\mathbf{R}^E)^{\mathbf{1} \times \mathbf{p}} \arrow[l, "F \mapsto \boldsymbol{f}_F"]
\end{tikzcd}$$

Fig. 2.12 – Image transform.

Proof – Let $c_1, c_2, ..., c_p$ be the column matrices of A and let $(FA)_i \triangleq Fc_i$ for $i \in \mathbf{p}$, then $FA = [(FA)_1, ..., (FA)_p]$. For any x in E, we have,

$$\begin{aligned}
[\boldsymbol{f}_{FA}(x)]_{\mathcal{E}} &= [(\boldsymbol{f}_{FA}(x))_j^{\mathcal{E}}] && \text{(definition of } [\cdot]_{\mathcal{V}}) \\
&= [((FA)_1(x), ..., (FA)_p(x))_j^{\mathcal{E}}] && \text{(definition of } \boldsymbol{f}_F) \\
&= [(FA)_j(x)] && \text{(Exercise 2.37)} \\
&= [Fc_j(x)] && \text{(definition of } (FA)_i) \\
&= [f_1(x), ..., f_p(x)][c_j] && \text{(extension definition)} \\
&= [f_1(x), ..., f_p(x)]\, A && \text{(definition of the } c_i\text{'s)} \\
&= [(f_1(x), ..., f_p(x))_i^{\mathcal{E}}]\, A && \text{(Exercise 2.37)} \\
&= [(\boldsymbol{f}_F(x))_i^{\mathcal{E}}]\, A && \text{(definition of } \boldsymbol{f}_F) \\
&= [\boldsymbol{f}_F(x)]_{\mathcal{E}}\, A. && \text{(definition of } [\cdot]_{\mathcal{V}})
\end{aligned}$$

□

Let $\boldsymbol{f}$ be an element of $(\mathbf{R}^p)^E$ and let A be any $p \times p$, recalling that $F \mapsto \boldsymbol{f}_F$ is a bijection, a consequence of Proposition 2.34 is, for any x in E,

$$[\boldsymbol{f}_{F_{\boldsymbol{f}}A}(x)]_{\mathcal{E}} = [\boldsymbol{f}(x)]_{\mathcal{E}}\, A. \tag{2.4}$$

Expression (2.4) shows that the mapping that transforms the image $\boldsymbol{f}$ into $\boldsymbol{f}_{F_f A}$ acts linearly on each pixel values according to matrix A.

Exercise 2.35 (transformed image) – Let $\boldsymbol{f} = \begin{bmatrix} (1,5) & (5,6) \\ (2,2) & (4,3) \end{bmatrix}$ and $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, using Expression (2.4) show that we have $\boldsymbol{f}_{F_f A} = \frac{1}{\sqrt{2}}\begin{bmatrix} (6,4) & (11,1) \\ (4,0) & (7,-1) \end{bmatrix}$. □

When A is an orthogonal matrix, the norm and the inner product of the pixel values are preserved (Cagnac et al., 1965, p. 41), that is, for any x and y in E,

$$\| \boldsymbol{f}(x) \| = \| \boldsymbol{f}_{F_f A}(x) \| \quad \text{and} \quad < \boldsymbol{f}(x), \boldsymbol{f}(y) > = < \boldsymbol{f}_{F_f A}(x), \boldsymbol{f}_{F_f A}(y) > .$$

When $p = 2$, the mapping that transforms the image $\boldsymbol{f}$ into $\boldsymbol{f}_{F_f A}$ acts on each pixel values like a rotation. Figure 2.13 illustrates the rotation effect on the pixel values. On the left part of the figure we have drawn a second pair of axes corresponding to the column vectors of A (obtained by using the canonical basis of $\mathbf{R}^2$): $(1/\sqrt{2})(1,1)$ and $(1/\sqrt{2})(-1,1)$ (by Proposition 2.51, we know that these vectors are orthonormal).

The linear transform characterized by A, rotates any vector (or point in the plane) in such a way that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ coincide, respectively, with the vectors $\boldsymbol{e}_1 = (1,0)$ and $\boldsymbol{e}_2 = (0,1)$ of the canonical basis of $\mathbf{R}^2$ (see Proposition 2.52). The right part of Figure 2.13 shows the pixel values of the transformed image, that is, the new pixel values after the rotation.

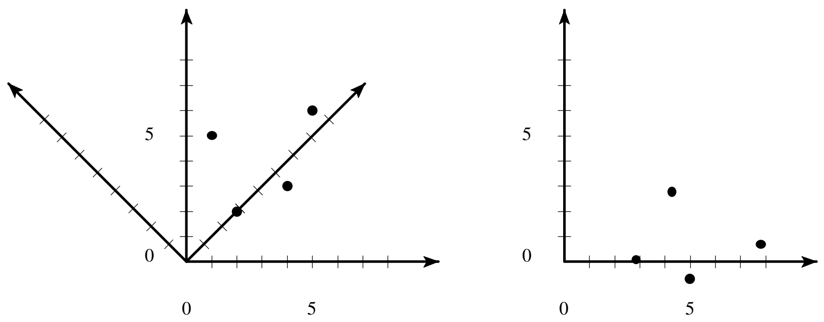


Fig. 2.13 – Pixel value rotation.

Finally, as a consequence of the orthogonal matrix properties, for any x in E, the coordinates of the pixel value $\boldsymbol{f}_{F_f A}(x)$ in the canonical basis of $\mathbf{R}^p$ are the coordinates of the pixel value $\boldsymbol{f}(x)$ in the basis formed by the column vectors of A (obtained by using the canonical basis of $\mathbf{R}^p$). This is illustrated again in the left part of Figure 2.13.

2.4 Some mathematical definitions and properties

To help the understanding of the definitions given in the previous sections, we recall some important mathematical definitions such as mapping restriction, operations, linear vector space, extended operations, and we recall some of their properties.

Mapping restriction

Let g be a mapping from U to V. Let A be a subset of U and B an element in $[g(A), V]$. A mapping f from A to B is said to be the *restriction of g to A and B* iff $f(a) = g(a)$ for any a in A. The mapping g is said to be an *extension of f* (generally, f may have more than one extension). We denote by $g/A, B$ the restriction of g to A and B; if $B = V$, this restriction is called simply restriction of g to A and is denoted by g/A.

Figure 2.14 shows a mapping $f: A \to B$ restriction of a mapping $g: U \to V$.

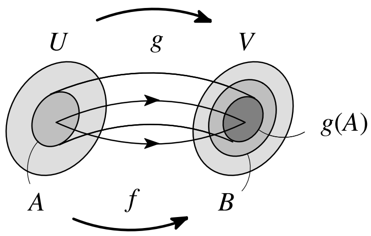


Fig. 2.14 – Mapping restriction.

A simple example of useful restriction mapping is the so–called insertion which is the restriction of the identity mapping (Mac Lane & Birkhoff, 1967, p. 5). More precisely, let A be a subset of U, the *insertion* $i_A: A \to U$ is the mapping $i_A \triangleq 1_U/A$.

Exercise 2.36 (relationship between restriction and composition) – By using the insertion mapping definition prove that for any mapping $g: U \to V$ and any subset A of U,

$$g/A = g \circ i_A. \qquad \square$$

From the notion of mapping we can recall the notion of operation.

Operations

Let B and C be two non–empty sets and let $I \triangleq \{1\}$ be the standard set with just one element.

A mapping from $C \times B$ to B is called a *binary operation* on B. If $C = B$, the operation is *internal* otherwise, it is *external* with operand in C.

A mapping from B to B is called a *unary operation* (or *operator*) on B.

A mapping from I to B is called a *nullary operation* on B. We are used to denoting by the same letter the nullary operation and the selected element in B. For example, let e be an element of B, then we denote by e the nullary operation $1 \mapsto e$.

Figure 2.15 illustrates by mean of block diagrams the above four types of operation. In this figure, the nullary operation is denoted e, the unary operation h, the internal binary operation $\square$ and the external binary operation $\cdot$.

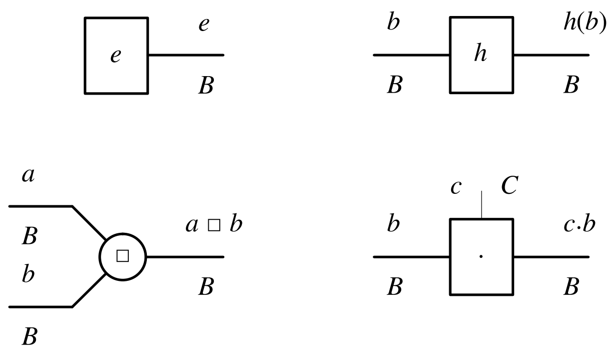


Fig. 2.15 – Nullary, unary and binary operations.

Let $\square$ be an internal binary operation on a set B. This operation is said *commutative* iff, for any a and b in B,

$$a \square b = b \square a$$

and it is said *associative* iff, for any a, b and c in B,

$$(a \square b) \square c = a \square (b \square c).$$

Figure 2.16 illustrates graphically the commutativity and associativity properties for a binary operation $\square$.

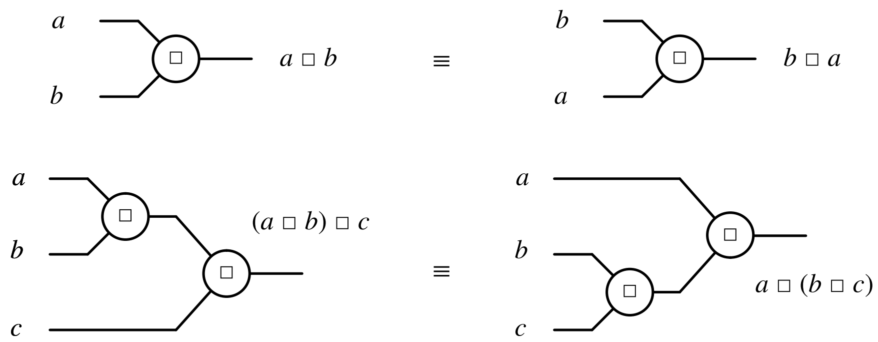


Fig. 2.16 – Commutativity and associativity of a binary operation.

An element u of B is a (two–sided) *unit element for the internal binary operation* $\square$, iff, for any b in B,

$$u \square b \;=\; b \;=\; b \square u.$$

If $\square$ has a unit element, then this element is **unique** since if e_1 and e_2 are two units, then

$$\begin{aligned} e_1 &= e_1 \square e_2 && (e_2 \text{ being an unit}) \\ &= e_2. && (e_1 \text{ being an unit}) \end{aligned}$$

Figure 2.17 illustrates graphically the existence of a unit element u for a binary operation $\square$.

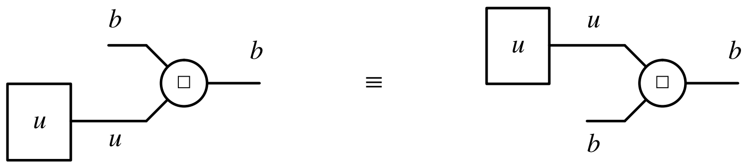


Fig. 2.17 – Unit element for a binary operation.

Let $*$ be another internal operation on the set B. This operation is said *distributive over the operation* $\square$ iff, for any a, b and c in B,

$$a * (b \square c) \;=\; (a * b) \square (a * c).$$

Figure 2.18 illustrates graphically the distributivity property over the binary operation $\square$ for a binary operation $*$.

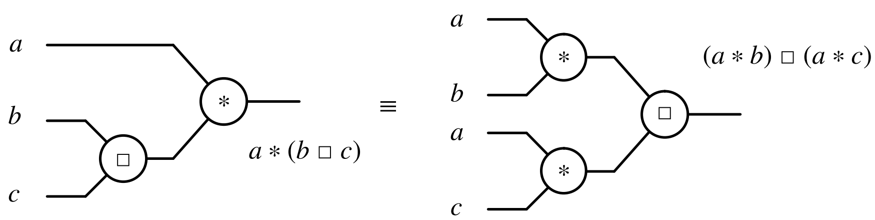


Fig. 2.18 – Distributivity of a binary operation over another one.

From the definitions of operation we can recall the axiomatic definition of linear vector space.

Linear vector space and inner product

Let $\mathbf{R}$ be the field of real numbers (with additive unit 0 and multiplicative unit 1). For the definition of a field, see for example Royden (1968, p. 29).

A set V equipped with an internal binary operation $+$, an external binary operation $\cdot$ with operand in $\mathbf{R}$ and a nullary operation $\boldsymbol{o}$ which satisfy the following properties (Cagnac et al.,1963; Royden, 1968, p. 181; Luenberger, 1969, p.12) for any $\boldsymbol{x}$, $\boldsymbol{y}$, and z in V, and α and β in $\mathbf{R}$,

$$\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x} \qquad \text{(commutativity)}$$

$$(\boldsymbol{x} + \boldsymbol{y}) + z = \boldsymbol{x} + (\boldsymbol{y} + z) \qquad \text{(associativity)}$$

$$\exists \boldsymbol{o} \in V,\ \boldsymbol{x} + \boldsymbol{o} = \boldsymbol{x} \qquad \text{(unit element)}$$

$$\alpha \cdot (\boldsymbol{x} + \boldsymbol{y}) = (\alpha \cdot \boldsymbol{x}) + (\alpha \cdot \boldsymbol{y}) \qquad \text{(distributivity over } + \text{ on } V)$$

$$(\alpha + \beta) \cdot \boldsymbol{x} = (\alpha \cdot \boldsymbol{x}) + (\beta \cdot \boldsymbol{x}) \qquad \text{(distributivity over } + \text{ on } \mathbf{R})$$

$$\alpha(\beta \cdot \boldsymbol{x}) = (\alpha\beta) \cdot \boldsymbol{x} \qquad \text{(mixt associativity)}$$

$$1 \cdot x = x \qquad \text{(unit element)}$$

is a *linear vector space over the field* $\mathbf{R}$ *of real numbers*. This algebraic system is denoted $(V, +, \cdot, \boldsymbol{o})$ or simply V when no confusion is likely to arise.

From the above axioms and the properties of the real numbers we can prove (Cagnac et al., 1963, p. 282), that $0 \cdot \boldsymbol{x} = \boldsymbol{o}$ and $\alpha \cdot \boldsymbol{o} = \boldsymbol{o}$.

We call $+$ addition and $\cdot$ multiplication by scalars. The elements of V are called *vectors*. The element $(-1) \cdot \boldsymbol{x}$ is called the *negative of* $\boldsymbol{x}$ and is denoted by $-\boldsymbol{x}$.

The set $\mathbf{R}$ of real numbers with their operations of addition and multiplication is a linear vector space over itself.

The set $\mathbf{R}^p$ of p–tuple of real numbers, with the extension to $\mathbf{R}^p$ of the addition on $\mathbf{R}$, the external extension to $\mathbf{R}^p$ of the multiplication on $\mathbf{R}$ and the nullary operation extension on $\mathbf{R}^p$ of the nullary operation 0 on $\mathbf{R}$, is a linear vector space on $\mathbf{R}$.

The addition, multiplication and nullary extensions we are referring to are recalled below.

If $\boldsymbol{a}$ is p–tuple, its i^{th} element is denoted by a_i. Conversely, if a_i denotes the i^{th} element of a p–tuple, then the p–tuple is denoted by (a_i). That is we have $\boldsymbol{a} = (a_i)$.

The addition $+$ on $\mathbf{R}^p$, extended from the addition on $\mathbf{R}$, is defined by, for any $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbf{R}^p$,

$$\boldsymbol{a} + \boldsymbol{b} \triangleq (a_i + b_i).$$

The external binary operation $\cdot$ on $\mathbf{R}^p$ with operand in $\mathbf{R}$, called scalar multiplication, extended from the multiplication on $\mathbf{R}$, is defined by, for any $\boldsymbol{a}$ in $\mathbf{R}^p$ and any α in $\mathbf{R}$,

$$\alpha \cdot \boldsymbol{a} \triangleq (\alpha a_i).$$

The nullary extension on $\mathbf{R}^p$ of the nullary operation 0 on $\mathbf{R}$, produces the p–tuple (0_i).

The set of vectors of the plane with their usual operations of addition and multiplication by scalars is a linear vector space over $\mathbf{R}$. Figure 2.19 illustrates these operations. Each arrow is a representative for a vector.

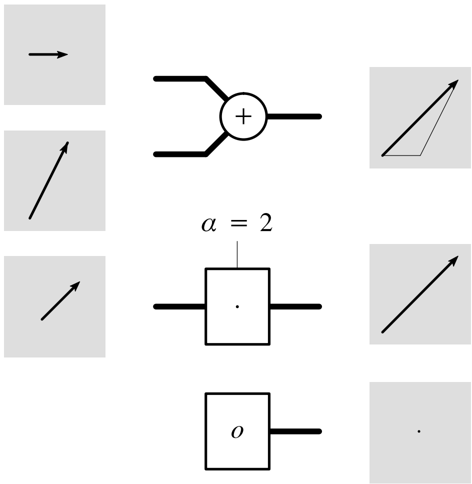


Fig. 2.19 – Operations on vectors of the plane.

A *linear combination of p vectors* $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_p$ in a linear vector space over **R** is a vector of the form $\alpha_1 \cdot \boldsymbol{v}_1 + \alpha_2 \cdot \boldsymbol{v}_2 + \ldots + \alpha_p \cdot \boldsymbol{v}_p$, where $\alpha_1, \alpha_2, \ldots, \alpha_p$ are any real numbers.

A set of p vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_p$ is *linearly independent* iff, for any α_i in **R**, $i = 1, 2, \ldots, p$,

$$\alpha_1 \cdot \boldsymbol{v}_1 + \alpha_2 \cdot \boldsymbol{v}_2 + \ldots + \alpha_p \cdot \boldsymbol{v}_p = o \Rightarrow (\alpha_1, \alpha_2, \ldots, \alpha_p) = (0, 0, \ldots, 0).$$

If a set of vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_p$ is **not** linearly independent, we say that it is *linearly dependent*, that is, for some α_i in **R**, $i = 1, 2, \ldots, p$,

$$(\alpha_1, \alpha_2, \ldots, \alpha_p) \neq (0, 0, \ldots, 0) \quad \text{and} \quad \alpha_1 \cdot \boldsymbol{v}_1 + \alpha_2 \cdot \boldsymbol{v}_2 + \ldots + \alpha_p \cdot \boldsymbol{v}_p = o\,.$$

A finite set of vectors in a linear vector space V over **R** is a *basis for V* iff they are linearly independent and any vector in V is a linear combination of them. We use the script letters, like $\mathcal{V}$, to denote a basis in a linear vector space.

If a linear vector space V has a basis, the α_i's that determine a vector $\boldsymbol{x}$ in V are unique (Cagnac et al., 1963, p. 291) and are called the *coordinates of* $\boldsymbol{x}$ *in the basis*.

For any $i = 1, 2, \ldots, p$, let $\boldsymbol{e}_i$ be the vector in $\mathbf{R}^p$ defined by,

$$\boldsymbol{e}_i \triangleq (0, \ldots, 0, 1, 0, \ldots, 0) \text{ with 1 at the } i^{\text{th}} \text{ position.}$$

The set of $\boldsymbol{e}_i$'s is an example of basis for $\mathbf{R}^p$, this basis is called the *canonical basis for* $\mathbf{R}^p$, and will be denoted $\mathcal{E}$.

Exercise 2.37 (p–tuple coordinates in the canonical basis) – Let $\boldsymbol{a}$ be a vector in the linear vector space $\mathbf{R}^p$, and let $\alpha_1, \alpha_2, \ldots, \alpha_p$ be its coordinates in the canonical basis. Prove that for any $i = 1, 2, \ldots, p$, $\alpha_i = a_i$. □

Solution – Under the assumptions,

$$
\begin{aligned}
(a_1, a_2, ..., a_p) &= \alpha_1 \cdot e_1 + \alpha_2 \cdot e_2 + ... + \alpha_p \cdot e_p && \text{(coordinate definition)} \\
&= \alpha_1 \cdot (1, 0, ..., 0) + ... + \alpha_p \cdot (0, 0, ..., 1) && \text{(def. of the } \boldsymbol{e}_i\text{'s)} \\
&= (\alpha_1, 0, ..., 0) + ... + (0, 0, ..., \alpha_p) && \text{(def. of the } \cdot) \\
&= (\alpha_1, \alpha_2, ..., \alpha_p), && \text{(definition of } +)
\end{aligned}
$$

that is, for any $i = 1, 2, ..., p$, $a_i = \alpha_i$ by p–tuple equality. □

We denote by $(\boldsymbol{x})_i^{\mathcal{V}}$ the i^{th} coordinate of a vector $\boldsymbol{x}$ in a basis $\mathcal{V}$.

With this notation, Exercise 2.37 becomes $(\boldsymbol{a})_i^{\mathcal{E}} = a_i$ for any $i = 1, 2, ..., p$.

We denote by $[\boldsymbol{x}]_{\mathcal{V}}$ the row matrix of coordinates in a basis $\mathcal{V}$ of a vector $\boldsymbol{x}$,

$$[\boldsymbol{x}]_{\mathcal{V}} \triangleq [(\boldsymbol{x})_1^{\mathcal{V}}, (\boldsymbol{x})_2^{\mathcal{V}}, ..., (\boldsymbol{x})_p^{\mathcal{V}}],$$

and we denote by $[\boldsymbol{x}]_{\mathcal{V}}{}^{\mathrm{t}}$ its column matrix of coordinates,

$$[\boldsymbol{x}]_{\mathcal{V}}{}^{\mathrm{t}} \triangleq \begin{bmatrix} (\boldsymbol{x})_1^{\mathcal{V}} \\ (\boldsymbol{x})_2^{\mathcal{V}} \\ ... \\ (\boldsymbol{x})_p^{\mathcal{V}} \end{bmatrix}.$$

From Exercise 2.37 we have,

$$[\boldsymbol{a}]_{\mathcal{E}} = [a_1 \; a_2 \; ... \; a_p].$$

If a linear vector space V has a basis of n vectors, then all the basis of V have the **same** number of vectors and this number is called the *dimension of* V (Cagnac et al., 1963, p. 293).

Let us recall now the definitions of an inner product and Euclidian vector space.

Definition 2.38 (inner product) – Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two vectors of a linear vector space V over the field **R** real numbers, the element of **R** denoted by $< \boldsymbol{x}, \boldsymbol{y} >$ is a *inner product of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$, iff, for any $\boldsymbol{x}$, $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{y}$ in V and α in **R**,

$$
\begin{aligned}
< \boldsymbol{x}, \boldsymbol{y} > &= < \boldsymbol{y}, \boldsymbol{x} > && \text{(commutativity)} \\
< \boldsymbol{x}_1 + \boldsymbol{x}_2, \boldsymbol{y} > &= < \boldsymbol{x}_1, \boldsymbol{y} > + < \boldsymbol{x}_2, \boldsymbol{y} > && \text{(dist. over + on } V) \\
< \alpha \cdot \boldsymbol{x}, \boldsymbol{y} > &= \alpha < \boldsymbol{x}, \boldsymbol{y} > && \text{(mixt associativity)} \\
\boldsymbol{x} \neq \boldsymbol{o} &\Rightarrow < \boldsymbol{x}, \boldsymbol{x} > > 0. && \text{(positiveness)}
\end{aligned}
$$

□

From the above definition we observe that, for any x in V, $< \boldsymbol{o}, \boldsymbol{x} > = 0$.

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two vectors in the plane. If l_x denotes the length of a vector $\boldsymbol{x}$ and $\theta_{\boldsymbol{x},\boldsymbol{y}}$ the angle between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, then product of $\boldsymbol{x}$ and $\boldsymbol{y}$ defined by

$$< \boldsymbol{x}, \boldsymbol{y} > \triangleq l_x l_y \cos \theta_{\boldsymbol{x},\boldsymbol{y}} \tag{2.5}$$

is an example of inner product.

The product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in the linear vector space $\mathbf{R}^p$ defined by

$$< \boldsymbol{a},\boldsymbol{b} > \triangleq a_1b_1 + a_2b_2 + ... + a_pb_p \tag{2.6}$$

is another example of inner product.

Definition 2.39 (Euclidian vector space) – A linear vector space equipped with the operation $(\boldsymbol{x},\boldsymbol{y}) \mapsto < \boldsymbol{x},\boldsymbol{y} >$ where $< \boldsymbol{x},\boldsymbol{y} >$ is an the inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$, is called an *Euclidian vector space*. □

The vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of an Euclidian vector space V are said *orthogonal* iff $< \boldsymbol{x},\boldsymbol{y} > = 0$.

We observe that if $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal then, for any α in $\mathbf{R}$, $\boldsymbol{x}$ and $\alpha{\cdot}\boldsymbol{y}$ are also orthogonal. In particular, any vector $\boldsymbol{x}$ is orthogonal to $\boldsymbol{o}$ and there is no other vector sharing this property.

For example, the vectors (1, 0) and (0, 1) of the Euclidian vector space $\mathbf{R}^2$ with the inner product defined by Expression (2.6) are orthogonal since $< (1,0),(0,1) > = 0$.

A set of vectors is *orthogonal* iff the vectors in this set are two by two orthogonal.

For any $\boldsymbol{x}$ in V, the *norm of* $\boldsymbol{x}$, denoted by $\| \boldsymbol{x} \|$, is the element of $\mathbf{R}$ given by

$$\| \boldsymbol{x} \| \triangleq \sqrt{< \boldsymbol{x},\boldsymbol{x} >}.$$

For the vectors in the plane, by using the inner product defined by Expression (2.5), we have $\| \boldsymbol{x} \| = 1_x$.

Exercise 2.40 (norm property) – Using an inner product axiom, prove that for any $\boldsymbol{x}$ in V,

$$\| \boldsymbol{x} \| = \| -\boldsymbol{x} \|.$$ □

We have the interesting following properties. For the proof of the Cauchy Schwarz inequality see Cagnac et al. (1965, p. 22 and p. 34).

Proposition 2.41 (norm properties) – For any $\boldsymbol{x}$ and $\boldsymbol{y}$ in V and α in $\mathbf{R}$, we have

$$\| \boldsymbol{x} \| = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{o},$$

$$| < \boldsymbol{x},\boldsymbol{y} > | \leq \| \boldsymbol{x} \| \| \boldsymbol{y} \|, \qquad \text{(Cauchy Schwarz inequality)}$$

$$\| \boldsymbol{x} + \boldsymbol{y} \|^2 = \| \boldsymbol{x} \|^2 + 2 < \boldsymbol{x},\boldsymbol{y} > + \| \boldsymbol{y} \|^2,$$

$$\| \alpha{\cdot}\boldsymbol{x} \|^2 = \alpha^2 \| \boldsymbol{x} \|^2.$$ □

We observe that $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal iff $\| \boldsymbol{x} + \boldsymbol{y} \|^2 = \| \boldsymbol{x} \|^2 + \| \boldsymbol{y} \|^2$.

Exercise 2.42 (norm property) – Using an inner product axiom and the norm properties of Proposition 2.41, prove that, for any $\boldsymbol{x}$ and $\boldsymbol{y}$ in V, and α and β in $\mathbf{R}$,

$$\| \alpha{\cdot}\boldsymbol{x} + \beta{\cdot}\boldsymbol{y} \|^2 = \alpha^2 \| \boldsymbol{x} \|^2 + 2\alpha\beta < \boldsymbol{x},\boldsymbol{y} > + \beta^2 \| \boldsymbol{y} \|^2.$$ □

A set of vectors is *orthonormal* iff it is orthogonal and each vector in the set has norm equal to **unity**.

One can prove (Cagnac et al., 1965, p. 12 and p. 37) that if an Euclidian vector space is of finite dimension, then it has **at least one** orthonormal basis.

Proposition 2.43 (coordinates of a vector in an orthonormal basis) – Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_p$ be a set of vectors forming an orthonormal basis of an Euclidian vector space. If α_i is the coordinate of $\boldsymbol{x}$ with respect to the vector $\boldsymbol{v}_i$, then for any $i = 1, 2, ..., p$, we have $\alpha_i = < \boldsymbol{x}, \boldsymbol{v}_i >$. □

Proof – Under the proposition assumptions, for any $i = 1, 2, ..., p$,

$$\begin{aligned} < \boldsymbol{x}, \boldsymbol{v}_i > &= < \alpha_1 \cdot \boldsymbol{v}_1 + \alpha_2 \cdot \boldsymbol{v}_2 + ... + \alpha_n \cdot \boldsymbol{v}_p, \boldsymbol{v}_i > && \text{(coordinate definition)} \\ &= < \alpha_1 \cdot \boldsymbol{v}_1, \boldsymbol{v}_i > + < \alpha_2 \cdot \boldsymbol{v}_2, \boldsymbol{v}_i > + ... + < \alpha_p \cdot \boldsymbol{v}_p, \boldsymbol{v}_i > && \text{(distributivity)} \\ &= \alpha_1 < \boldsymbol{v}_1, \boldsymbol{v}_i > + \alpha_2 < \boldsymbol{v}_2, \boldsymbol{v}_i > + ... + \alpha_p < \boldsymbol{v}_p, \boldsymbol{v}_i > && \text{(mixt assoc.)} \\ &= \alpha_i. && \text{(the } \boldsymbol{v}_i\text{'s form an orthonormal basis)} \end{aligned}$$

□

Let $\boldsymbol{v}_i$ be the i^{th} vector of an orthonormal basis $\mathcal{N}$, then from Proposition 2.43, we see that its coordinates are 0 except the i^{th} which is 1. In symbols,

$$[v_i]_{\mathcal{N}} = [0, ..., 0, 1, 0, ..., 0] \text{ with 1 at the } i^{\text{th}} \text{ position.}$$

Exercise 2.44 (expression of the inner product in an orthonormal basis) – Let $\alpha_1, \alpha_2, ..., \alpha_p$ be the coordinates of a vector $\boldsymbol{x}$, and let $\beta_1, \beta_2, ..., \beta_p$ be the coordinates of a vector $\boldsymbol{y}$ in an orthonormal basis of an Euclidian vector space. Show that

$$< \boldsymbol{x}, \boldsymbol{y} > = \alpha_1\beta_1 + \alpha_2\beta_2 + ... + \alpha_p\beta_p.$$ □

Using the notation of row matrix of coordinates the above expression can be written

$$< \boldsymbol{x}, \boldsymbol{y} > = [\boldsymbol{x}]_{\mathcal{N}}[\boldsymbol{y}]_{\mathcal{N}}^{t},$$

where $\mathcal{N}$ is any orthonormal basis.

The linear vector space $\mathbf{R}^p$ is called *Euclidian* iff its canonical basis is orthonormal (Cagnac et al., 1965).

As a consequence of Exercises 2.37 and 2.44, we have the following proposition.

Proposition 2.45 (expression of the inner product in an Euclidian vector space of p–tuples) – If $\mathbf{R}^p$ is Euclidian, then for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$< \boldsymbol{a}, \boldsymbol{b} > = a_1b_1 + a_2b_2 + ... + a_pb_p.$$ □

Extended operations

By using the concept of mapping, we can construct new operation from simpler one.

Let A be a non–empty set. From operations on B we can construct new operations on the set of mappings from A to B.

Let $\square$ be an internal binary operation on B, for any f and g in B^A, and any a in A, let $(f \square g)(a)$ be the expression defined by

$$(f \square g)(a) \triangleq f(a) \square g(a).$$

The internal binary operation on B^A, given by $(f, g) \mapsto f \,\square\, g$ is called the *extension to* B^A *of the internal binary operation* $\square$ on B. We will denote it by $\square_{B^A}$ or simply by $\square$ if no confusion is likely to arise.

Let $\cdot$ be an external binary operation on B with operand in C, for any c in C, f in B^A, and any a in A, let $(c{\cdot}f)(a)$ be the expression defined by

$$(c{\cdot}f)(a) \triangleq c{\cdot}f(a).$$

The external binary operation on B^A with operand in C, given by $(c,f) \mapsto c{\cdot}f$ is called the *external extension to* B^A *of the binary operation* $\cdot$ on B. We will denote it by $\cdot_{B^A}$ or simply by $\cdot$ if no confusion is likely to arise.

Let h be a unary operation on B. The unary operation from B^A to B^A, given by $f \mapsto h \circ f$ is called the *extension* to B^A *of the unary operation* h on B. We will denote it by h_{B^A} or simply by h if no confusion is likely to arise.

Let e be an element of B, let e be the corresponding nullary operation on B and let e (we use again the same notation) be the constant mapping $a \mapsto e$ in B^A. The nullary operation on B^A, given by $1 \mapsto e$ is called the *extension* to B^A *of the nullary operation* e on B. We will denote it again by the same symbol e. In other words, this extension e select in B^A the constant mapping assuming the value e in B.

Figure 2.20 illustrates by mean of block diagrams the above extensions for the four types of operation. Each gray block represents the resulting mapping of an operation, its interior shows how the mapping is obtained.

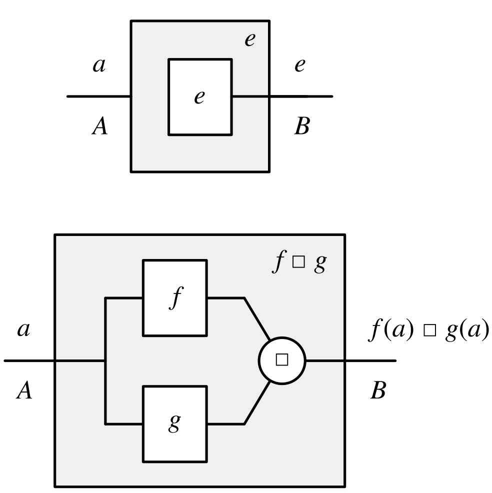


Fig. 2.20 – Operation extensions.

The above process of extension of operations on B to operations on B^A is attractive because it preserves the usual operation properties.

Proposition 2.46 (commutativity extension) – Let A be a non–empty set. If $\square$ is a commutative operation on B, then its extension to B^A is commutative. $\square$

Proof – For any f and g in B^A, and for any a in A,

$$\begin{aligned} (f \,\square\, g)(a) &= f(a) \,\square\, g(a) && \text{(extension definition)} \\ &= g(a) \,\square\, f(a) && \text{(commutativity of } \square \text{ on } B\text{)} \\ &= (g \,\square\, f)(a). && \text{(extension definition)} \end{aligned}$$

This prove the commutativity of the extension of $\square$ to B^A. $\square$

Proposition 2.47 (associativity extension) – Let A be a non–empty set. If $\square$ is an associative operation on B, then its extension to B^A is associative. $\square$

Proof – For any f, g and h in B^A, and for any a in A,

$$\begin{aligned} ((f \,\square\, g) \,\square\, h)(a) &= (f \,\square\, g)(a) \,\square\, h(a) && \text{(extension definition)} \\ &= (f(a) \,\square\, g(a)) \,\square\, h(a)), && \text{(extension definition)} \\ &= f(a) \,\square\, (g(a) \,\square\, h(a)), && \text{(associativity of } \square \text{ on } B\text{)} \\ &= f(a) \,\square\, (g \,\square\, h)(a), && \text{(extension definition)} \\ &= (f \,\square\, (g \,\square\, h))(a). && \text{(extension definition)} \end{aligned}$$

This prove the associativity of the extension to B^A. $\square$

Proposition 2.48 (unit element extension) – Let A be a non–empty set. If a nullary operation u on B selects the unit element of an internal binary operation $\square$ on B, then the extension to B^A of u selects the unit element of the extension to B^A of $\square$ on B. $\square$

Proof – For any f in B^A, and for any a in A,

$$\begin{aligned} (f \,\square\, u)(a) &= f(a) \,\square\, u(a) && \text{(extension definition)} \\ &= f(a) \,\square\, u && \text{(definition of the extension of } u\text{)} \\ &= f(a). && \text{(unit element of } \square \text{ on } B\text{)} \end{aligned}$$

This prove that u is the unit element of the extension to B^A of $\square$ on B. $\square$

By combining extended operations and composition we can get the following useful result (see Chapter 4).

Exercise 2.49 (distributivity of the composition over an extended binary operation) – Let A and B be two non–empty sets.

1. Prove that if $\square$ is an internal binary operation on C, then its extensions to C^A and C^B satisfy, for any f in B^A, and any g_1 and g_2 in C^B,

$$(g_1 \,\square_{C^B}\, g_2) \circ f = (g_1 \circ f) \,\square_{C^A}\, (g_2 \circ f).$$

2. Prove that if $\cdot$ is an external binary operation on C (with operand in D), then its extensions to C^A and C^B satisfy, for any f in B^A, any g in C^B and any α in D,

$$(\alpha \cdot_{C^B} g) \circ f = \alpha \cdot_{C^A} (g \circ f)\ . \qquad \square$$

As a direct consequence of Exercise 2.49 we have the following result.

Exercise 2.50 (restriction property) – Let B and C be two non–empty sets, and let A be a subset of B. Prove that for any g_1 and g_2 in C^B,

$$(g_1 \,\square_{C^B}\, g_2)/A \;=\; (g_1/A) \,\square_{C^A}\, (g_2/A),$$

(Hint: combine the results of the Exercises 2.36 and 2.49). □

Orthogonal matrix and diagonal form

A real square matrix A is *orthogonal* iff its inverse is equal to its transpose (rows and columns are interchanged), that is,

$$A^{-1} \;=\; A^{\mathrm{t}},$$

or, equivalently,

$$A^{\mathrm{t}}A \;=\; AA^{\mathrm{t}} \;=\; I, \tag{2.7}$$

where I is the identity matrix (i.e., a square matrix with 1 along its diagonal and 0 outside).

If A is an orthogonal matrix p by p, then from Expression (2.7), we have $\sum_{j\in\mathbf{p}} a_{j\,k}{}^2 \;=\; 1$ for any $k \in \mathbf{p}$.

The matrix $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is an example of orthogonal matrix.

The orthogonal matrices play an important role in the diagonalizing process of a symmetric matrix.

A matrix A is *symmetric* iff it is equal to its transpose, that is,

$$A \;=\; A^{\mathrm{t}}.$$

The matrix $\begin{bmatrix} 10 & 3 \\ 3 & 10 \end{bmatrix}$ is an example of symmetric matrix.

Let A be a $p \times p$ real matrix, then a *column vector of A with respect to a given basis* of $\mathbf{R}^p$ is a vector whose coordinates in that basis are the elements of a column of A.

For any symmetric matrix Σ, one can prove (Stein, 1967; Graybill, 1969, p. 48) that there exists an orthogonal matrix A such that

$$A^{\mathrm{t}}(\Sigma A) \;=\; \Lambda,$$

where Λ is a diagonal matrix (i.e., a square matrix with 0 outside the diagonal). The diagonal elements of Λ are the *eigenvalues* of Σ (i.e., the roots λ's of $|\Sigma - \lambda I| \;=\; 0$, where $|A|$ means determinant of A) and the column vectors of A in a basis $\mathscr{V}$ are the corresponding *eigenvectors* (i.e., the solutions $\boldsymbol{a}$'s of $\Sigma[\boldsymbol{a}]_{\mathscr{V}}{}^{\mathrm{t}} \;=\; \lambda{\cdot}[\boldsymbol{a}]_{\mathscr{V}}{}^{\mathrm{t}}$).

Furthermore the traces of Σ (i.e., the sum of the diagonal elements) and Λ are equal.

In (Bolch & Huang, 1974, pp. 38–40) one can find a routine to compute the eigenvalues and eigenvectors (in the canonical basis) of a symmetric matrix.

We now recall two interesting properties of the orthogonal matrices.

Proposition 2.51 (orthogonal matrix column property) – The column vectors in an orthonormal basis $\mathcal{N}$ of the vector space $\mathbf{R}^p$ of an orthogonal $p \times p$ real matrix form an orthonormal set of vectors. □

Proof – Let A be a orthogonal $p \times p$ real matrix and let $\boldsymbol{a}_1, \boldsymbol{a}_2, ..., \boldsymbol{a}_p$ be its column vectors in an orthonormal basis $\mathcal{N}$ of the vector space $\mathbf{R}^p$. We have the following equalities,

$$\begin{aligned}
[< \boldsymbol{a}_i, \boldsymbol{a}_j >] &= \begin{bmatrix} [\boldsymbol{a}_1]_{\mathcal{N}} \\ [\boldsymbol{a}_2]_{\mathcal{N}} \\ \cdots \\ [\boldsymbol{a}_p]_{\mathcal{N}} \end{bmatrix} \begin{bmatrix} [\boldsymbol{a}_1]_{\mathcal{N}}{}^{\mathrm{t}} \, [\boldsymbol{a}_2]_{\mathcal{N}}{}^{\mathrm{t}} \cdots [\boldsymbol{a}_p]_{\mathcal{N}}{}^{\mathrm{t}} \end{bmatrix} && \text{(Exercise 2.44)} \\
&= A^{\mathrm{t}}A && \text{(definition of the } \boldsymbol{a}_j\text{'s)} \\
&= I. && \text{(orthogonality of } A\text{)}
\end{aligned}$$

Therefore, $< \boldsymbol{a}_i, \boldsymbol{a}_j >$ is 0 for any $i \neq j$ and 1 otherwise. □

Proposition 2.52 (linear transform of the column vectors of an orthogonal matrix) – Let $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p$ be a set of vectors forming an orthonormal basis $\mathcal{N}$ of a the linear vector space $\mathbf{R}^p$. Let A be an orthogonal $p \times p$ real matrix and let $\boldsymbol{a}_1, \boldsymbol{a}_2, ..., \boldsymbol{a}_p$ be its column vectors in $\mathcal{N}$. For any $i = 1, 2, ..., p$, the transform of $\boldsymbol{a}_i$ through the linear mapping characterized by A is $\boldsymbol{v}_i$, that is,

$$[\boldsymbol{a}_i]_{\mathcal{N}} A = [\boldsymbol{v}_i]_{\mathcal{N}}.$$ □

Proof – We have the following equalities,

$$\begin{aligned}
\begin{bmatrix} [\boldsymbol{a}_1]_{\mathcal{N}} \\ [\boldsymbol{a}_2]_{\mathcal{N}} \\ \cdots \\ [\boldsymbol{a}_p]_{\mathcal{N}} \end{bmatrix} A &= A^{\mathrm{t}}A && \text{(definition of the } \boldsymbol{a}_i\text{'s)} \\
&= I && \text{(orthogonality of } A\text{)} \\
&= \begin{bmatrix} [\boldsymbol{v}_1]_{\mathcal{N}} \\ [\boldsymbol{v}_2]_{\mathcal{N}} \\ \cdots \\ [\boldsymbol{v}_p]_{\mathcal{N}} \end{bmatrix}. && \text{(Proposition 2.43)}
\end{aligned}$$

□

Chapter 3

Pointwise enhancement

Sometimes, the image brightness and contrast have to be adjusted in order to achieve visual enhancement. This adjustment is done by pointwise operators or by spatially invariant pointwise operators. A spatially invariant pointwise operator is characterized by a so–called look–up–table, or lut for short. Its properties depend on the properties of its lut. So we begin to study the luts.

3.1 Look–up–table

In Chapter 1, we have seen that a gray–scale with p gray–levels, together with the binary relation "*is darker or equal than*" is isomorphic to the interval $[0, p-1]$ with the usual ordering on the natural numbers.

In Chapter 2, we have considered the interval $[0, p-1]$ as a subset of the set $\mathbf{R}$ of real numbers and we have defined, in this way, the notion of real image.

In this chapter, the sets $[0, p-1]$ and $\mathbf{R}$ will be treated separately and will be called, respectively, *discrete gray–scale* and *continuous gray–scale*. We will use the letter K to represent generically a gray–scale or sometimes to represent an interval of natural numbers.

Definition 3.1 (look–up–table) – A *look–up–table* or *lut* or *palette* is a mapping between two gray–scales which can be the same or not. □

In what follows, we will consider the luts between two identical gray–scales forming with an internal binary operation and a nullary operation a commutative monoid. Moreover, we will characterize the luts which are morphisms of that binary operation viewed as an external binary operation.

Let $(K, \cdot, e')$ be a commutative monoid. From the morphism definition, a lut l from K to K is a morphism for the operation $\cdot$ viewed as an external binary operation (with operand in K) iff, for any s and β in K,

$$l(\beta \cdot s) = \beta \cdot l(s).$$

Let l be a mapping from K to K, we denote by α_l the expression defined by

$$\alpha_l \triangleq l(e').$$

Let α and s be two elements in K, we denote by $l_\alpha(s)$ the expression defined by

$$l_\alpha(s) \triangleq s \cdot \alpha.$$

Proposition 3.2 (characterization of the luts which are morphisms of an external operation) – Let $(K, \cdot, e')$ be a monoid and let $\mathrm{M}_\cdot(K, K)$ be the set of luts from K to K which are morphisms for the operation $\cdot$ viewed as an external binary operation (with operand in K), then the mapping $l \mapsto \alpha_l$ from $\mathrm{M}_\cdot(K, K)$ to K is a bijection and its inverse is $\alpha \mapsto l_\alpha$. □

Proof – 1. For any l in $\mathrm{M}_\cdot(K, K)$, α_l is an element of K.

2. For any α in K, the mapping $s \mapsto l_\alpha(s)$ is in $\mathrm{M}_\cdot(K, K)$ as we can see below. For any s and β in K,

$$\begin{aligned} l_\alpha(\beta \cdot s) &= (\beta \cdot s) \cdot \alpha && \text{(definition of } l_\alpha(s)) \\ &= \beta \cdot (s \cdot \alpha) && \text{(associativity)} \\ &= \beta \cdot l_\alpha(s). && \text{(definition of } l_\alpha(s)) \end{aligned}$$

3. Let l be in $\mathrm{M}_\cdot(K, K)$, for any s in K

$$\begin{aligned} l_{\alpha_l}(s) &= s \cdot \alpha_l && \text{(definition of } l_\alpha(s)) \\ &= s \cdot l(e') && \text{(definition of } \alpha_l) \\ &= l(s \cdot e') && (l \text{ is in } \mathrm{M}_\cdot(K, K)) \\ &= l(s), && (e' \text{ is a unit element of } \cdot) \end{aligned}$$

that is, $l \mapsto \alpha_l$ is an injection.

4. Let α be in K,

$$\begin{aligned} \alpha_{l_\alpha} &= l_\alpha(e') && \text{(definition of } \alpha_l) \\ &= e' \cdot \alpha && \text{(definition of } l_\alpha(s)) \\ &= \alpha && (e' \text{ is a unit element of } \cdot) \end{aligned}$$

that is, $l \mapsto \alpha_l$ is a surjection. □

The characterization presented in Proposition 3.2 is important because it shows that each lut which is a morphism as described in this proposition reduces to a unique parameter, the value of which is a gray–level.

Additionally, if K is equipped with another internal binary operation $\square$ and another nullary operation e, then, under some specific conditions, we have the following interesting consequence of Proposition 3.2.

Exercise 3.3 (characterization of the luts which are morphisms of two binary operations and of one nullary operation) – Let $(K, \square, \cdot, e, e')$ be a set K equipped with two internal binary operations $\square$ and $\cdot$, and two nullary operation e and e', and let $\mathrm{M}_{\square,\cdot,e}(K, K)$ be the set of luts from K to K which are morphisms for the binary operation $\square$, the nullary operation e and the binary operation $\cdot$ viewed as an external binary operation (with operand in K). Prove that if $\cdot$ is distributive over $\square$, $(K, \cdot, e')$ is a commutative monoid, and $e{\cdot}s = e$ (for any s in K), then the mapping $l \mapsto \alpha_l$ from $\mathrm{M}_{\square,\cdot,e}(K, K)$ to K is a bijection and its inverse is $\alpha \mapsto l_\alpha$. (Hint: prove that $\mathrm{M}_{\square,\cdot,e}(K, K)$ is $\mathrm{M}_{\cdot}(K, K)$). □

The result stated in Exercise 3.3 can be applied to the luts which preserve the linear vector space structure or the luts which preserve some commutative monoid structures in the lattice structure.

We begin defining and characterizing the luts which are linear mappings, that is the luts that preserve the linear vector space structure.

We know that the set $(\mathbf{R}, +, \cdot, 0)$ of real numbers is a linear vector space. Therefore, we can apply Definition 3.60 to the luts.

Definition 3.4 (linear luts) – A lut l from $(\mathbf{R}, +, \cdot, 0)$ to $(\mathbf{R}, +, \cdot, 0)$, is *linear* iff, for any s, t and α in $\mathbf{R}$,

$$l(s + t) = l(s) + l(t) \quad \text{and} \quad l(\alpha s) = \alpha l(s).$$ □

From Exercise 3.61, a linear lut l verifies $l(0) = 0$.

Let l be a mapping from $\mathbf{R}$ to $\mathbf{R}$, we denote by α_l the expression defined by

$$\alpha_l \triangleq l(1).$$

Let α and s be two elements in $\mathbf{R}$, we denote by $l_\alpha(s)$ the expression defined by

$$l_\alpha(s) \triangleq s{\cdot}\alpha.$$

Proposition 3.5 (characterization of the linear luts) – The mapping $l \mapsto \alpha_l$ from $\mathrm{L}(\mathbf{R}, \mathbf{R})$ to $\mathbf{R}$ is a bijection and its inverse is $\alpha \mapsto l_\alpha$. □

Proof – The algebraic structure $(\mathbf{R}, +, \cdot, 0, 1)$ and the set $\mathrm{L}(\mathbf{R}, \mathbf{R})$ of linear luts, satisfy the assumptions of Exercise 3.3, therefore, the result of that exercise applies to the linear luts. □

Figure 3.1 shows two linear luts l_1 and l_2 with their parameters α_{l_1} and α_{l_2}.

We go on defining and characterizing the luts which are dilations, erosions, anti–dilation and anti–erosion, that is the luts that preserve some commutative monoid structures in the lattice structure.

In the discrete case, we can consider the general case where the luts are mappings between two different gray–scales. Later on, we will return to the case of two identical gray–scales.

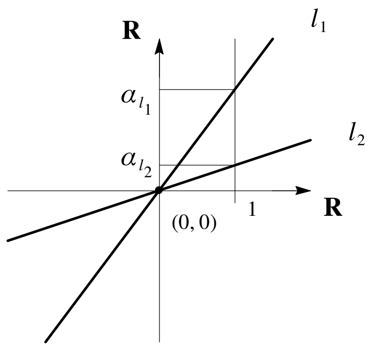


Fig. 3.1 – Two linear luts.

Let $K_1 = [0, k_1] \subset \mathbf{Z}$ and $K_2 = [0, k_2] \subset \mathbf{Z}$. Let $(K_1, \leq)$ and $(K_2, \leq)$ be the poset under the usual partial order on the set $\mathbf{Z}$ of natural numbers.

By Proposition 3.43, $(K_1, \leq)$ and $(K_2, \leq)$ are two lattices. For each one, we denote, respectively, the union $\vee_{\leq}$ and intersection $\wedge_{\leq}$ with respect to $\leq$, simply by $\vee$ and $\wedge$. For example, if $K_1 = K_2 = [0, 3]$, then $1 \vee 2 = 2$ and $1 \wedge 2 = 1$.

Furthermore, by Proposition 3.49, the $\vee$'s on K_1 and K_2 have 0 as unit elements and $\wedge$ on K_1 has k_1 as unit element, and $\wedge$ on K_2 has k_2 as unit element.

In other words, in the lattice $(K_1, \vee, \wedge)$ we can identify the two idempotent and commutative monoids $(K_1, \vee, 0)$ and $(K_1, \wedge, k_1)$. In the same way, in the lattice $(K_2, \vee, \wedge)$ we can identify the two idempotent and commutative monoids $(K_2, \vee, 0)$ and $(K_2, \wedge, k_2)$.

Therefore, we can apply Definition 3.62 to the luts.

Definition 3.6 (elementary morphological luts) – A lut l from $(K_1, \vee, 0)$ to $(K_2, \vee, 0)$ is a *dilation* iff, for any s and t in K_1,

$$l(s \vee t) = l(s) \vee l(t) \quad \text{and} \quad l(0) = 0;$$

a lut from $(K_1, \wedge, k_1)$ to $(K_2, \wedge, k_2)$ is an *erosion* iff for any s and t in K_1,

$$l(s \wedge t) = l(s) \wedge l(t) \quad \text{and} \quad l(k_1) = k_2;$$

a lut from $(K_1, \vee, 0)$ to $(K_2, \wedge, k_2)$ is an *anti–dilation* iff for any s and t in K_1,

$$l(s \vee t) = l(s) \wedge l(t) \quad \text{and} \quad l(0) = k_2;$$

a lut from $(K_1, \wedge, k_1)$ to $(K_2, \vee, 0)$ is an *anti–erosion* iff for any s and t in K_1,

$$l(s \wedge t) = l(s) \vee l(t) \quad \text{and} \quad l(k_1) = 0.$$

The luts which are dilations, erosions, anti–dilations and anti–erosions are called *elementary morphological luts*. □

Proposition 3.66 applies directly to the characterization of the elementary morphological luts.

Proposition 3.7 (characterization of the elementary morphological luts) – Let K_1 and K_2 be, respectively, two discrete gray–scales $[0, k_1]$ and $[0, k_2]$. Let $(K_1, \leq)$ and $(K_2, \leq)$ be the lattices corresponding to the usual partial order on the set **N** of natural numbers. Let l a mapping from K_1 to K_2, then we have the following statements:

$$l \text{ is a dilation} \Leftrightarrow l \text{ is increasing and } l(0) = 0$$

$$l \text{ is an erosion} \Leftrightarrow l \text{ is increasing and } l(k_1) = k_2$$

$$l \text{ is an anti–dilation} \Leftrightarrow l \text{ is decreasing and } l(0) = k_2$$

$$l \text{ is an anti–erosion} \Leftrightarrow l \text{ is decreasing and } l(k_1) = 0. \qquad \square$$

Figure 3.2 shows two samples in each class of elementary morphological luts from $K_1 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ to $K_2 = \{0, 1, 2, 3\}$. We see that the graph of the dilation (resp., erosion, anti–dilation, anti–erosion) contains the point $(0, 0)$ (resp., $(7, 3)$, $(0, 3)$, $(7, 0)$).

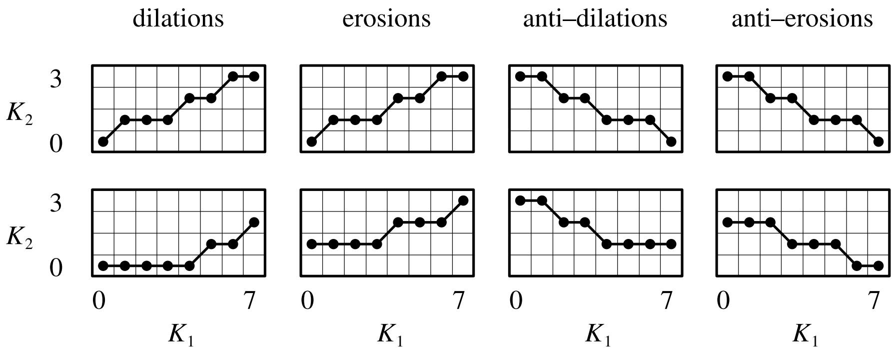


Fig. 3.2 – Some elementary morphological luts.

The next proposition gives the number of elementary morphological luts.

Proposition 3.8 (number of elementary morphological luts) – Let K_1 and K_2 be two finite chains, and let $k_1 = (\#K_1) - 1$ and $k_2 = (\#K_2) - 1$. Let $N(k_1, k_2)$ be the number of luts in a given class of elementary morphological luts from K_1 to K_2, then for any $k_1 \geq 1$ and $k_2 \geq 1$,

$$N(k_1, k_2) = \frac{(k_1 + k_2)!}{k_1! k_2!}. \qquad \square$$

Proof – See Proposition 1.6 of Banon (1995). $\square$

Because of the very large number of elementary morphological luts, it is interesting to define some parametric classes. We present some examples from the literature in the case $K_1 = K_2 = K = [0, k]$.

For any z in K, let $\mathcal{C}^+$ and $\mathcal{C}^-$ be the classes of binary operations from K^2 to K defined by, respectively,

$$\dotplus \in \mathcal{C}^+ \overset{\Delta}{\Leftrightarrow} \begin{cases} (t_1 \leq t_2 \Rightarrow t_1 \dotplus z \leq t_2 \dotplus z \ \ (t_1, t_2 \in K)) \text{ and} \\ (0 \dotplus z = 0) \end{cases}$$

and

$$- \in \mathcal{C}^- \overset{\Delta}{\Leftrightarrow} \begin{cases} (s_1 \leq s_2 \Rightarrow s_1 - z \leq s_2 - z \;\; (s_1, s_2 \in K)) \text{ and} \\ (k - z = k). \end{cases}$$

Exercise 3.9 (elementary morphological luts) – For any z in K, let l^+ be a mapping from K to K defined by, for any t in K, $l^+(t) \triangleq t + z$, with $+ \in \mathcal{C}^+$, and let l^- be a mapping from K to K defined by, for any s in K, $l^-(s) \triangleq s - z$, with $- \in \mathcal{C}^-$. Using Proposition 3.7, prove that l^+ is a dilation and l^- is an erosion. □

Some examples of binary operations belonging to $\mathcal{C}^+$ are given below, and Figure 3.3 shows the corresponding luts $t \mapsto t +^i z$ and $s \mapsto s -^i z$ for a given z.

1. Heijmans' addition for upward shift (1991, 1994)

$$+^1\!: \; (t, z) \mapsto t +^1 z \triangleq \begin{cases} 0 & \text{if } t = 0, \\ t + z & \text{if } t > 0 \text{ and } t + z \leq k, \\ k & \text{if } t + z > k. \end{cases}$$

2. Heijmans' addition for downward shift (1991, 1994)

$$+^2\!: \; (t, z) \mapsto t +^2 z \triangleq 0 \vee (t - z) = \begin{cases} 0 & \text{if } t - z < 0, \\ t - z & \text{if } 0 \leq t - z. \end{cases}$$

3. Bloch and Maître (1992), and Baets et al. (1994)

$$+^3\!: \; (t, z) \mapsto t +^3 z \triangleq t \wedge z.$$

4. Banon (1995)

$$+^4\!: \; (t, z) \mapsto t +^4 z \triangleq \begin{cases} 0 & \text{if } t - z \leq 0, \\ t & \text{if } 0 < t - z. \end{cases}$$

Some examples of binary operations belonging to $\mathcal{C}^-$ are given below.

1. Heijmans' subtraction for downward shift (1991, 1994)

$$-^1\!: \; (s, z) \mapsto s -^1 z \triangleq \begin{cases} 0 & \text{if } s < k \text{ and } s - z < 0, \\ s - z & \text{if } s < k \text{ and } 0 \leq s - z \leq k, \\ k & \text{if } s = k. \end{cases}$$

2. Sinha and Dougherty (1991), and Heijmans' subtraction for upward shift (1991, 1994)

$$-^2\!: \; (s, z) \mapsto s -^2 z \triangleq k \wedge (s + z) = \begin{cases} s + z & \text{if } s + z \leq k, \\ k & \text{if } s + z > k. \end{cases}$$

3. Banon (1995)

$$-^3\!: \; (s, z) \mapsto s -^3 z \triangleq \begin{cases} s & \text{if } s - z < 0, \\ k & \text{if } 0 \leq s - z. \end{cases}$$

4. Bloch and Maître (1992), and Baets et al. (1994)

$$-^4\!: \; (s, z) \mapsto s -^4 z \triangleq s \vee z.$$

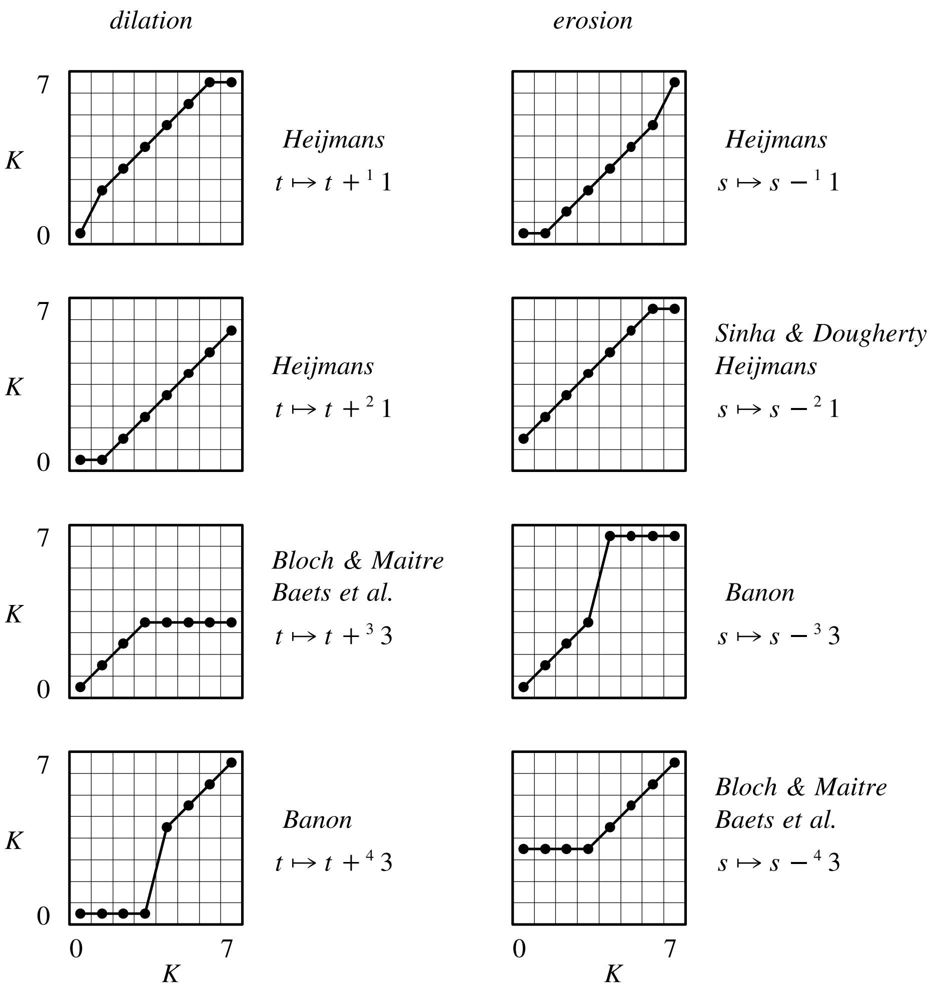


Fig. 3.3 – Some compagnion luts.

We say that $+ \in \mathcal{C}^+$ and $- \in \mathcal{C}^-$ are *companions* iff for any $s, t, z \in K$,

$$t + z \leq s \Leftrightarrow t \leq s - z.$$

For example, the above operations $+^i$ and $-^i$ are companions for any $i = 1, 2, 3, 4$.

For a given $z \in K$, we say that the luts $t \mapsto t + z$ and $s \mapsto s - z$ are *companions* iff $+ \in \mathcal{C}^+$ and $- \in \mathcal{C}^-$ are companions.

For example, for a given $i \in \{1, 2, 3, 4\}$ and $z \in K$, the luts $t \mapsto t +^i z$ and $s \mapsto s -^i z$ are companions.

In Figure 3.3, the dilations and erosions on each row are companions.

The result of Exercise 3.3 also applies to the case of a discrete gray–scale when we consider the set of luts which are at the same time dilations and morphisms for the external operation of intersection or at the same time erosions and morphisms for the external operation of union.

Let d and e be mappings from K to K, we denote by α_d and α_e the expressions defined by

$$\alpha_d \triangleq d(k) \quad \text{and} \quad \alpha_e \triangleq e(0).$$

Let α, s and t be elements in K, we denote by $d_\alpha(t)$ and $e_\alpha(s)$ the expressions defined by

$$d_\alpha(t) \triangleq t \wedge \alpha \quad \text{and} \quad e_\alpha(s) \triangleq s \vee \alpha.$$

Proposition 3.10 (characterization of the luts which are dilations and morphisms for the external operation of intersection) – Let $K = [0, k] \subset \mathbf{Z}$. Let $\vee$ and $\wedge$ be the operations of union and intersection derived from the usual order between the natural numbers. Let $\mathrm{M}_{\vee,\wedge,0}(K, K)$ be the set of dilations which are morphisms for the operation $\wedge$ viewed as an external binary operation (with operand in K). Then the mapping $d \mapsto \alpha_d$ from $\mathrm{M}_{\vee,\wedge,0}(K, K)$ to K is a bijection and its inverse is $\alpha \mapsto d_\alpha$. □

Proof – The algebraic structure $(K, \vee, \wedge, 0, k)$ and the set $\mathrm{M}_{\vee,\wedge,0}(K, K)$ of luts which are dilations and morphisms for the operation $\wedge$ viewed as an external binary operation (with operand in K), satisfy the assumptions of Exercise 3.3, therefore, the result of that exercise applies to the luts which are dilations and morphisms for the external operation of intersection. □

Hence, from Proposition 3.10, the luts from K to K of the form $t \mapsto t \wedge \alpha$ (with α in K) are dilations. We observe that these luts are the luts presented above using the operation $+^3$.

In a similar way, we can characterize the luts which are erosions and morphisms for the external operation of union.

Proposition 3.11 (characterization of the luts which are erosions and morphisms for the external operation of union) – Let $K = [0, k] \subset \mathbf{Z}$. Let $\vee$ and $\wedge$ be the operations of union and intersection derived from the usual order between the natural numbers. Let $\mathrm{M}_{\wedge,\vee,k}(K, K)$ be the set of erosions which are morphisms for the operation $\vee$ viewed as an external binary operation (with operand in K). Then the mapping $e \mapsto \alpha_e$ from $\mathrm{M}_{\wedge,\vee,k}(K, K)$ to K is a bijection and its inverse is $\alpha \mapsto e_\alpha$. □

Hence, from Proposition 3.11, the luts from K to K of the form $s \mapsto s \vee \alpha$ (with α in K) are erosions. We observe that these luts are the luts presented above using the operation $-^4$.

Before ending this section we want to introduce some operations on the luts.

In the continuous case, the addition $+$ on $\mathbf{R}$ can be extended to $\mathbf{R}^{\mathbf{R}}$ (see Section 2.4), resulting in an internal binary operation on the set of luts, that we call addition and we denote by $+$. In the same way, the multiplication on $\mathbf{R}$ can be extended to $\mathbf{R}^{\mathbf{R}}$, resulting in an external binary operation on the set of luts, with operand in $\mathbf{R}$, that we call multiplication by a scalar and we denote by $\cdot$.

Figure 3.4 illustrates the result of these operations on linear luts.

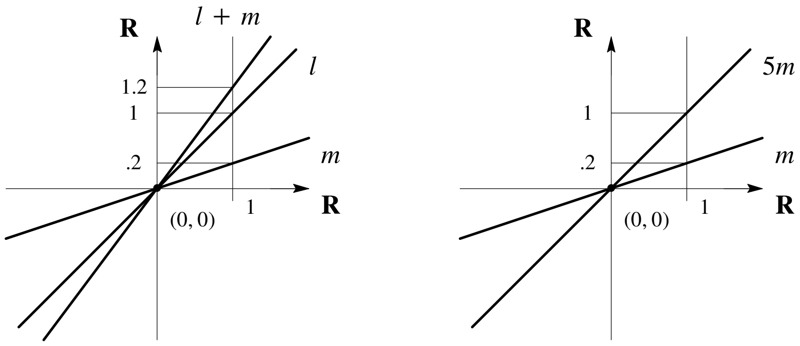


Fig. 3.4 – Sum of two linear luts and product of a linear lut by a scalar.

The nullary operation 0 on $\mathbf{R}$ (that selects 0), can be extended to $\mathbf{R}^{\mathbf{R}}$, resulting in a nullary operation that selects the constant lut $s \mapsto 0$. We denote this lut by o.

As in the case of the real images of Chapter 2, the algebraic structure $(\mathbf{R}^{\mathbf{R}}, +, \cdot, o)$ consisting of the set of luts equipped with the above three operations is a linear vector space.

Exercise 3.12 (subspace of linear luts) – Prove that the algebraic structure $(\mathrm{L}(\mathbf{R}, \mathbf{R}), +, \cdot, o)$ is a subspace of the linear vector space $(\mathbf{R}^{\mathbf{R}}, +, \cdot, o)$. □

Solution – Parts of Propositions 3.67 and 3.68 apply to the set of luts, in other words, $\mathrm{M}_{+}(\mathbf{R}, \mathbf{R})$ and $\mathrm{M}_{\cdot}(\mathbf{R}, \mathbf{R})$ are $+_{\mathbf{R}^{\mathbf{R}}}$–closed and $\cdot_{\mathbf{R}^{\mathbf{R}}}$–closed. Consequently, by Proposition 3.55 their intersection which is $\mathrm{L}(\mathbf{R}, \mathbf{R})$ is also $+_{\mathbf{R}^{\mathbf{R}}}$–closed and $\cdot_{\mathbf{R}^{\mathbf{R}}}$–closed. Hence, by Proposition 3.53, $(\mathrm{L}(\mathbf{R}, \mathbf{R}), +, \cdot, o)$ is a subspace of $(\mathbf{R}^{\mathbf{R}}, +, \cdot, o)$. □

In the discrete case, by Propositions 3.72, the poset $({K_2}^{K_1}, \leq_{\leq})$ or simply $({K_2}^{K_1}, \leq)$, is a lattice and we can introduce the following four operations on the set ${K_2}^{K_1}$.

The *lut union*, denoted by $\vee$, is the union on $({K_2}^{K_1}, \leq)$, or equivalently, the extension to ${K_2}^{K_1}$ of the union $\vee$ on K_2.

The *lut intersection*, denoted by $\wedge$, is the intersection on $({K_2}^{K_1}, \leq)$, or equivalently, the extension to ${K_2}^{K_1}$ of the intersection $\wedge$ on K_2.

The *least lut selection*, denoted by o, is the extension to ${K_2}^{K_1}$ of the nullary operation on K_2 which selects the value 0. This operation selects the *least lut*, that is, the constant lut, $s \mapsto 0$. We denote by o this lut.

The *greatest lut selection*, denoted by i, is the extension to ${K_2}^{K_1}$ of the nullary operation on K_2 which selects the value k_2. This operation selects the *greatest lut*, that is, the constant lut $s \mapsto k_2$. We denote by i this lut.

Figure 3.5 shows two distinct luts l and m from $K_1 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ to $K_2 = \{0, 1, 2, 3\}$, and their supremum and infimum

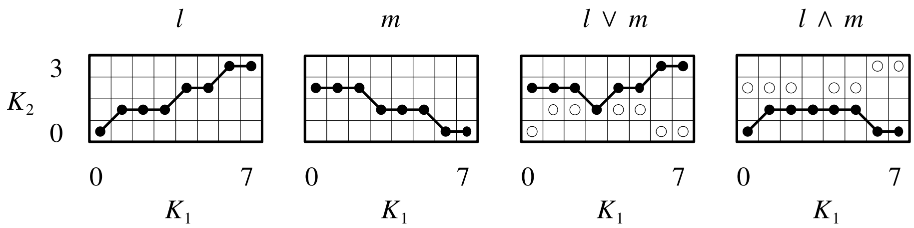


Fig. 3.5 – Supremum and infimum of two luts.

Figure 3.6 shows the constant luts o and i from $K_1 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ to $K_2 = \{0, 1, 2, 3\}$.

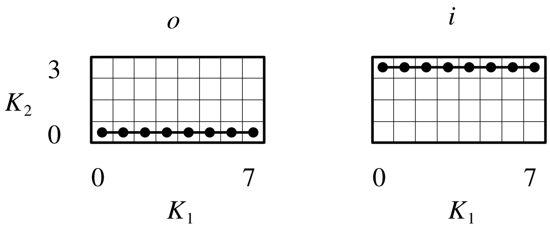


Fig. 3.6 – Two constant luts.

In the lattice $({K_2}^{K_1}, \vee, \wedge)$, we can identify the two idempotent and commutative monoids $({K_2}^{K_1}, \vee, o)$ and $({K_2}^{K_1}, \wedge, i)$.

Exercise 3.13 (submonoids of the morphological luts) – Prove that the algebraic structures $(\mathrm{D}(K_1, K_2), \vee, o)$ and $(\mathrm{E^a}(K_1, K_2), \vee, o)$ are submonoids of $({K_2}^{K_1}, \vee, o)$ and that the algebraic structures $(\mathrm{E}(K_1, K_2), \wedge, i)$ and $(\mathrm{D^a}(K_1, K_2), \wedge, i)$ are submonoids of $({K_2}^{K_1}, \wedge, i)$. □

Solution – Parts of Propositions 3.67 and 3.69 apply to the set of luts which are dilations, in other words, the sets $\mathrm{M}_\vee(K_1, K_2)$ and $\mathrm{M}_0(K_1, K_2)$ are $\vee_{K_2^{K_1}}$–closed and $o_{K_2^{K_1}}$–closed. Consequently, by Propositions 3.55 their intersection which is $\mathrm{D}(K_1, K_2)$ is also $\vee_{K_2^{K_1}}$–closed and $o_{K_2^{K_1}}$–closed. Hence, by Proposition 3.52, $(\mathrm{D}(K_1, K_2), \vee, o)$ is a submonoid of $(K_2^{K_1}, \vee, o)$. The same arguments can be used for the others algebraic structures. □

3.2 Spatially invariant pointwise operator

Let K_1 and K_2 be two gray–scales (continuous or discrete – actually, in the first part of this section we do not assume any algebraic structure on these two gray–scales), and let K_1^E and K_2^E be two sets of images whose domain is E.

Any mapping ψ from K_1^E to K_2^E transforms any image f in K_1^E into a unique image $\psi(f)$ in K_2^E. We call ψ an *image operator* or simply an *operator*. We denote by $\mathrm{Op}(K_1^E, K_2^E)$, or sometimes simply by Op, the set of operators from K_1^E to K_2^E. Figure 3.7 shows an image operator.

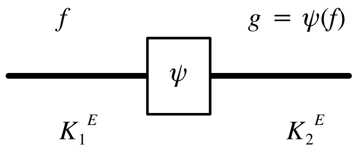


Fig. 3.7 – An image operator.

One of the simplest and yet useful class of image operators is that of pointwise operators.

Definition 3.14 (pointwise operator) – An operator ψ from K_1^E to K_2^E is *pointwise* iff, for any y in E, and f and g in K_1^E,

$$f(y) = g(y) \Rightarrow \psi(f)(y) = \psi(g)(y).$$ □

In other words, ψ is pointwise iff the value of the pixel of the transformed image at position y depends only on the value of the pixel of the original image at position y.

Let $\beta \in \mathbf{R}$, the mapping $f \mapsto f + \beta \cdot i$ is an example of pointwise operator from $\mathbf{R}^E$ to $\mathbf{R}^E$.

It is worthwhile characterizing the pointwise operators, i.e., finding a way to construct them.

Let ψ be an operator from ${K_1}^E$ to ${K_2}^E$. For any y in E, and s in K_1, we denote by $p_\psi(y)(s)$ the expression defined by

$$p_\psi(y)(s) \triangleq \psi(f)(y),$$

where f is any image of ${K_1}^E$ such that $f(y) = s$.

Let p be a mapping from E to ${K_2}^{K_1}$. For any f in ${K_1}^E$, and y in E, we denote by $\psi_p(f)(y)$ the expression defined by

$$\psi_p(f)(y) \triangleq (p(y) \circ f)(y).$$

Proposition 3.15 (pointwise operator characterization) – The mapping $\psi \mapsto p_\psi$ from the set of pointwise operators from ${K_1}^E$ to ${K_2}^E$ to the set of mappings from E to ${K_2}^{K_1}$ is a bijection. Its inverse is $p \mapsto \psi_p$. □

Proof – 1. For any pointwise operator ψ from ${K_1}^E$ to ${K_2}^E$, by definition, p_ψ is a mapping from E to ${K_2}^{K_1}$.

2. For any p from E to ${K_2}^{K_1}$, by definition, ψ_p is an operator from ${K_1}^E$ to ${K_2}^E$, furthermore, for any y in E, and f and g in ${K_1}^E$ such that $f(y) = g(y)$,

$$\begin{aligned}
\psi_p(f)(y) &= (p(y) \circ f)(y) && (\psi_p \text{ definition})\\
&= p(y)(f(y)) && (\text{composition definition})\\
&= p(y)(g(y)) && (\text{hypothesis})\\
&= (p(y) \circ g)(y) && (\text{composition definition})\\
&= \psi_p(g)(y), && (\psi_p \text{ definition})
\end{aligned}$$

that is, ψ_p is pointwise.

3. Let ψ be a pointwise operator from ${K_1}^E$ to ${K_2}^E$, for any y in E, and f in ${K_1}^E$,

$$\begin{aligned}
\psi_{p_\psi}(f)(y) &= (p_\psi(y) \circ f)(y) && (\psi_p \text{ definition})\\
&= p_\psi(y)(f(y)) && (\text{composition definition})\\
&= \psi(f)(y), && (p_\psi \text{ definition})
\end{aligned}$$

that is, $\psi \mapsto p_\psi$ is an injection.

4. Let p be a mapping from E to ${K_2}^{K_1}$, for any y in E, s in K_1, and f in ${K_1}^E$ such that $f(y) = s$,

$$\begin{aligned}
p_{\psi_p}(y)(s) &= \psi_p(f)(y) && (p_\psi \text{ definition})\\
&= (p(y) \circ f)(y) && (\psi_p \text{ definition})\\
&= p(y)(f(y)) && (\text{composition definition})\\
&= p(y)(s), && (\text{hypothesis})
\end{aligned}$$

that is, $\psi \mapsto p_\psi$ is a surjection. □

Most of the useful pointwise operators are spatially invariant.

Definition 3.16 (spatially invariant pointwise operator) – A pointwise operator ψ from ${K_1}^E$ to ${K_2}^E$ is *spatially invariant* iff for any y_1 and y_2 in E, and any f in ${K_1}^E$,

$$f(y_1) = f(y_2) \Rightarrow \psi(f)(y_1) = \psi(f)(y_2).$$

We denote by $\mathrm{IP}({K_1}^E, {K_2}^E)$ the set of spatially invariant pointwise operators from ${K_1}^E$ to ${K_2}^E$. □

The spatially invariant pointwise operators have the following equivalent definition.

Proposition 3.17 (equivalent definition of the spatially invariant pointwise operator) – A pointwise operator ψ from ${K_1}^E$ to ${K_2}^E$ is *spatially invariant* iff there exists a lut l from K_1 to K_2 such that for any y in E, $p_\psi(y) = l$. □

Proof – 1. Let us prove that if ψ from ${K_1}^E$ to ${K_2}^E$ is a spatially invariant pointwise operator then there exists a lut l from K_1 to K_2 such that for any y in E, $p_\psi(y) = l$.

Let s be any gray–level in K and let y be any point in E. We can find a point y' in E and an image g in ${K_1}^E$ such that $g(y') = s$. We can also find an image f in ${K_1}^E$ such that $f(y) = s$ and $f(y') = s$. For such s, let $l(s)$ be the expression given by,

$$l(s) \triangleq \psi(g)(y').$$

This expression defines a mapping l from K_1 to K_2. Let us show that $p_\psi(y) = l$.

For the above s and y, we have,

$$\begin{aligned} p_\psi(y)(s) &= \psi(f)(y) && (p_\psi \text{ definition}) \\ &= \psi(f)(y') && (\psi \text{ is spatially invariant}) \\ &= \psi(g)(y') && (\psi \text{ is pointwise}) \\ &= l(s). && (l(s) \text{ definition}) \end{aligned}$$

2. Let ψ be a pointwise operator from ${K_1}^E$ to ${K_2}^E$. Let us prove that if there exists a lut l from K_1 to K_2 such that for any y in E, $p_\psi(y) = l$ then ψ is spatially invariant.

Let y_1 and y_2 be any points in E and let f be in ${K_1}^E$ such that $f(y_1) = f(y_2)$. Then we have,

$$\begin{aligned} \psi(f)(y_1) &= p_\psi(y_1)(f(y_1)) && (p_\psi \text{ definition}) \\ &= l(f(y_1)) && (\text{hypothesis on } p_\psi) \\ &= l(f(y_2)) && (\text{hypothesis on } f) \\ &= p_\psi(y_2)(f(y_2)) && (\text{hypothesis on } p_\psi) \\ &= \psi(f)(y_2). && (p_\psi \text{ definition}) \end{aligned}$$

□

In the case of spatial invariance, the pointwise operator characterization is simpler.

Let ψ be an operator from $K_1{}^E$ to $K_2{}^E$. For any s in K_1, we denote by $l_\psi(s)$ the expression defined by

$$l_\psi(s) \triangleq \psi(f)(y),$$

where y is one particular element of E, and f is any image of $K_1{}^E$ such that $f(y) = s$.

Let l be a lut from K_1 to K_2. For any f in $K_1{}^E$, we denote by $\psi_l(f)$ the expression defined by

$$\psi_l(f) \triangleq l \circ f.$$

Proposition 3.18 (spatially invariant pointwise operator characterization) – The mapping $\psi \mapsto l_\psi$ from the set $\mathrm{IP}(K_1{}^E, K_2{}^E)$ of spatially invariant pointwise operators (from $K_1{}^E$ to $K_2{}^E$) to the set of luts from K_1 to K_2 is a bijection. Its inverse is $l \mapsto \psi_l$. □

Proof – 1. For any ψ in $\mathrm{IP}(K_1{}^E, K_2{}^E)$, by definition, l_ψ is a mapping from K_1 to K_2.

2. For any l in $K_2{}^{K_1}$, by definition, ψ_l is an operator from $K_1{}^E$ to $K_2{}^E$, furthermore, for any y in E, and f and g in $K_1{}^E$ such that $f(y) = g(y)$,

$$\begin{aligned}
\psi_l(f)(y) &= (l \circ f)(y) && (\psi_l \text{ definition})\\
&= l(f(y)) && (\text{composition definition})\\
&= l(g(y)) && (\text{hypothesis})\\
&= (l \circ g)(y) && (\text{composition definition})\\
&= \psi_l(g)(y), && (\psi_l \text{ definition})
\end{aligned}$$

that is, ψ_l is pointwise.

Furthermore, for any y in E, s in K_1, and f in $K_1{}^E$ such that $f(y) = s$,

$$\begin{aligned}
p_{\psi_l}(y)(s) &= \psi_l(f)(y) && (p_\psi \text{ definition})\\
&= (l \circ f)(y) && (\psi_l \text{ definition})\\
&= l(f(y)) && (\text{composition definition})\\
&= l(s), && (\text{hypothesis})
\end{aligned}$$

that is, by Proposition 3.17, ψ_l is pointwise and spatially invariant.

The rest of the proof is similar to Parts 3 and 4 of the proof of Proposition 3.15. □

Figure 3.8 shows three ways of representing a spatially invariant pointwise operator.

Exercise 3.19 (spatially invariant pointwise operator properties) – Prove that the class of spatially invariant pointwise operators from K^E to K^E is closed under the composition. By Proposition 3.18, it is sufficient to prove that for any luts l and m from K to K,

$$\psi_l \circ \psi_m = \psi_{l \circ m}\,.$$

□

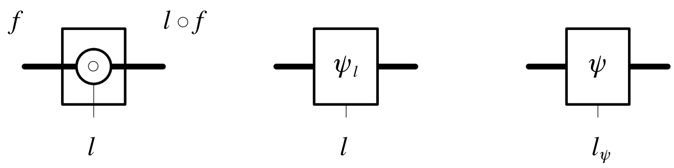


Fig. 3.8 – A spatially invariant pointwise operator.

Before ending this section, we will state two propositions that will be useful for the next section. The first proposition is about morphism properties of the construction $l \mapsto \psi_l$ and the second one is about closure properties of the spatially invariant pointwise operators.

In order to be able to talk about the morphism properties of $l \mapsto \psi_l$, we now assume that K_2 is equipped with some operations and we extend them to the set $\mathrm{Op}({K_1}^E, {K_2}^E)$ of image operators. For the moment, we do not need to make any algebraic assumption on the gray–scale K_1.

Let $\square$ be the extension to ${K_2}^E$ of an internal binary operation $\square$ on K_2. We denote by $\square_{\mathrm{Op}}$ or simply by $\square$ when no confusion is likely to arise, the extension to the operators from ${K_1}^E$ to ${K_2}^E$ of the internal binary operation $\square$ on the images in ${K_2}^E$.

In the same way, let $\cdot$ be the extension to ${K_2}^E$ of an external binary operation $\cdot$ on K_2, with operand in set C. We denote by $\cdot_{\mathrm{Op}}$ or simply by $\cdot$ when no confusion is likely to arise, the extension to the operators from ${K_1}^E$ to ${K_2}^E$ of the external binary operation $\cdot$ on the images in ${K_2}^E$, with operand in C.

Finally, let e be the extension on ${K_2}^E$ of a nullary operation e on K_2. We denote by $\varepsilon_{\mathrm{Op}}$ or simply by ε the extension to the operators from ${K_1}^E$ to ${K_2}^E$ of the nullary operation e on the images in ${K_2}^E$. This operation selects the operator ε which produces the constant image e.

Figure 3.9 shows the above internal and external binary operations on operators.

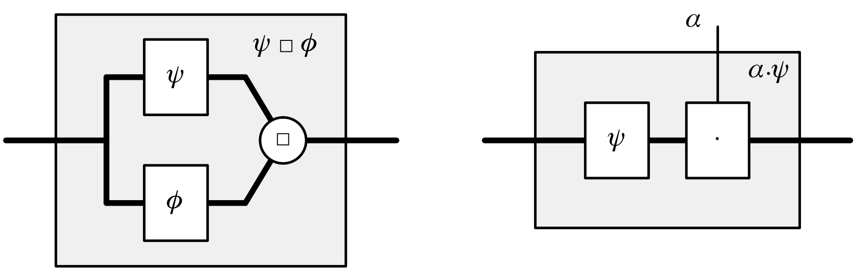


Fig. 3.9 – Internal and external binary operations on image operators.

Proposition 3.20 (morphisms between luts and operators) – Let E and K_1 be two non–empty sets.

1. If $(K_2, \square)$ is a set equipped with an internal binary operation, then the mapping $l \mapsto \psi_l$ from the set of luts from K_1 to K_2 to the set of operators from ${K_1}^E$ to ${K_2}^E$ is a morphism from $({K_2}^{K_1}, \square)$ to $(\mathrm{Op}({K_1}^E, {K_2}^E), \square)$. That is, we have, for any luts l and m from K_1 to K_2,

$$\psi_{l \,\square\, m} = \psi_l \,\square\, \psi_m.$$

2. If $(K_2, \cdot)$ is a set equipped with an external binary operation with operand in C, then the mapping $l \mapsto \psi_l$ is a morphism from $({K_2}^{K_1}, \cdot)$ to $(\mathrm{Op}({K_1}^E, {K_2}^E), \cdot)$. That is, we have, for any lut l from K_1 to K_2, and any α in C,

$$\psi_{\alpha . l} = \alpha \cdot \psi_l.$$

3. If (K_2, e) is a set equipped with a nullary operation, then the mapping $l \mapsto \psi_l$ is a morphism from $({K_2}^{K_1}, e)$ to $(\mathrm{Op}({K_1}^E, {K_2}^E), \varepsilon)$. That is, we have,

$$\psi_e = \varepsilon. \qquad \square$$

Proof – 1. For any mapping l and m from K_1 to K_2, any f in ${K_1}^E$, and any y in E,

$$\begin{aligned}
\psi_{l \,\square\, m}(f)(y) &= ((l \,\square\, m) \circ f)(y) && (\psi_l \text{ definition})\\
&= (l \,\square\, m)(f(y)) && (\text{composition definition})\\
&= l(f(y)) \,\square\, m(f(y)) && (\text{extension definition})\\
&= (l \circ f)(y) \,\square\, (m \circ f)(y) && (\text{composition def.})\\
&= \psi_l(f)(y) \,\square\, \psi_m(f)(y) && (\psi_l \text{ definition})\\
&= (\psi_l(f) \,\square\, \psi_m(f))(y) && (\text{extension definition})\\
&= (\psi_l \,\square\, \psi_m)(f)(y), && (\text{extension definition})
\end{aligned}$$

that is, for any mapping l and m from K_1 to K_2, $\psi_{l \,\square\, m} = \psi_l \,\square\, \psi_m$.

2. For any mapping l from K_1 to K_2, any α in C, any f in ${K_1}^E$, and any y in E,

$$\begin{aligned}
\psi_{\alpha . l}(f)(y) &= ((\alpha \cdot l) \circ f)(y) && (\psi_l \text{ definition})\\
&= (\alpha \cdot l)(f(y)) && (\text{composition definition})\\
&= \alpha \cdot (l(f(y)) && (\text{extension definition})\\
&= \alpha \cdot (l \circ f)(y) && (\text{composition definition})\\
&= \alpha \cdot \psi_l(f)(y) && (\psi_l \text{ definition})\\
&= (\alpha \cdot \psi_l(f))(y) && (\text{extension definition})\\
&= (\alpha \cdot \psi_l)(f)(y) && (\text{extension definition})
\end{aligned}$$

that is, for any mapping l from K_1 to K_2, and any α in C, $\psi_{\alpha . l} = \alpha \cdot \psi_l$.

3. For any f in ${K_1}^E$, and any y in E,

$$\begin{aligned}
\psi_e(f)(y) &= (e \circ f)(y) && (\psi_l \text{ definition})\\
&= e(f(y)) && (\text{composition definition})\\
&= e_2 && (\text{extension def., } e \text{ is a lut})\\
&= e(y) && (\text{extension def., } e \text{ is an image})\\
&= \varepsilon(f)(y), && (\text{extension def., } \varepsilon \text{ is an operator})
\end{aligned}$$

that is, $\psi_e = \varepsilon$. □

Figure 3.10 illustrates graphically the equality $\psi_{l \,\square\, m} = \psi_l \,\square\, \psi_m$ of Proposition 3.20. We observe that the right handside implementation is simpler than the left handside one.

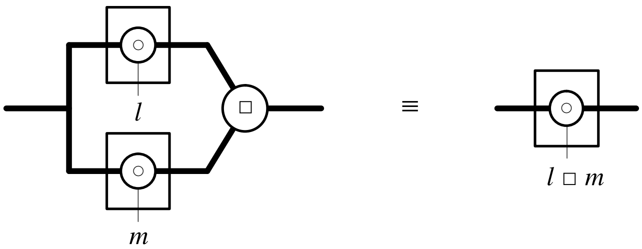


Fig. 3.10 – Morphism between luts and operators.

As a consequence of the previous proposition we have the following result.

Proposition 3.21 (closed subset of spatially invariant pointwise operators) – Let E and K_1 be two non–empty sets.

1. If $(K_2, \square)$ is a set equipped with an internal binary operation, then the subset $\mathrm{IP}({K_1}^E, {K_2}^E)$ is $\square_{\mathrm{Op}}$–closed.

2. If $(K_2, \cdot)$ is a set equipped with an external binary operation with operand in a set C, then the subset $\mathrm{IP}({K_1}^E, {K_2}^E)$ $\cdot_{\mathrm{Op}}$–closed.

3. If (K_2, e) is a set equipped with a nullary operation, then the subset $\mathrm{IP}({K_1}^E, {K_2}^E)$ $\varepsilon_{\mathrm{Op}}$–closed. □

Proof – By Proposition 3.18, the image of ${K_2}^{K_1}$ through $l \mapsto \psi_l$ is $\mathrm{IP}({K_1}^E, {K_2}^E)$.

1. By Part 1 of Proposition 3.20, $l \mapsto \psi_l$ is a morphism from $({K_2}^{K_1}, \square)$ to $(\mathrm{Op}({K_1}^E, {K_2}^E), \square)$. Therefore, from Part 1 of Exercise 3.57, $\mathrm{IP}({K_1}^E, {K_2}^E)$ is $\square_{\mathrm{Op}}$–closed.

2. By Part 2 of Proposition 3.20, $l \mapsto \psi_l$ is a morphism from $({K_2}^{K_1}, \cdot)$ to $(\mathrm{Op}({K_1}^E, {K_2}^E), \cdot)$. Therefore, from Part 2 of Exercise 3.57, $\mathrm{IP}({K_1}^E, {K_2}^E)$ is $\cdot_{\mathrm{Op}}$–closed.

3. By Proposition 3.20, $\psi_e = \varepsilon$. This prove that, ε is in $\mathrm{IP}({K_1}^E, {K_2}^E)$. □

3.3 Spatially invariant pointwise morphism

In this section we will characterize the spatially invariant pointwise operators which are morphisms. To do so we now need to consider some algebraic structures on **both** gray–scales K_1 and K_2.

Let $(K_1, \square)$ and $(K_2, \square)$ be two sets equipped with internal binary operations. We denote by $\mathrm{IPM}_\square({K_1}^E, {K_2}^E)$, the set of spatially invariant pointwise operators which are morphisms for the extensions of the $\square$'s to ${K_1}^E$ and ${K_2}^E$.

In the same way, let $(K_1, \cdot)$ and $(K_2, \cdot)$ be two sets equipped with external binary operations with operand in C. We denote by $\mathrm{IPM}_\cdot({K_1}^E, {K_2}^E)$, the set of spatially invariant pointwise operators which are morphisms for the extensions of the $\cdot$'s to ${K_1}^E$ and ${K_2}^E$.

Finally, let (K_1, e_1) and (K_2, e_2) be two sets equipped with nullary operations. We denote by $\mathrm{IPM}_e({K_1}^E, {K_2}^E)$, the set of spatially invariant pointwise operators which are morphisms for the extensions of e_1 and e_2 to, respectively, ${K_1}^E$ and ${K_2}^E$.

Proposition 3.22 (characterization of the spatially invariant pointwise operators which are morphisms) – Let E be a non–empty set.

1.1. Let $(K_1, \square)$ be a set equipped with an internal binary operation. If $(K_2, \square)$ is a commutative semigroup and if $(K_2, \cdot)$ is a set equipped with an external binary operation with operand in C which is distributive over $\square$, then the mapping $\psi \mapsto l_\psi$ is an isomorphism from $(\mathrm{IPM}_\square({K_1}^E, {K_2}^E), \square, \cdot)$ to $(\mathrm{M}_\square(K_1, K_2), \square, \cdot)$.

1.2. Let $(K_1, \square)$ be a set equipped with an internal binary operation. If $(K_2, \square, e_2)$ is a commutative monoid, then the mapping $\psi \mapsto l_\psi$ is an isomorphism from $(\mathrm{IPM}_\square({K_1}^E, {K_2}^E), \square, \varepsilon)$ to $(\mathrm{M}_\square(K_1, K_2), \square, e)$.

2. Let $(C, \cdot)$ be a set equipped with a commutative internal binary operation and let $(K_1, \cdot)$ and $(K_2, \cdot)$ be two sets equipped with external binary operations with operand in C. If $(K_2, \cdot)$ satisfies, for any t in K_2, and any α and β in C,

$$\alpha \cdot (\beta \cdot t) = (\alpha \cdot \beta) \cdot t, \qquad \text{(mixt associativity)}$$

and $(K_2, \square)$ is a set equipped with an internal binary operation such that $\cdot$ on K_2 is distributive over $\square$, then the mapping $\psi \mapsto l_\psi$ is an isomorphism from $(\mathrm{IPM}_\cdot({K_2}^E, {K_2}^E), \square, \cdot)$ to $(\mathrm{M}_\cdot(K_1, K_2), \square, \cdot)$.

3. Let (K_1, e_1) and (K_2, e_2) be two sets equipped with nullary operations. If $(K_2, \square, e_2)$ is a monoid, then the mapping $\psi \mapsto l_\psi$ is an isomorphism from $(\mathrm{IPM}_e({K_1}^E, {K_2}^E), \square, \varepsilon)$ to $(\mathrm{M}_e(K_1, K_2), \square, e)$. □

Proof – Let us divide the proof into four main steps.

1.1. Let ψ be in $\mathrm{IPM}_\square({K_1}^E, {K_2}^E)$. Let us prove that l_ψ is in $\mathrm{M}_\square(K_1, K_2)$. For any s and t in K_1, let y be any element of E, and let f and g be any images of ${K_1}^E$ such that $f(y) = s$ and $g(y) = t$, then

$$
\begin{aligned}
l_\psi(s \mathbin{\square} t) &= \psi(f \mathbin{\square} g)(y) && \text{(extension def. and } l_\psi \text{ def.)}\\
&= (\psi(f) \mathbin{\square} \psi(g))(y) && (\psi \text{ is a morphism})\\
&= \psi(f)(y) \mathbin{\square} \psi(g)(y) && \text{(extension definition)}\\
&= l_\psi(s) \mathbin{\square} l_\psi(t). && (l_\psi \text{ definition})
\end{aligned}
$$

1.2. Let ψ be in $\mathrm{IPM}_{\cdot}(K_1{}^E, K_2{}^E)$. Let us prove that l_ψ is in $\mathrm{M}_{\cdot}(K_1, K_2)$. For any s in K_1, let y be any element of E, and let f be any image of $K_1{}^E$ such that $f(y) = s$, then, for any α in C,

$$
\begin{aligned}
l_\psi(\alpha \cdot s) &= \psi(\alpha \cdot f)(y) && \text{(extension def. and } l_\psi \text{ def.)}\\
&= (\alpha \cdot \psi(f))(y) && (\psi \text{ is a morphism})\\
&= \alpha \cdot \psi(f)(y) && \text{(extension definition)}\\
&= \alpha \cdot l_\psi(s). && (l_\psi \text{ definition})
\end{aligned}
$$

1.3. Let ψ be in $\mathrm{IPM}_{e}(K_1{}^E, K_2{}^E)$. Let us prove that l_ψ is in $\mathrm{M}_e(K_1, K_2)$. Let y be any element of E, then

$$
\begin{aligned}
l_\psi(e_1) &= \psi(e_1)(y) && (l_\psi \text{ definition})\\
&= e_2(y) && (\psi \text{ is a morphism})\\
&= e_2. && \text{(extension definition)}
\end{aligned}
$$

2.1. Let l be in $\mathrm{M}_{\square}(K_1, K_2)$. Let us prove that ψ_l is in $\mathrm{IPM}_{\square}(K_1{}^E, K_2{}^E)$. For any images f and g of $K_1{}^E$, and any y in E,

$$
\begin{aligned}
\psi_l(f \mathbin{\square} g)(y) &= (l \circ (f \mathbin{\square} g))(y) && (\psi_l \text{ definition})\\
&= l((f \mathbin{\square} g)(y)) && \text{(composition definition)}\\
&= l(f(y) \mathbin{\square} g(y)) && \text{(extension definition)}\\
&= l(f(y)) \mathbin{\square} l(g(y)) && (l \text{ is a morphism})\\
&= (l \circ f)(y) \mathbin{\square} (l \circ g)(y) && \text{(composition definition)}\\
&= \psi_l(f)(y) \mathbin{\square} \psi_l(g)(y) && (\psi_l \text{ definition})\\
&= (\psi_l(f) \mathbin{\square} \psi_l(g))(y). && \text{(extension definition)}
\end{aligned}
$$

2.2. Let l be in $\mathrm{M}_{\cdot}(K_1, K_2)$. Let us prove that ψ_l is in $\mathrm{IPM}_{\cdot}(K_1{}^E, K_2{}^E)$. For any image f of $K_1{}^E$, any y in E, and any α in C,

$$
\begin{aligned}
\psi_l(\alpha \cdot f)(y) &= (l \circ (\alpha \cdot f))(y) && (\psi_l \text{ definition})\\
&= l((\alpha \cdot f)(y)) && \text{(composition definition)}\\
&= l(\alpha \cdot f(y)) && \text{(extension definition)}
\end{aligned}
$$

$$= \alpha\cdot l(f(y)) \qquad (l \text{ is a morphism})$$

$$= \alpha\cdot(l \circ f)(y) \qquad (\text{composition definition})$$

$$= \alpha\cdot\psi_l(f)(y) \qquad (\psi_l \text{ definition})$$

$$= (\alpha\cdot\psi_l(f))(y). \qquad (\text{extension definition})$$

2.3. Let l be in $\mathrm{M}_e(K_1, K_2)$. Let us prove that ψ_l is in $\mathrm{IPM}_{e.}({K_1}^E, {K_2}^E)$. For any y in E,

$$\psi_l(e_1)(y) = (l \circ e_1)(y) \qquad (\psi_l \text{ definition})$$

$$= l(e_1(y)) \qquad (\text{composition definition})$$

$$= l(e_1) \qquad (\text{extension definition})$$

$$= e_2 \qquad (l \text{ is a morphism})$$

$$= e_2(y). \qquad (\text{extension definition})$$

3.1. From Parts 1.1 and 2.1 above, and by Exercise 1.26 and Proposition 3.18, $\psi \mapsto l_\psi$ from $\mathrm{IPM}_\square({K_1}^E, {K_2}^E)$ to $\mathrm{M}_\square(K_1, K_2)$ is a bijection.

3.2. From Parts 1.2 and 2.2 above, and by Exercise 1.26 and Proposition 3.18, $\psi \mapsto l_\psi$ from $\mathrm{IPM}_\cdot({K_1}^E, {K_2}^E)$ to $\mathrm{M}_\cdot(K_1, K_2)$ is a bijection.

3.3. From Parts 1.3 and 2.3 above, and by Exercise 1.26 and Proposition 3.18, $\psi \mapsto l_\psi$ from $\mathrm{IPM}_e({K_1}^E, {K_2}^E)$ to $\mathrm{M}_e(K_1, K_2)$ is a bijection.

4.1.1. The set $\mathrm{IPM}_\square({K_1}^E, {K_2}^E)$ is $\square_{\mathrm{Op}}$–closed and $\cdot_{\mathrm{Op}}$–closed since it is the intersection of the subsets $\mathrm{IP}({K_1}^E, {K_2}^E)$ and $\mathrm{M}_\square({K_1}^E, {K_2}^E)$ which are, respectively, $\square_{\mathrm{Op}}$–closed and $\cdot_{\mathrm{Op}}$–closed by Propositions 3.21 and 3.67. The set $\mathrm{M}_\square(K_1, K_2)$ is $\square_{{K_2}^{K_1}}$–closed and $\cdot_{{K_2}^{K_1}}$–closed by Proposition 3.67. Finally, by Proposition 3.20, $l \mapsto \psi_l$ is a morphism from $({K_2}^{K_1}, \square, \cdot)$ to $(\mathrm{Op}({K_1}^E, {K_2}^E), \square, \cdot)$, therefore, by Part 3.1, $l \mapsto \psi_l$ from $\mathrm{M}_\square(K_1, K_2)$ to $\mathrm{IPM}_\square({K_1}^E, {K_2}^E)$ is an isomorphism from $(\mathrm{M}_\square(K_1, K_2), \square, \cdot)$ to $(\mathrm{IPM}_\square({K_1}^E, {K_2}^E), \square, \cdot)$, that is (as in Exercise 3.59), its inverse $\psi \mapsto l_\psi$ is an isomorphism from $(\mathrm{IPM}_\square({K_1}^E, {K_2}^E), \square, \cdot)$ to $(\mathrm{M}_\square(K_1, K_2), \square, \cdot)$.

4.1.2. The set $\mathrm{IPM}_\square({K_1}^E, {K_2}^E)$ is $\square_{\mathrm{Op}}$–closed and $\varepsilon_{\mathrm{Op}}$–closed since it is the intersection of the subsets $\mathrm{IP}({K_1}^E, {K_2}^E)$ and $\mathrm{M}_\square({K_1}^E, {K_2}^E)$ which are, respectively, $\square_{\mathrm{Op}}$–closed and $\varepsilon_{\mathrm{Op}}$–closed by Propositions 3.21 and 3.67. The set $\mathrm{M}_\square(K_1, K_2)$ is $\square_{{K_2}^{K_1}}$–closed and $e_{{K_2}^{K_1}}$–closed by Proposition 3.67. Finally, by Proposition 3.20, $l \mapsto \psi_l$ is a morphism from $({K_2}^{K_1}, \square, e)$ to $(\mathrm{Op}({K_1}^E, {K_2}^E), \square, \varepsilon)$, therefore, by Part 3.1, $l \mapsto \psi_l$ from $\mathrm{M}_\square(K_1, K_2)$ to $\mathrm{IPM}_\square({K_1}^E, {K_2}^E)$ is an isomorphism from $(\mathrm{M}_\square(K_1, K_2), \square, e)$ to $(\mathrm{IPM}_\square({K_1}^E, {K_2}^E), \square, \varepsilon)$, that is (as in Exercise 3.59), its inverse $\psi \mapsto l_\psi$ is an isomorphism from $(\mathrm{IPM}_\square({K_1}^E, {K_2}^E), \square, \varepsilon)$ to $(\mathrm{M}_\square(K_1, K_2), \square, e)$.

4.2. The set $\mathrm{IPM}_{\cdot}({K_1}^E, {K_2}^E)$ is $\square_{\mathrm{Op}}$–closed and $\cdot_{\mathrm{Op}}$–closed since it is the intersection of the subsets $\mathrm{IP}({K_1}^E, {K_2}^E)$ and $\mathrm{M}_{\cdot}({K_1}^E, {K_2}^E)$ which are, respectively, $\square_{\mathrm{Op}}$–closed and $\cdot_{\mathrm{Op}}$–closed by Propositions 3.21 and 3.68. The set $\mathrm{M}_{\cdot}(K_1, K_2)$ is $\square_{{K_2}^{K_1}}$–closed and $\cdot_{{K_2}^{K_1}}$–closed by Proposition 3.68. Finally, by Proposition 3.20, $l \mapsto \psi_l$ is a morphism from $({K_2}^{K_1}, \square, \cdot)$ to $(\mathrm{Op}({K_1}^E, {K_2}^E), \square, \cdot)$, therefore, by Part 3.2, $l \mapsto \psi_l$ is an isomorphism from $(\mathrm{M}_{\cdot}(K_1, K_2), \square, \cdot)$ to $(\mathrm{IPM}_{\cdot}({K_1}^E, {K_2}^E), \square, \cdot)$, that is, its inverse $\psi \mapsto l_\psi$ is an isomorphism from $(\mathrm{IPM}_{\cdot}({K_1}^E, {K_2}^E), \square, \cdot)$ to $(\mathrm{M}_{\cdot}(K_1, K_2), \square, \cdot)$.

4.3. The set $\mathrm{IPM}_e({K_1}^E, {K_2}^E)$ is $\square_{\mathrm{Op}}$–closed and e_{Op}–closed since it is the intersection of the subsets $\mathrm{IP}({K_1}^E, {K_2}^E)$ and $\mathrm{M}_e({K_1}^E, {K_2}^E)$ which are, respectively, $\square_{\mathrm{Op}}$–closed and e_{Op}–closed by Propositions 3.21 and 3.69. The set $\mathrm{M}_e(K_1, K_2)$ is $\square_{{K_2}^{K_1}}$–closed and $e_{{K_2}^{K_1}}$–closed by Proposition 3.69. Finally, by Proposition 3.20, $l \mapsto \psi_l$ is a morphism from $({K_2}^{K_1}, \square, e)$ to $(\mathrm{Op}({K_1}^E, {K_2}^E), \square, \varepsilon)$, therefore, by Part 3.3, $l \mapsto \psi_l$ is an isomorphism from $(\mathrm{M}_e(K_1, K_2), \square, e)$ to $(\mathrm{IPM}_e({K_1}^E, {K_2}^E), \square, \varepsilon)$, that is, its inverse $\psi \mapsto l_\psi$ is an isomorphism from $(\mathrm{IPM}_e({K_1}^E, {K_2}^E), \square, \varepsilon)$ to $(\mathrm{M}_e(K_1, K_2), \square, e)$. $\square$

Figure 3.11 shows the isomorphism $\psi \mapsto l_\psi$ of Proposition 3.22 for the internal binary operations on the set of operators and on the set of luts.

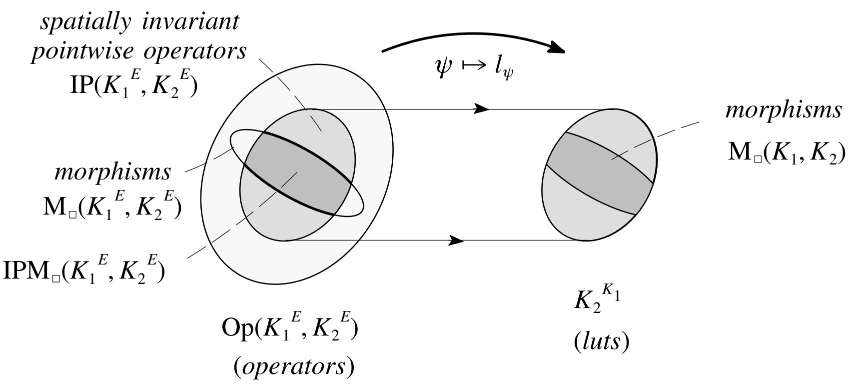


Fig. 3.11 – Isomorphism.

Figure 3.12 illustrates graphically an aspect of Part 1.2 of Proposition 3.22. From Part 1.2 we know that if l is a lut which is a morphism of internal binary operations on K_1 and K_2, then the spatially invariant pointwise operator constructed with this lut is also a morphism of internal binary operations on ${K_1}^E$ and ${K_2}^E$. This means that we have the two equivalent implementations shown in Figure 3.12. We observe that the right handside implementation is simpler than the left handside one.

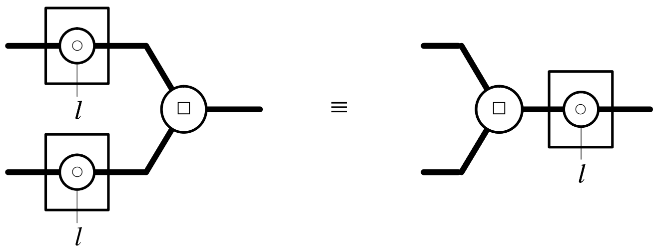


Fig. 3.12 – Two equivalent implementations when the lut is a morphism.

Recalling (see Part 1 of Proposition 3.67) that if the luts l and m are two morphisms for the internal binary operations $\square$'s on K_1 and K_2 then $l \square m$ is also such a morphism, we can combine the equivalence of Figure 3.12 with the one shown in Figure 3.10, to get the equivalence between the two implementations shown in Figure 3.13. We observe that the right handside implementations is much simpler than the left handside one.

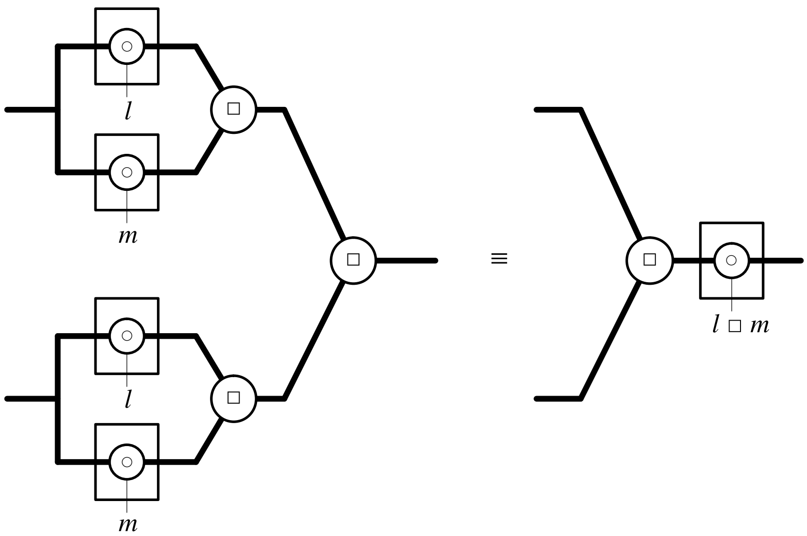


Fig. 3.13 – Two equivalent implementations when the luts are morphisms.

Before ending this section, we apply Proposition 3.22 to the continuous and discrete cases. Let consider first the continuous case, that is the case of the linear operators

In Section 2.1, we have introduced the image addition and multiplication by a scalar. By Proposition 2.1, we know that the set $(\mathbf{R}^E, +, \cdot, o)$ of real images is a linear vector space. Therefore, we can apply Definition 3.60 to the image operators.

Definition 3.23 (linear operator) – An operator ψ from $(\mathbf{R}^E, +, \cdot, o)$ to $(\mathbf{R}^E, +, \cdot, o)$, is *linear* (not necessarily spatially invariant) iff, for any f, g in $\mathbf{R}^E$ and α in $\mathbf{R}$,

$$\psi(f + g) = \psi(f) + \psi(g) \quad \text{and} \quad \psi(\alpha.f) = \alpha.\psi(f). \qquad \square$$

From Exercise 3.61, a linear operator ψ verifies $\psi(o) = o$.

The algebraic structure $(\mathrm{Op}(\mathbf{R}^E, \mathbf{R}^E), +, \cdot, o)$ consisting of the set of operators equipped with the extended operations from the operations on $\mathbf{R}$, is again a linear vector space.

Furthermore, Propositions 3.67 and 3.68 apply to the set of operators, in other words, $\mathrm{M}_+(\mathbf{R}^E, \mathbf{R}^E)$ and $\mathrm{M}_\cdot(\mathbf{R}^E, \mathbf{R}^E)$ are $+_{\mathrm{Op}}$–closed and $\cdot_{\mathrm{Op}}$–closed. Consequently, by Proposition 3.55 their intersection which is $\mathrm{L}(\mathbf{R}^E, \mathbf{R}^E)$ is also $+_{\mathrm{Op}}$–closed and $\cdot_{\mathrm{Op}}$–closed. Hence, by Proposition 3.53, $(\mathrm{L}(\mathbf{R}^E, \mathbf{R}^E), +, \cdot, o)$ is a subspace of $(\mathrm{Op}(\mathbf{R}^E, \mathbf{R}^E), +, \cdot, o)$.

Now combining linearity with spatial invariance, we have the following proposition.

Proposition 3.24 (characterization of the spatially invariant pointwise operators which are linear) – Let E be a non–empty set and let $(\mathbf{R}, +, \cdot, 0)$ be the linear vector space of the real numbers. The mapping $\psi \mapsto l_\psi$ from the subspace $(\mathrm{IPL}(\mathbf{R}^E, \mathbf{R}^E), +, \cdot, o)$ of spatially invariant pointwise operators which are linear, to the subspace $(\mathrm{L}(\mathbf{R}, \mathbf{R}), +, \cdot, o)$ of luts which are linear, is an isomorphism of linear vector spaces. Its inverse is $l \mapsto \psi_l$. In particular, we have, for any ψ and ϕ in $\mathrm{IPL}(\mathbf{R}^E, \mathbf{R}^E)$, and α in $\mathbf{R}$,

$$l_{\psi+\phi} = l_\psi + l_\phi\,, \quad l_{\alpha.\psi} = \alpha.l_\psi \quad \text{and} \quad l_o = o,$$

and, for any l and m in $\mathrm{L}(\mathbf{R}, \mathbf{R})$, and α in $\mathbf{R}$,

$$\psi_{l+m} = \psi_l + \psi_m\,, \quad \psi_{\alpha.l} = \alpha.\psi_l \quad \text{and} \quad \psi_o = o. \qquad \square$$

Proof – We divide the proof in four steps.

1. Parts of Propositions 3.67 and 3.68 apply to the set of operators, in other words, $\mathrm{IPM}_+(\mathbf{R}^E, \mathbf{R}^E)$ and $\mathrm{IPM}_\cdot(\mathbf{R}^E, \mathbf{R}^E)$ are $+_{\mathrm{Op}}$–closed and $\cdot_{\mathrm{Op}}$–closed. Consequently, by Proposition 3.55 their intersection which is $\mathrm{IPL}(\mathbf{R}^E, \mathbf{R}^E)$ is also $+_{\mathrm{Op}}$–closed and $\cdot_{\mathrm{Op}}$–closed. Hence, by Proposition 3.53, $(\mathrm{IPL}(\mathbf{R}^E, \mathbf{R}^E), +, \cdot, o)$ is a subspace of the linear vector space $(\mathrm{Op}(\mathbf{R}^E, \mathbf{R}^E), +, \cdot, o)$.

2. By Exercise 3.12, $(\mathrm{L}(\mathbf{R}, \mathbf{R}), +, \cdot, o)$ is a subspace of the linear vector space $(\mathbf{R}^{\mathbf{R}}, +, \cdot, o)$.

3. From Proposition 3.18, $\psi \mapsto l_\psi$ is a bijection and from Proposition 3.22, $\mathrm{M}_+(\mathbf{R}, \mathbf{R})$ (resp. $\mathrm{M}_\cdot(\mathbf{R}, \mathbf{R})$) is the image of $\mathrm{IPM}_+(\mathbf{R}^E, \mathbf{R}^E)$ (resp. $\mathrm{IPM}_\cdot(\mathbf{R}^E, \mathbf{R}^E)$) through $\psi \mapsto l_\psi$, therefore, by Exercise 1.30, $\mathrm{L}(\mathbf{R}, \mathbf{R})$ is the image of $\mathrm{IPL}(\mathbf{R}^E, \mathbf{R}^E)$ through $\psi \mapsto l_\psi$.

4. By Parts 1.1 and 2 of Proposition 3.22, $\psi \mapsto l_\psi$ is an isomorphism from $(\mathrm{IPM}_+(\mathbf{R}^E, \mathbf{R}^E), +, \cdot)$ to $(\mathrm{M}_+(\mathbf{R}, \mathbf{R}), +, \cdot)$ and an isomorphism from $(\mathrm{IPM}_\cdot(\mathbf{R}^E, \mathbf{R}^E), +, \cdot)$ to $\mathrm{M}_\cdot(\mathbf{R}, \mathbf{R}), +, \cdot)$. As we see above in Part 1, $\mathrm{IPL}(\mathbf{R}^E, \mathbf{R}^E)$ is $+_{\mathrm{Op}}$–closed and $\cdot_{\mathrm{Op}}$–closed, therefore by Exercise 3.58 and Part 3 above, $\psi \mapsto l_\psi$ restricted to $\mathrm{IPL}(\mathbf{R}^E, \mathbf{R}^E)$ and $\mathrm{L}(\mathbf{R}, \mathbf{R})$ is an isomorphism from $(\mathrm{IPL}(\mathbf{R}^E, \mathbf{R}^E), +, \cdot)$ to $(\mathrm{L}(\mathbf{R}, \mathbf{R}), +, \cdot)$. $\square$

We now consider the discrete case.

Let $(K, \leq)$ be the interval $[0, k]$ of natural numbers equipped with the usual order relation. By Propositions 3.72, the poset $(K^E, \leq_{\leq})$ is a lattice and we can introduce the following four operations on the set K^E.

The *image union*, denoted by $\vee$, is the union on $(K^E, \leq_{\leq})$, or equivalently (see Proposition 3.72), the extension to K^E of the union $\vee$ on K.

The *image intersection*, denoted by $\wedge$, is the intersection on $(K^E, \leq_{\leq})$, or equivalently (see Proposition 3.72), the extension to K^E of the intersection $\wedge$ on K.

The *black image selection*, denoted by o, is the nullary operation on $(K^E, \leq_{\leq})$ which selects the least image $x \mapsto 0$ called the *black image* (denoted by o), or equivalently (see Proposition 3.73), the extension to K^E of the nullary operation on K which select the value 0.

The *white image selection*, denoted by i, is the nullary operation on $(K^E, \leq_{\leq})$ which selects the greatest image $x \mapsto k$ called the *white image* (denoted by i), or equivalently (see Proposition 3.73), the extension to K^E of the nullary operation on K which select the value k.

Figure 3.14 shows through block diagrams the above four operations on images of size 2 by 2 with the gray–scale $K = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

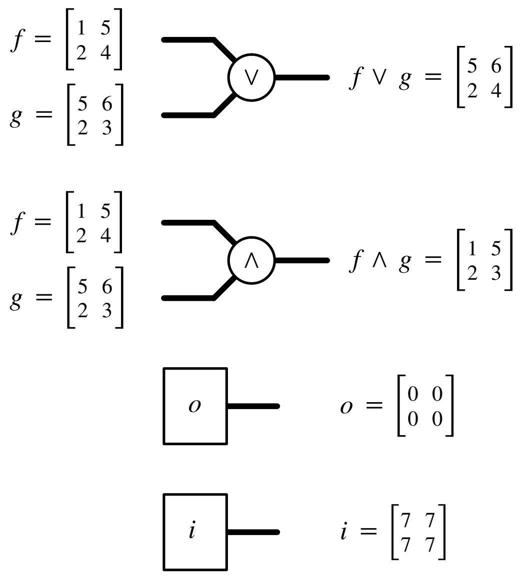


Fig. 3.14– Operations on images.

In the lattice of images $(K^E, \vee, \wedge)$ we can identify two idempotent and commutative monoids, namely $(K^E, \vee, o)$ and $(K^E, \wedge, i)$.

The same can be done for the two lattices of images $(K_1{}^E, \vee, \wedge)$ and $(K_2{}^E, \vee, \wedge)$. Therefore, we can apply Definition 3.62 to the image operators.

Definition 3.25 (elementary morphological operators) – An operator ψ from $(K_1{}^E, \vee, o)$ to $(K_2{}^E, \vee, o)$ is a *dilation* iff, for any f and g in $K_1{}^E$,

$$\psi(f \vee g) = \psi(f) \vee \psi(g) \quad \text{and} \quad \psi(o) = o;$$

an operator ψ from $(K_1{}^E, \wedge, i)$ to $(K_2{}^E, \wedge, i)$ is an *erosion* iff for any f and g in $K_1{}^E$,

$$\psi(f \wedge g) = \psi(f) \wedge \psi(g) \quad \text{and} \quad \psi(i) = i\,;$$

an operator ψ from $(K_1{}^E, \vee, o)$ to $(K_2{}^E, \wedge, i)$ is an *anti–dilation* iff for any f and g in $K_1{}^E$,

$$\psi(f \vee g) = \psi(f) \wedge \psi(g) \quad \text{and} \quad \psi(o) = i\,;$$

an operator ψ from $(K_1{}^E, \wedge, i)$ to $(K_2{}^E, \vee, o)$ is an *anti–erosion* iff for any f and g in $K_1{}^E$,

$$\psi(f \wedge g) = \psi(f) \vee \psi(g) \quad \text{and} \quad \psi(i) = o.$$

The operators which are dilations, erosions, anti–dilations and anti–erosions are called *elementary morphological operators*. □

Going one level up, by Propositions 3.72, the poset $(\mathrm{Op}(K_1{}^E, K_2{}^E), \leq)$ is in turn a lattice and we can introduce the following four operations on the set $\mathrm{Op}(K_1{}^E, K_2{}^E)$.

The *operator union*, denoted by $\vee$, is the union on $(\mathrm{Op}(K_1{}^E, K_2{}^E), \leq)$, or equivalently, the extension to $\mathrm{Op}(K_1{}^E, K_2{}^E)$ of the union $\vee$ on $K_2{}^E$.

The *operator intersection*, denoted by $\wedge$, is the intersection on $(\mathrm{Op}(K_1{}^E, K_2{}^E), \leq)$, or equivalently, the extension to $\mathrm{Op}(K_1{}^E, K_2{}^E)$ of the intersection $\wedge$ on $K_2{}^E$.

The *least operator selection*, denoted by o, is the extension to $\mathrm{Op}(K_1{}^E, K_2{}^E)$ of the nullary operation on $K_2{}^E$ which select the black image o. This operation selects the *least operator*, that is, the constant operator $f \mapsto o$. We denote by o this operator.

The *greatest operator selection*, denoted by ι, is the extension to $\mathrm{Op}(K_1{}^E, K_2{}^E)$ of the nullary operation on $K_2{}^E$ which select the white image i. This operation selects the *greatest operator*, that is, the constant operator $f \mapsto i$. We denote by ι this operator.

In the lattice $(\mathrm{Op}(K_1{}^E, K_2{}^E), \vee, \wedge)$, we can identify the two idempotent and commutative monoids $(\mathrm{Op}(K_1{}^E, K_2{}^E), \vee, o)$ and $(\mathrm{Op}(K_1{}^E, K_2{}^E), \wedge, \iota)$.

Furthermore, parts of Propositions 3.67 and 3.69 apply to the set of operators which are dilations, in other words, the sets $\mathrm{M}_{\vee}(K_1{}^E, K_2{}^E)$ and $\mathrm{M}_o(K_1{}^E, K_2{}^E)$ are $\vee_{\mathrm{Op}}$–closed and o_{Op}–closed. Consequently, by Propositions 3.55 and 3.52, $(\mathrm{D}(K_1{}^E, K_2{}^E), \vee, o)$ is a submonoid of $(\mathrm{Op}(K_1{}^E, K_2{}^E), \vee, o)$.

Using the same arguments, $(\mathrm{E}^{\mathrm{a}}(K_1{}^E, K_2{}^E), \vee, o)$ is a submonoid of $(\mathrm{Op}(K_1{}^E, K_2{}^E), \vee, o)$, and $(\mathrm{E}(K_1{}^E, K_2{}^E), \wedge, \iota)$ and $(\mathrm{D}^{\mathrm{a}}(K_1{}^E, K_2{}^E), \wedge, \iota)$ are submonoids of $(\mathrm{Op}(K_1{}^E, K_2{}^E), \wedge, \iota)$.

Now combining morphological properties with spatial invariance, we have the following proposition.

Proposition 3.26 (characterization of the spatially invariant pointwise elementary morphological operators) – Let E be a non–empty set and let $(K_1, \leq)$ and $(K_2, \leq)$ be, respectively, the intervals $[0, k_1]$ and $[0, k_2]$ of natural numbers equipped with the usual order relation.

1. The mapping $\psi \mapsto l_\psi$ from the submonoid $(\mathrm{IPD}({K_1}^E, {K_2}^E), \vee, o)$ (resp. $(\mathrm{IPE^a}({K_1}^E, {K_2}^E), \vee, o)$) of spatially invariant pointwise operators which are dilations (resp. anti–erosions), to the submonoid $(\mathrm{D}(K_1, K_2), \vee, o)$ (resp. $(\mathrm{E^a}(K_1, K_2), \vee, o)$) of luts which are dilations (resp. anti–erosions), is an isomorphism of monoids. Its inverse is $l \mapsto \psi_l$. In particular, we have, for any ψ and ϕ in $\mathrm{IPD}({K_1}^E, {K_2}^E)$ (resp. $\mathrm{IPE^a}({K_1}^E, {K_2}^E)$),

$$l_{\psi \vee \phi} = l_\psi \vee l_\phi \quad \text{and} \quad l_o = o,$$

and, for any l and m in $\mathrm{D}(K_1, K_2)$ (resp. $\mathrm{E^a}(K_1, K_2)$),

$$\psi_{l \vee m} = \psi_l \vee \psi_m \quad \text{and} \quad \psi_o = o.$$

2. The mapping $\psi \mapsto l_\psi$ from the submonoid $(\mathrm{IPE}({K_1}^E, {K_2}^E), \wedge, \iota)$ (resp. $(\mathrm{IPD^a}({K_1}^E, {K_2}^E), \wedge, \iota)$) of spatially invariant pointwise operators which are erosions (resp. anti–dilations), to the submonoid $(\mathrm{E}(K_1, K_2), \wedge, i)$ (resp. $(\mathrm{D^a}(K_1, K_2), \wedge, i)$) of luts which are erosions (resp. anti–dilations), is an isomorphism of monoids. Its inverse is $l \mapsto \psi_l$. In particular, we have, for any ψ and ϕ in $\mathrm{IPE}({K_1}^E, {K_2}^E)$ (resp. $\mathrm{IPD^a}({K_1}^E, {K_2}^E)$),

$$l_{\psi \wedge \phi} = l_\psi \wedge l_\phi \quad \text{and} \quad l_\iota = i,$$

and, for any l and m in $\mathrm{E}(K_1, K_2)$ (resp. $\mathrm{D^a}(K_1, K_2)$),

$$\psi_{l \wedge m} = \psi_l \wedge \psi_m \quad \text{and} \quad \psi_i = \iota. \qquad \square$$

Proof – Let us prove the result for the dilations. We divide the proof in four steps.

1. Parts of Propositions 3.67 and 3.69 apply to the set of operators, in other words, $\mathrm{IPM}_\vee({K_1}^E, {K_2}^E)$ and $\mathrm{IPM}_o({K_1}^E, {K_2}^E)$ are $\vee_{\mathrm{Op}}$–closed and o_{Op}–closed. Consequently, by Proposition 3.55 their intersection which is $\mathrm{IPD}({K_1}^E, {K_2}^E)$ is also $\vee_{\mathrm{Op}}$–closed and o_{Op}–closed. Hence, by Proposition 3.52, $(\mathrm{IPD}({K_1}^E, {K_2}^E), \vee, o)$ is a submonoid of $(\mathrm{Op}({K_1}^E, {K_2}^E), \vee, o)$.

2. By Exercise 3.13, $(\mathrm{D}(K_1, K_2), \vee, o)$ is a submonoid of $({K_2}^{K_1}, \vee, o)$.

3. From Proposition 3.18, $\psi \mapsto l_\psi$ is a bijection and from Proposition 3.22, $\mathrm{M}_\vee(K_1, K_2)$ (resp. $\mathrm{M}_0(K_1, K_2)$) is the image of $\mathrm{IPM}_\vee({K_1}^E, {K_2}^E)$ (resp. $\mathrm{IPM}_o({K_1}^E, {K_2}^E)$) through $\psi \mapsto l_\psi$, therefore, by Exercise 1.30, $\mathrm{D}(K_1, K_2)$ is the image of $\mathrm{IPD}({K_1}^E, {K_2}^E)$ through $\psi \mapsto l_\psi$.

4. By Parts 1.2 and 3 of Proposition 3.22, $\psi \mapsto l_\psi$ is an isomorphism from $(\mathrm{IPM}_\vee({K_1}^E, {K_2}^E), \vee, o)$ to $(\mathrm{M}_\vee(K_1, K_2), \vee, o)$ and an isomorphism from $(\mathrm{IPM}_o({K_1}^E, {K_2}^E), \vee, o)$ to $(\mathrm{M}_0(K_1, K_2), \vee, o)$. As we see above in Part 1, $\mathrm{IPD}({K_1}^E, {K_2}^E)$ is $\vee_{\mathrm{Op}}$–closed and o_{Op}–closed, therefore by Exercise 3.58 and Part 3 above, $\psi \mapsto l_\psi$ restricted to $\mathrm{IPD}({K_1}^E, {K_2}^E)$ and $\mathrm{D}(K_1, K_2)$ is an isomorphism from $(\mathrm{IPD}({K_1}^E, {K_2}^E), \vee, o)$ to $(\mathrm{D}(K_1, K_2), \vee, o)$.

The results for the erosions, the anti–dilations and the anti–erosions can be proved in a similar manner. □

3.4 Enhancement technique

Let us introduce a simple enhancement technique. For any $\alpha, \beta \in \mathbf{R}$, we consider the operator on $\mathbf{R}^E$ given by $f \mapsto \alpha \cdot f + \beta \cdot i$ and shown in Figure 3.15.

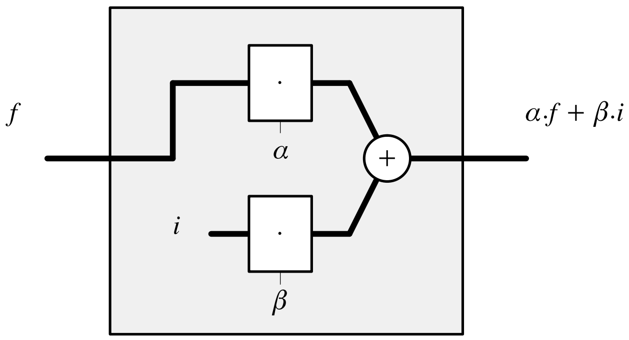


Fig. 3.15 – An enhancement operator.

By using the properties of the mean and the variance given in Section 2.2, we get the following equalities:

$$\mathrm{m}(\alpha \cdot f + \beta \cdot i)) = \alpha \mathrm{m}(f) + \beta$$

$$\mathrm{var}(\alpha \cdot f + \beta \cdot i)) = \alpha^2 \, \mathrm{var}(f).$$

From this equalities, we can find the parameters α and β in such a way that the above operator transforms a given image f into another one with desired mean μ and variance σ^2. These parameters must satisfy the following system of equations,

$$\mu = \alpha \mathrm{m}(f) + \beta$$

$$\sigma^2 = \alpha^2 \, \mathrm{var}(f).$$

By solving this system, we find that

$$\alpha = \frac{\sigma}{\sqrt{\mathrm{var}(f)}} \quad \text{and} \quad \beta = \mu - \frac{\sigma}{\sqrt{\mathrm{var}(f)}}\mathrm{m}(f).$$

We observe that the enhancement operator $f \mapsto \alpha.f + \beta.i$ is not linear because the output depends on the internal image i. Furthermore, when α and β depend on f, it is not pointwise.

An important class of luts is the class of mappings h from $\mathbf{R}$ to $K = [0, k] \subset \mathbf{Z}$. An important example of such luts is the mapping c given by

$$c(s) \triangleq (\lfloor s \rfloor \vee 0) \wedge k \quad (s \in \mathbf{R}),$$

where $\lfloor \cdot \rfloor$ stands for the "integer part of", $\vee$ and $\wedge$ are the union and intersection on $\mathbf{Z}$.

The lut c is used to convert the continuous positive values of the real pixels to discrete values (i.e., to convert from $\mathbf{R}$ to K). The graph of this lut is shown in Figure 3.16. The resulting spatially invariant pointwise operator, that we call an *continuous to discrete converter,* is also shown along with its effect on a particular numerical image.

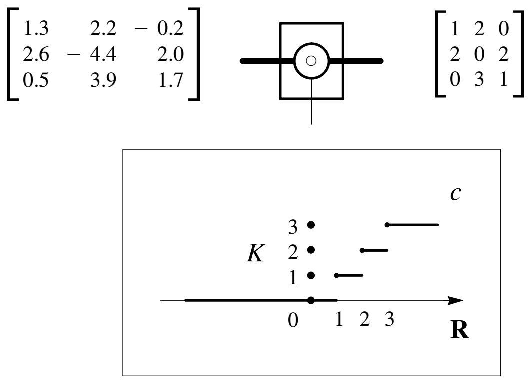


Fig. 3.16 – An analog to digital converter.

Another important lut is the function n from K to K given by

$$n(s) \triangleq k - s \quad (s \in K).$$

The lut n is used to transform an image f in K^E into another image in K^E, called its *negative* and given by, for any $x \in E$, $f(x) = k - f(x)$. The resulting spatially invariant pointwise operator that we call the *negation* is also shown in Figure 3.17 along with its effect on a particular image.

$$
\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&1&1&1&1&0&0\\
0&0&1&0&1&1&3&1&0&0&0\\
0&0&1&1&1&2&3&1&0&0&0\\
0&0&1&1&1&2&3&1&0&0&0\\
0&0&1&0&1&1&3&1&0&0&0\\
0&0&1&0&1&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
\qquad
\begin{bmatrix}
3&3&3&3&3&3&3&3&3&3&3\\
3&3&3&3&3&3&3&3&3&3&3\\
3&3&2&2&2&2&2&2&2&3&3\\
3&3&2&3&2&2&0&2&3&3&3\\
3&3&2&2&2&1&0&2&3&3&3\\
3&3&2&2&2&1&0&2&3&3&3\\
3&3&2&3&2&2&0&2&3&3&3\\
3&3&2&3&2&2&2&2&2&3&3\\
3&3&3&3&3&3&3&3&3&3&3\\
3&3&3&3&3&3&3&3&3&3&3
\end{bmatrix}
$$

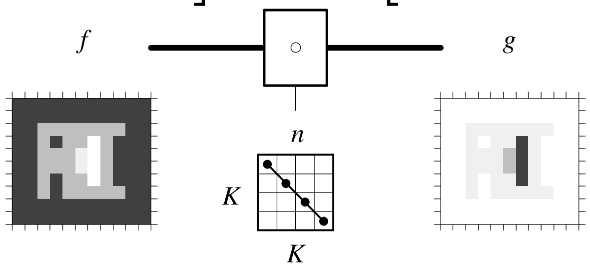


Fig. 3.17 – The negation.

Exercise 3.27 (spatially invariant pointwise operator) – Draw the graph of the lut h from $\mathbf{R}$ to K given by $h \triangleq n \circ c$, where $K = \{0, 1, 2, 3\}$. Find the numerical image $\psi_h(f)$ for

$$f = \begin{bmatrix} 1.3 & 2.2 & -0.2 \\ 2.6 & -4.4 & 2.0 \\ 0.5 & 3.9 & 1.7 \end{bmatrix}. \qquad \square$$

3.5 Some mathematical definitions and properties

To help the understanding of the definitions given in the previous sections, we recall some important mathematical definitions such as supremum and infimum, lattice, monoid, closed subset, morphism, extended binary relation and power lattice, and we recall some of their properties.

Supremum and infimum

Let (A, R_1) and (A, R_2) be two posets, if R_1 and R_2 are mutually converse, then the posets are said *mutually dual*, and any two definitions (statements, properties, propositions, etc.) equal up to a change between R_1 and R_2 are said *mutually dual*.

Proposition 3.28 (duality principle) – If two statements S_1 and S_2 are mutually dual then S_1 is true iff S_2 is true. $\square$

Proof – If S_1 is true based on the set of axioms for R_1, then S_2 is true based on the set of axioms for R_2 since R_1 and R_2 have the same set of axioms (see Exercise 1.37). $\square$

The duality principle was first introduced by Schröder in 1890 (Szász, 1971, p.26).

We often denote by $(\leq, \geq)$ a pair of mutually dual partial orderings.

We now recall some important concepts related to partial ordering.

Definition 3.29 (lower and upper bounds of a set) – Let X be a subset of a poset $(A, \leq)$. An element l of A is a *lower bound*, l.b. for short, *of* X iff $l \leq x$ for all x in X. An element u of A is an *upper bound*, u.b. for short, *of* X iff $x \leq u$ for all x in X. If X is empty then all the elements of A are both an l.b. and a u.b.. □

The lower and upper bound definitions are mutually dual.

The upper left illustration of Figure 3.18 shows the upper bounds (in a poset A) of a certain subset X.

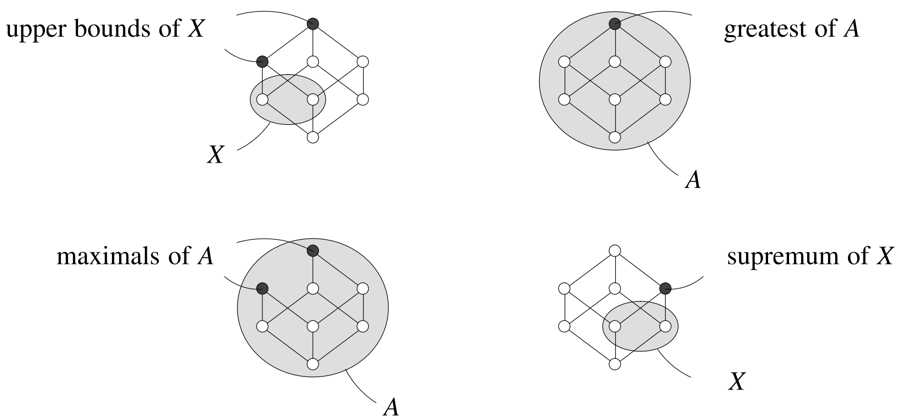


Fig. 3.18 – Some special elements of a poset.

We observe that if $u \leq u'$ and u is an u.b. of X, then u' is an u.b. of X as well, and if $l' \leq l$ and l is a l.b. of X, then l' is a l.b. of X as well.

Definition 3.30 (least and greatest elements of a poset) – Let $(A, \leq)$ be a poset. An element o of A is the *least element of* A iff o is a l.b. of A. An element i of A is the *greatest element of* A iff i is a u.b. of A. □

The least and greatest element definitions are mutually dual.

The upper right illustration of Figure 3.18 shows the greatest element of a certain poset A.

A poset $(A, \leq)$ can contain at most one least element. For, if o_1 and o_2 are two such elements, then $o_1 \leq o_2$ (considering o_1 as a least element) and $o_2 \leq o_1$ (considering o_2 as a least element) whence, by $\leq$ antisymmetry, $o_1 = o_2$. The same is true for the greatest element.

When it exists, we denote the least element of a poset $(A, \leq)$ by $\min A$ and the greatest element by $\max A$.

Definition 3.31 (minimal and maximal elements of a poset) – Let $(A, \leq)$ be a poset. An element m of A is *minimal* iff for any a in A, $a \leq m$ implies that $a = m$. An element m of A is *maximal* iff for any a in A, $m \leq a$ implies that $a = m$. □

The minimal and maximal element definitions are mutually dual.

The lower left illustration of Figure 3.18 shows the maximal elements of a certain poset A.

A least element is a minimal element and a greatest element is a maximal element, but the converse is not true. Let us prove that a least element is a minimal element. If o is the least element of $(A, \leq)$ then $o \leq a$ for any a in A, that is, for any a in A, $a \leq o$ implies $a = o$ by antisymmetry of $\leq$.

Proposition 3.32 (minimal and maximal element existence) – Any finite poset has at least one minimal element and one maximal element. □

Proof – Let $(A, \leq)$ be a finite poset, since A is finite, let say of cardinality n, there exists a bijection from **n** to A that we denote by $i \mapsto a_i$. Let $m_1 = a_1$, and let m_k be a_k if $a_k \leq m_{k-1}$ and m_{k-1} otherwise. Then for any a in A $a \leq m_k$ implies $a = m_k$, that is, m_k is minimal. The rest of the proposition follows by duality. □

Proposition 3.33 (finite chain property) – Any finite chain has a least element and a greatest element. □

Proof – Let $(A, \leq)$ be a finite chain, by Proposition 3.32, it has an element m such that for any a in A, $a \leq m$ implies that $a = m$, that is, for any a in A, $a \neq m$ implies that $m \leq a$ by chain definition. In other words, m is a l.b. of A, that is, it is the least element of A. The rest of the proposition follows by duality. □

Exercise 3.34 (lower and upper bound property) – Let X and Y be two subsets of a poset $(A, \leq)$. Prove that if $X \subset Y$ and u is a l.b. (resp., u.b.) of Y, then u is a l.b. (resp., u.b.) of X. □

Solution – For any X and Y subsets of A such that $X \subset Y$,

$$\begin{aligned} l \text{ is a l.b. of } Y &\Leftrightarrow l \leq y \text{ for all } y \text{ in } Y && \text{(l.b. definition)} \\ &\Rightarrow l \leq x \text{ for all } x \text{ in } X && (X \subset Y) \\ &\Leftrightarrow l \text{ is a l.b. of } X. \end{aligned}$$

By duality, the same result applies to the u.b.. □

Definition 3.35 (supremum and infimum) – Let X be a subset of a poset $(A, \leq)$. An element u of A is the *supremum of* X (in $(A, \leq)$) iff, for any x in A,

$$x \text{ is an u.b. of } X \text{ iff } u \leq x.$$

An element l of A is the *infimum of* X (in $(A, \leq)$) iff, for any x in A,

$$x \text{ is a l.b. of } X \text{ iff } x \leq l.$$ □

The supremum and infimum definitions are mutually dual.

The lower right illustration of Figure 3.18 shows the supremum (in a poset A) of a certain subset X.

Proposition 3.36 (supremum an infimum equivalent definition) – Let X be a subset of a poset $(A, \leq)$. An element u of A is the supremum of X iff u is the least upper bound of X, that is, among the subset of all the u.b. of X, it is the least element. An element l of A is the infimum of X iff l is the greatest lower bound of X, that is, among the subset of all the l.b. of X, it is the greatest element. □

Proof – 1. Let u be the supremum of X. The "if" condition, in the supremum definition, shows, when x is u, that the supremum of X is an u.b. of X. This result together with the "only if" condition show that the supremum of X is the least u.b. of X.

2. Let u be the least u.b. of X. If x is an u.b. of X, then $u \leq x$ since u is the least u.b. of X. Conversely, if $u \leq x$, then x is an u.b. of X since u is an u.b. of X.

By duality, the same result applies to the infimum. □

Given a subset X of a poset $(A, \leq)$, there is at most one supremum of X since the supremum is the least element of some subset of A.

When it exists, we denote the supremum of X by $\sup X$, and the infimum by $\inf X$.

Exercise 3.37 (supremum and infimum property) – Let X and Y be two subsets of a poset $(A, \leq)$. Prove that if X and Y have a supremum, and $X \subset Y$, then $\sup X \leq \sup Y$. Prove that if X and Y have a infimum, and $X \subset Y$, then $\inf \mathrm{Y} \leq \inf X$. □

Solution – Let X and Y be any subsets of A such that $X \subset Y$. By Proposition 3.36, $\sup Y$ is an u.b. of Y. Since $X \subset Y$, by Exercise 3.34, $\sup Y$ is an u.b. of X. By Proposition 3.36, $\sup X$ is the least upper bound of X, that is, $\sup X \leq \sup Y$. By duality, the same result applies to the infimum. □

Proposition 3.38 (sufficient condition to have a supremum and an infimum) – Let X be a subset of a poset $(A, \leq)$. If X has a greatest element, then X has a supremum and $\sup X = \max X$. If X has a least element, then X has an infimum and $\inf X = \min X$. □

Proof – First, let u be the greatest element of X, then, by definition, u is an u.b. of X and consequently $\sup X \leq u$; furthermore, u being in X and $\sup X$ being an u.b. of X (Proposition 3.36), we have $u \leq \sup X$; that is, by $\leq$ antisymmetry, $u = \sup X$. Second, by definition $u = \max X$, that is, by $=$ transitivity, $\sup X = \max X$. By duality, the same result applies to the infimum. □

Exercise 3.39 (supremum and infimum recurrence) – Let $(A, \leq)$ be a poset. Prove that, for any distinct elements a, b, and c in A,

$$\sup\{a, b, c\} = \sup\{\sup\{a, b\}, c\},$$

whenever the suprema exist, and

$$\inf\{a, b, c\} = \inf\{\inf\{a, b\}, c\},$$

whenever the infima exist. □

Solution – Let a, b, and c be any distinct elements of A. Let $u_1 = \sup\{\sup\{a, b\}, c\}$ and let $u_2 = \sup\{a, b, c\}$. We want to prove that $u_1 = u_2$. First,

$$u_1 = \sup\{\sup\{a, b\}, c\} \Rightarrow \sup\{a, b\} \leq u_1 \text{ and } c \leq u_1 \qquad \text{(supremum is an u.b.)}$$

$$\Rightarrow \begin{cases} a \leq \sup\{a, b\} \leq u_1 \text{ and} \\ b \leq \sup\{a, b\} \leq u_1 \text{ and} \\ c \leq u_1 \end{cases} \qquad \text{(supremum is an u.b.)}$$

$$
\begin{aligned}
&\Rightarrow\ a \le u_1 \text{ and } b \le u_1 \text{ and } c \le u_1 && (\le \text{ transitivity})\\
&\Rightarrow\ u_1 \text{ is an u.b. of } \{a, b, c\} && (\text{u.b. definition})\\
&\Leftrightarrow\ u_2 \le u_1. && (\text{supremum definition})
\end{aligned}
$$

Second,

$$
\begin{aligned}
u_2 = \sup\{a,b,c\}\ &\Rightarrow\ \sup\{a,b\} \le u_2 \text{ and } c \le u_2 && (\text{Exercise 3.37 and sup is an u.b.})\\
&\Leftrightarrow\ u_2 \text{ is an u.b. of } \{\sup\{a, b\}, c\} && (\text{u.b. definition})\\
&\Leftrightarrow\ u_1 \le u_2. && (\text{supremum definition})
\end{aligned}
$$

That is, by $\le$ antisymmetry, $u_1 = u_2$.

The result for the infimum follows by duality. □

Exercise 3.39 shows that if any two elements have a supremum then any three elements have a supremum.

Lattice

Definition 3.40 (lattice) – A non–empty set A together with two binary operations $\square_1$ and $\square_2$ satisfying the axioms below is a *lattice*; we denote it by $(A, \square_1, \square_2)$. For any a, b and c in A,

$$a \square_1 b = b \square_1 a \quad \text{and} \quad a \square_2 b = b \square_2 a \qquad (\text{commutativity})$$

$$(a \square_1 b) \square_1 c = a \square_1 (b \square_1 c) \quad \text{and} \quad (a \square_2 b) \square_2 c = a \square_2 (b \square_2 c) \qquad (\text{associativity})$$

$$a \square_2 (a \square_1 b) = a \quad \text{and} \quad a \square_1 (a \square_2 b) = a. \qquad (\text{absorption})$$

□

Since the operations $\square_1$ and $\square_2$ play the same role in the lattice axioms, $(A, \square_1, \square_2)$ is a lattice iff $(A, \square_2, \square_1)$ is a lattice. These two lattices are called *mutually dual*.

From the lattice definition we can derive the following property called *idempotence*. Let $(A, \square_1, \square_2)$ be a lattice, then for any a in A, $a \square_1 a = a$ and $a \square_2 a = a$. The idempotence follows from the absorption as shown in Birkhoff (1967, p. 21). Figure 3.19 illustrates graphically the idempotence property.

Fig. 3.19 – Idempotence property in a lattice.

A non–empty set A together with a binary operation $\square$ which is idempotent, commutative and associative is called a *semilattice*.

Exercise 3.41 (implication) – Let $(A, \square_1, \square_2)$ be a lattice, prove that

$$a \square_1 b = b \quad \text{iff} \quad a \square_2 b = a.$$

□

The operations $\square_1$ and $\square_2$ can be characterized in terms of an ordering relation through a relationship between some posets and the lattices.

Let $(A, \leq)$ be a poset. For any a and b in A, we denote by $a \vee_{\leq} b$ and $a \wedge_{\leq} b$ the expressions defined by

$$a \vee_{\leq} b \triangleq \sup\{a, b\} \quad \text{and} \quad a \wedge_{\leq} b \triangleq \inf\{a, b\}$$

whenever the supremum and the infimum exist.

Proposition 3.42 (lattice from poset) – Let $(A, \leq)$ be a poset in which any two elements have a supremum and an infimum. Then $(A, \vee_{\leq}, \wedge_{\leq})$ is a lattice. □

Proof – 1. Since for any a and b in A, the expressions $a \vee_{\leq} b$ and $a \wedge_{\leq} b$ are well defined elements of A, $\vee_{\leq}$ and $\wedge_{\leq}$ are two binary operations on A.

2. The commutativity of $\vee_{\leq}$ and $\wedge_{\leq}$ is an inheritance of the commutativity of the conjunctions "and" which appears in the definition of supremum and infimum.

3. Let a, b, and c be any elements of A. Let $u_1 = \sup\{\sup\{a, b\}, c\}$, let $u_2 = \sup\{a, b, c\}$ and let $u_3 = \sup\{a, \sup\{b, c\}\}$. To prove the associativity of $\vee_{\leq}$, we have to prove that $u_1 = u_3$. By Exercise 3.39, $u_1 = u_2$ and $u_2 = u_3$, therefore, by $=$ transitivity, $u_1 = u_3$. By duality, the associativity also apply to $\wedge_{\leq}$.

4. Let a and b be any elements of A. First,

$$\begin{aligned} a &\leq \sup\{a, b\} && \text{(sup is an u.b.)} \\ &= a \vee_{\leq} b, && \text{(definition of } \vee_{\leq}) \end{aligned}$$

that is, $a \leq a \vee_{\leq} b$.

Second, if $a = a \vee_{\leq} b$ then by definition of $\vee_{\leq}$, $a \wedge_{\leq} (a \vee_{\leq} b) = a$, otherwise

$$\begin{aligned} a \wedge_{\leq} (a \vee_{\leq} b) &= \inf\{a, (a \vee_{\leq} b)\} && \text{(definition of } \wedge_{\leq}) \\ &= a, && \text{(first part and Proposition 3.38)} \end{aligned}$$

that is, in both case, $a \wedge_{\leq} (a \vee_{\leq} b) = a$. By duality, we also have $a \vee_{\leq} (a \wedge_{\leq} b) = a$. This prove the absorption. □

From their definition, the operations $\vee_{\leq}$ and $\wedge_{\leq}$ are idempotent, therefore $(A, \vee_{\leq})$ and $(A, \wedge_{\leq})$ are semilattices. The operations $\vee_{\leq}$ and $\wedge_{\leq}$ are called, respectively, *union* and *intersection* with respect to $\leq$.

Whenever any two (distinct) elements of a poset have both a supremum and an infimum, the poset is called a *lattice*. This definition of lattice should not be confused with the lattice definition 3.40, where lattice is an algebraic structure with two binary operations. The use of the word "lattice" for some poset is partially justified by Proposition 3.42. and completely justified by Proposition 3.47 below.

Proposition 3.43 (chain property) – If $(A, \leq)$ be a chain, then $(A, \vee_{\leq}, \wedge_{\leq})$ is a lattice. Furthermore for any a and b in A,

$$a \vee_{\leq} b = \max\{a, b\} \quad \text{and} \quad a \wedge_{\leq} b = \min\{a, b\}. \qquad \square$$

Proof – Let $(A, \leq)$ be a chain. By chain definition, in any subchain of $(A, \leq)$, with two elements, one element is the least element and the other is the greatest element. That is, by Proposition 3.38, any subchain of $(A, \leq)$, with two elements, has an infimum and a supremum. By Proposition 3.42, this means that $(A, \vee_{\leq}, \wedge_{\leq})$ is a lattice. The rest of the proposition follows from Proposition 3.38 again. □

If $(A, \leq)$ is chain, then, for any a and b in A,

$$(a \vee_{\leq} b = a \text{ and } a \wedge_{\leq} b = b) \quad \text{or} \quad (a \vee_{\leq} b = b \text{ and } a \wedge_{\leq} b = a).$$

Exercise 3.44 (first consistency property) – Prove that, for any a and b in the lattice $(A, \leq)$,

$$a \leq b \quad \text{iff} \quad a \vee_{\leq} b = b \quad \text{and} \quad a \leq b \quad \text{iff} \quad a \wedge_{\leq} b = a. \qquad \square$$

Solution – For any a and b in the lattice $(A, \leq)$,

$$\begin{aligned} a \vee_{\leq} b = b \quad &\Leftrightarrow \quad b = \sup\{a, b\} && (\text{definition of } \vee_{\leq}) \\ &\Leftrightarrow \quad a \leq b. && (\Rightarrow \text{sup is an u.b.}, \Leftarrow \text{Proposition 3.38}) \end{aligned}$$

The rest of the result follows by duality. $\square$

Now, instead of starting from a poset (which is lattice), we start from a lattice.

Let $(A, \square_1, \square_2)$ be a lattice. For any a and b in A, we denote by $a \leq_{1,2} b$ and $a \leq_{2,1} b$ the binary relations defined by

$$a \leq_{1,2} b \overset{\Delta}{\Leftrightarrow} a \square_1 b = b \quad (\text{or equivalently } a \square_2 b = a).$$

and

$$a \leq_{2,1} b \overset{\Delta}{\Leftrightarrow} a \square_1 b = a \quad (\text{or equivalently } a \square_2 b = b).$$

The next proposition, which is the converse of Proposition 3.42, shows that behind any lattice there is a poset.

Proposition 3.45 (poset from lattice) – Let $(A, \square_1, \square_2)$ be a lattice, then $(A, \leq_{1,2})$ and $(A, \leq_{2,1})$ are two posets, mutually dual. $\square$

Proof – First we prove that $\leq_{1,2}$ is a partial ordering. 1. The reflexivity of $\leq_{1,2}$ is a consequence of the idempotence of $\square_1$ and $\square_2$.

2. The anti–symmetry of $\leq_{1,2}$ is a consequence of the $=$ transitivity, since, for any a and b in A,

$$\begin{aligned} b &= a \square_1 b && (a \leq_{1,2} b) \\ &= b \square_1 a && (\text{commutativity of } \square_1) \\ &= a. && (b \leq_{1,2} a) \end{aligned}$$

3. The transitivity of $\leq_{1,2}$ is a consequence of the $\square_1$ and $\square_2$ associativity, since for any a, b and c in A,

$$\begin{aligned} a \square_1 c &= a \square_1 (b \square_1 c) && (b \leq_{1,2} c) \\ &= (a \square_1 b) \square_1 c && (\text{associativity of } \vee) \\ &= b \square_1 c && (a \leq_{1,2} b) \\ &= c, && (b \leq_{1,2} c) \end{aligned}$$

that is, by $=$ transitivity, $a \square_1 c = c$, and by $\leq_{1,2}$ definition, $a \leq_{1,2} c$.

Second, for any a and b in A,

$$\begin{aligned} a \le_{1,2} b &\Leftrightarrow a \,\square_1\, b = b && (\le_{1,2} \text{ definition}) \\ &\Leftrightarrow b \,\square_1\, a = b && (\text{commutativity}) \\ &\Leftrightarrow b \le_{2,1} a, && (\le_{2,1} \text{ definition}) \end{aligned}$$

that is, $\le_{1,2}$ and $\le_{2,1}$ are mutually converse (which also prove that $\le_{2,1}$ is a partial ordering.) □

If $(A, \square_1, \square_2)$ is lattice, then, for any a and b in A, we have

$$(a \text{ and } b \le_{1,2} a \,\square_1\, b) \quad \text{and} \quad (a \,\square_2\, b \le_{1,2} a \text{ and } b).$$

and

$$(a \,\square_1\, b \le_{2,1} a \text{ and } b) \quad \text{and} \quad (a \text{ and } b \le_{2,1} a \,\square_2\, b).$$

Proposition 3.45 shows that we can associate a poset to any lattice, but the converse is not true.

Exercise 3.46 (second consistency properties) – Prove that, for any elements a and b in the lattice $(A, \square_1, \square_2)$,

$$a \,\square_1\, b = \sup_{\le_{1,2}} \{a, b\} \quad \text{and} \quad a \,\square_2\, b = \inf_{\le_{1,2}} \{a, b\}$$

and

$$a \,\square_1\, b = \inf_{\le_{2,1}} \{a, b\} \quad \text{and} \quad a \,\square_2\, b = \sup_{\le_{2,1}} \{a, b\}.$$ □

Solution – First, for any elements a and b in $(A, \square_1, \square_2)$,

$$\begin{aligned} u = \sup_{\le_{1,2}} \{a, b\} &\Leftrightarrow (a \le_{1,2} u \text{ and } b \le_{1,2} u) && (\text{sup is u.b.}) \\ &\Leftrightarrow (a \,\square_1\, u = u \text{ and } b \,\square_1\, u = u) && (\le_{1,2} \text{ definition}) \\ &\Rightarrow a \,\square_1\, (b \,\square_1\, u) = u && (\text{substitution}) \\ &\Leftrightarrow (a \,\square_1\, b) \,\square_1\, u = u && (\text{associativity}) \\ &\Leftrightarrow (a \,\square_1\, b) \le_{1,2} u, && (\le_{1,2} \text{ definition}) \end{aligned}$$

that is, $a \,\square_1\, b \le_{1,2} \sup_{\le_{1,2}} \{a, b\}$.

Second, for any elements a and b in A,

$$\begin{aligned} u = \sup_{\le_{1,2}} \{a, b\} &\Leftrightarrow ((a \le_{1,2} x \text{ and } b \le_{1,2} x) \Rightarrow u \le_{1,2} x) && (\text{sup is least u.b.}) \\ &\Rightarrow u \le_{1,2} (a \,\square_1\, b), && (a \text{ and } b \le_{1,2} a \,\square_1\, b) \end{aligned}$$

that is, $\sup_{\le_{1,2}} \{a, b\} \le_{1,2} a \,\square_1\, b$. By $\le_{1,2}$ antisymmetry, $a \,\square_1\, b = \sup_{\le_{1,2}} \{a, b\}$. The rest of the result follows by duality. □

The first and second consistency properties (Exercises 3.44 and 3.46) are useful to prove the existence of a one–to–one relationship between the set of posets which are lattice and the set of lattices.

Proposition 3.47 (relationship between the set of posets which are lattice and the set of lattices) – The mapping $\leq \mapsto (\vee_{\leq}, \wedge_{\leq})$ from the set of posets which are lattice to the set of lattices is a bijection. Its inverse is $(\square_1, \square_2) \mapsto \leq_{1,2}$. □

Proof – 1. Proposition 3.42 shows that the image of a poset which is a lattice, through $\leq \mapsto (\vee_{\leq}, \wedge_{\leq})$, is a lattice.

2. Proposition 3.45 shows that the image of a lattice, through $(\square_1, \square_2) \mapsto \leq_{1,2}$, is a poset. Furthermore, by Exercise 3.46, this poset is a lattice.

3. Let $\leq$ be a partial order on A, for any a and b in A,

$$a \leq_{\vee_{\leq}} b \Leftrightarrow a \vee_{\leq} b = b \qquad (\leq_{1,2} \text{ definition})$$

$$\Leftrightarrow a \leq b, \qquad \text{(Exercises 3.44)}$$

that is, $\leq \mapsto (\vee_{\leq}, \wedge_{\leq})$ is injective.

4. Let $\square_1$ and $\square_2$ be two lattices operations on A, for any a and b in A,

$$a \vee_{\leq_{1,2}} b = \sup_{\leq_{1,2}} \{a, b\} \qquad (\vee_{\leq} \text{ definition})$$

$$\Leftrightarrow a \,\square_1\, b, \qquad \text{(Exercises 3.46)}$$

the same arguments apply to $\square_2$, that is, $\leq \mapsto (\vee_{\leq}, \wedge_{\leq})$ is surjective. □

Based on Proposition 3.47, Figure 3.20 illustrate the relationship between the posets and the lattices.

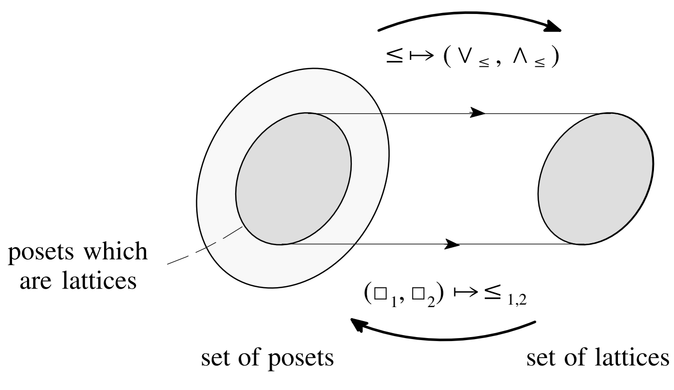


Fig. 3.20 – Relationship between posets and lattices.

Lattices which have a least element and a greatest element have the following important property.

Proposition 3.48 (least and greatest elements property for a lattice) – If the lattice $(A, \leq)$ have a least element (denoted by o) and a greatest element (denoted by i), then the operations $\vee_\leq$ and $\wedge_\leq$ have a unit element, more precisely, for any a in A,

$$a \ \vee_\leq \ o = a \quad \text{and} \quad a \ \wedge_\leq \ i = a. \qquad \square$$

Proof – Let o be the least element of $(A, \leq)$, then for any a in A,

$$\begin{aligned} \text{true} &\Leftrightarrow o \leq a && \text{(least element definition)} \\ &\Leftrightarrow a \ \vee_\leq \ o = a. && \text{(Exercise 3.44)} \end{aligned}$$

The rest of the proposition follows by duality. $\square$

The above property applies to finite lattice.

Proposition 3.49 (finite lattice property) – If $(A, \leq)$ is a finite lattice, then A has a least element and a greatest element. $\square$

Proof – Let m be a minimal element of $(A, \leq)$. By Proposition 3.32 this element exists. For any a in A,

$$\begin{aligned} \text{true} &\Leftrightarrow \inf\{a, m\} \leq m && \text{(inf is a l.b.)} \\ &\Rightarrow \inf\{a, m\} = m && \text{(m is minimal)} \\ &\Rightarrow m \leq a, && \text{(inf is a l.b.)} \end{aligned}$$

that is, m is a l.b. of A, in other words, it is the least element of $(A, \leq)$.

The rest of the proposition follows by duality. $\square$

In the finite lattice structure we can identify two monoid structures.

Monoid

Definition 3.50 (semigroup and monoid) – A *semigroup* $(B, \square)$ is a set B equipped with an internal binary operation $\square$ which is associative. A *monoid* $(B, \square, e)$ is a semigroup $(B, \square)$ with an element e in B which a unit for $\square$. $\square$

In a monoid there is only **one** unit (in Section 2.4 we have proved that the unit element is unique).

By construction, in a vector space $(V, +, \cdot, o)$, $(V, +, o)$ is a commutative monoid.

If $(L, \leq)$ is a lattice which have a least element o and a greatest element i, then, from Proposition 3.48, $(L, \ \vee_\leq, o)$ and $(L, \ \wedge_\leq, i)$ are two commutative and idempotent monoids, where o and i are, respectively, the unit elements of $\vee_\leq$ and $\wedge_\leq$.

Let $(B, \square, e)$ be monoid. The set B^A equipped with the extension to B^A of $\square$ on B, and the extension to B^A of e of B, denoted by $(B^A, \square, e)$, is called the *extension* of $(B, \square, e)$ to the mappings whose domain is A.

From Propositions 2.46 to 2.48, we can deduce the next proposition.

Proposition 3.51 (commutative monoid of mappings) – If $(B, \square, e)$ is a commutative monoid, then its extension $(B^A, \square, e)$ to the mappings whose domain is A, is a commutative monoid. $\square$

Closed subset

Let B and C be two non–empty sets and let $I \triangleq \{1\}$ be the standard set with just one element.

A subset X of the set B equipped with an internal binary operation $\square$, is *closed under* $\square$, or $\square$*–closed*, iff for any a and b in X, $a \square b$ is in X. Figure 3.21 illustrates a non closed subset X under $\square$.

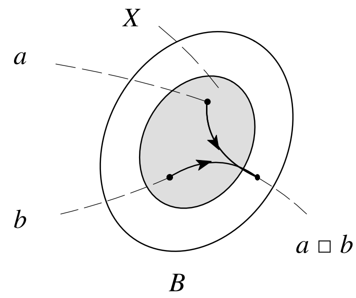


Fig. 3.21 – Non closed subset.

A subset X of the set B equipped with an external binary operation $\cdot$ with operand in C, is *closed under* $\cdot$, or $\cdot$*–closed*, iff for any c in C and b in X, $c \cdot b$ is in X.

A subset X of the set B equipped with a unary operation h, is *closed under h*, or *h–closed*, iff for any b in X, $h(b)$ is in X.

A subset X of the set B equipped with a nullary operation e, is *closed under e*, or *e–closed*, iff the selected element by e, $e(1)$ is in X.

To each of the above operation on B it corresponds its *restrictions to a closed subset X*.

The notion of closed operation under an operation is important to define notions like submonoid, subspace of a linear vector subspace, or sublattice

Proposition 3.52 (submonoid) – Let $(B, \square, e)$ be a monoid and let X be a subset of B which is $\square$–closed and e–closed, then the restrictions of $\square$ and e on X confer on X the monoid structure. In this case, X is called a *submonoid of B*. □

Proof – The subset X is e–closed and $x \square e = x$ for any x in X, therefore e is the unit element of $\square$ restricted to X. □

Proposition 3.53 (subspace of a linear vector space) – Let $(V, +, \cdot, o)$ be a linear vector space and let X be a subset of V which is $+$–closed and $\cdot$–closed, then the restrictions of $+$, $\cdot$ and o on X confer on X the linear vector space structure. In this case, X is called a *subspace of the linear vector space V.* □

For the proof see Cagnac et al. (1963, p. 284).

Proposition 3.54 (sublattice) – Let $(A, \square_1, \square_2)$ be a lattice and let X be a subset of A which is $\square_1$–closed and $\square_2$–closed, then the restrictions of $\square_1$ and $\square_2$ on X confer on X the lattice structure. In this case, X is called a *sublattice of A*. □

For the proof see Birkhoff (1967, p. 7).

Proposition 3.55 (intersection of closed subsets) – The intersection of two closed subsets under the same operation is a closed subset under that operation. □

Proof – Let □ be an internal binary operation on B. Let X and Y are two subsets of B closed under □. For any a and b in $X \cap Y$, a and b are in X and Y (by set intersection definition), and therefore $a \,\square\, b$ is an element of X and Y (by definition of □–closed subset). Consequently $a \,\square\, b$ is an element of $X \cap Y$ which prove that $X \cap Y$ is □–closed. The same proof could be made for the other types of operation on B. □

For example, from Proposition 3.55, the intersection of two subspaces of a linear vector space is another subspace of that linear vector space.

We now recall the definition of some types of morphism.

Morphism

Let $\square_1$ be an internal binary operation on a set B_1 and let $\square_2$ be another internal binary operation on a set B_2. A mapping f from B_1 to B_2 is a *morphism of internal binary operations*, from $(B_1, \square_1)$ to $(B_2, \square_2)$, iff it carries $\square_1$ onto $\square_2$, in the sense that, for any a and b in B_1,

$$f(a \,\square_1\, b) = f(a) \,\square_2\, f(b).$$

We denote by $\mathrm{M}_{\square_1\square_2}(B_1, B_2)$, or sometimes simply by $\mathrm{M}_{\square}(B_1, B_2)$, the set of morphism of internal binary operations, from $(B_1, \square_1)$ to $(B_2, \square_2)$.

Let $\cdot_1$ be an external binary operation on a set B_1 with left operand in a set C, and let $\cdot_2$ be another external binary operation on a set B_2 with left operand in the same set C. A mapping f from B_1 to B_2 is a *morphism of external binary operations with operand in C*, from $(B_1, \cdot_1)$ to $(B_2, \cdot_2)$, iff it carries $\cdot_1$ onto $\cdot_2$, in the sense that, for any b in A_1 and c in C,

$$f(c \cdot_1 b) = c \cdot_2 f(b).$$

We denote by $\mathrm{M}_{\cdot_1 \cdot_2}(B_1, B_2)$, or sometimes simply by $\mathrm{M}_{\cdot}(B_1, B_2)$, the set of morphism of external binary operations, from $(B_1, \cdot_1)$ to $(B_2, \cdot_2)$.

Let h_1 be a unary operation on a set B_1 and h_2 another unary operation on a set B_2. A mapping f from B_1 to B_2 is a *morphism of unary operations*, from (B_1, h_1) to (B_2, h_2) iff it carries h_1 onto h_2, in the sense that,

$$f \circ h_1 = h_2 \circ f.$$

We denote by $\mathrm{M}_{h_1 h_2}(B_1, B_2)$, or sometimes simply by $\mathrm{M}_h(B_1, B_2)$, the set of morphism of unary operations, from (B_1, h_1) to (B_2, h_2).

Let e_1 be the element selected by a nullary operation e_1 on a set B_1, and e_2 be the element selected by a nullary operation e_2 on a set B_2. A mapping f from B_1 to B_2 is a *morphism of nullary operations*, from (B_1, e_1) to (B_2, e_2) iff it carries e_1 onto e_2, in the sense that,

$$f(e_1) = e_2.$$

We denote by $\mathrm{M}_{e_1 e_2}(B_1, B_2)$, or sometimes simply by $\mathrm{M}_e(B_1, B_2)$, the set of morphism of nullary operations, from (B_1, e_1) to (B_2, e_2).

Figures 3.22 and 3.23 illustrate graphically the above four types of morphism.

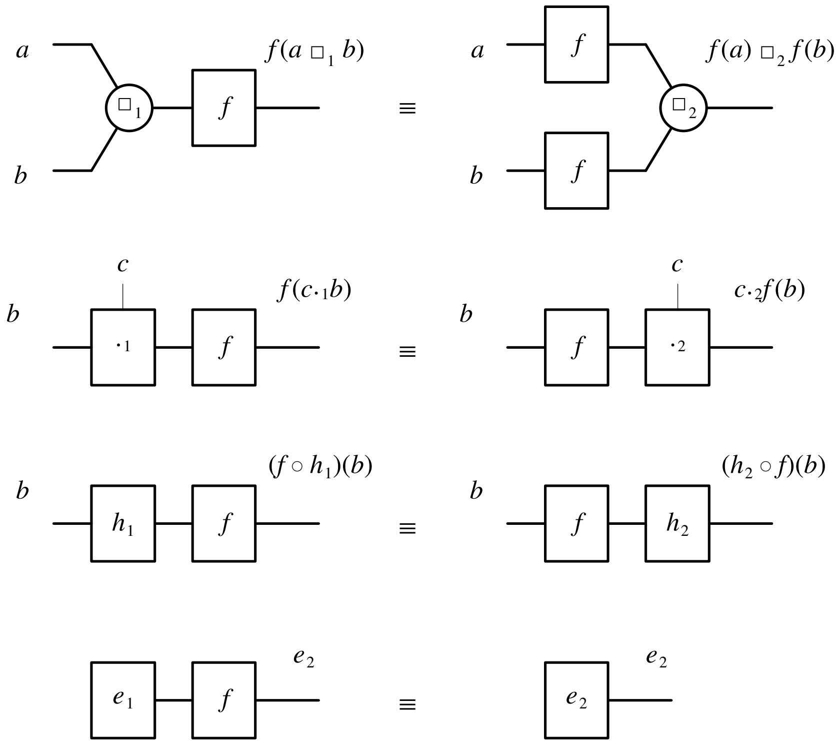


Fig. 3.22 – Four types of morphism.

If a morphism is a bijection it is called an *isomorphism*.

Exercise 3.56 (restriction of a morphism) – Let $\square_1$ be an internal binary operation on a set B_1 and let $\square_2$ be an internal binary operation on a set B_2. Let X_1 be a $\square_1$–closed subset of B_1 and X_2 be a subset of B_2 such that $X_2 \in [f(X_1), B_2]$, where f is a mapping from B_1 to B_2. Prove that if f is a morphism from $(B_1, \square_1)$ to $(B_2, \square_2)$ then $f/X_1, X_2$ is a morphism from $(X_1, \square_1)$ to $(X_2, \square_2)$. □

Exercise 3.57 (closure of the image of a closed subset through a morphism) – 1. Let $\square_1$ be an internal binary operation on a set B_1 and let $\square_2$ be an internal binary operation on a set B_2. Let f be a morphism from $(B_1, \square_1)$ to $(B_2, \square_2)$ and let X_1 be a $\square_1$–closed subset of B_1, prove that $f(X_1)$ is a $\square_2$–closed subset of B_2.

2. Let $\cdot_1$ be an external binary operation on a set B_1 with operand in a set C and let $\cdot_2$ be an external binary operation on a set B_2 with operand in C. Let f be a morphism from $(B_1, \cdot_1)$ to $(B_2, \cdot_2)$ and let X_1 be a $\cdot_1$–closed subset of B_1, prove that $f(X_1)$ is a $\cdot_2$–closed subset of B_2. □

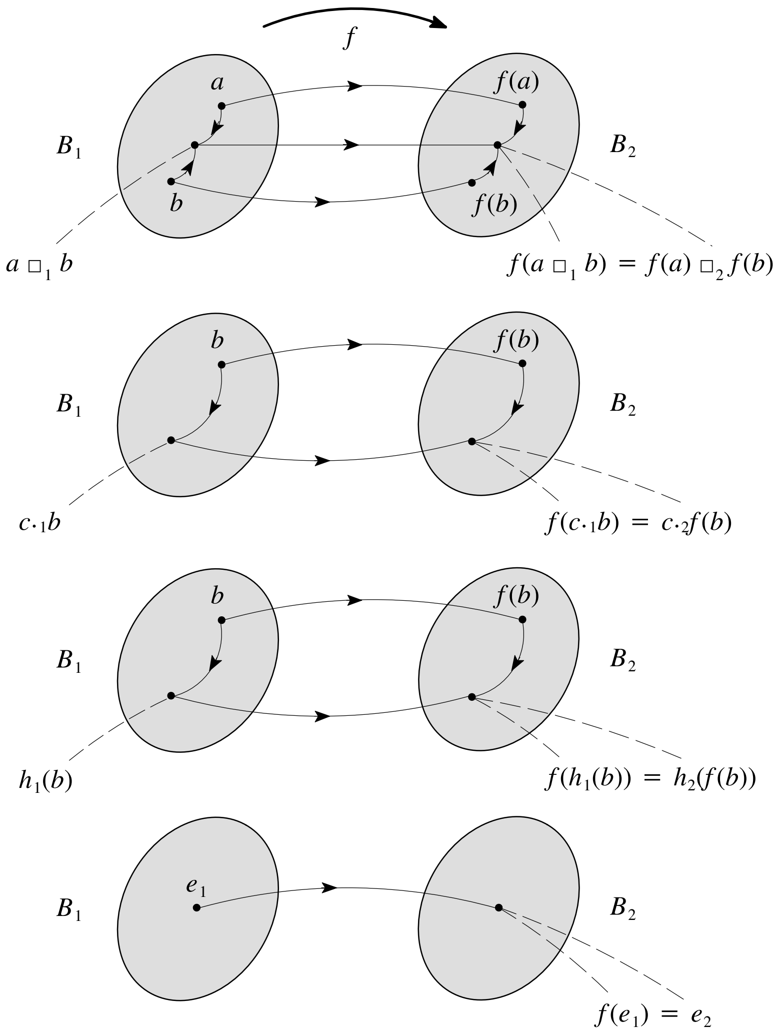


Fig. 3.23 – Four types of morphism.

Exercise 3.58 (restriction of an isomorphism) – Let $\square_1$ be an internal binary operation on a set B_1 and let $\square_2$ be an internal binary operation on a set B_2. Let X_1 be a $\square_1$–closed subset of B_1. Prove that if f is an isomorphism from $(B_1, \square_1)$ to $(B_2, \square_2)$ then $f/X_1, f(X_1)$ is an isomorphism from $(X_1, \square_1)$ to $(f(X_1), \square_2)$. □

Solution – The mapping $f/X_1, f(X_1)$ is injective (since f is injective) and is surjective by definition, therefore it is bijective. Since f is an isomorphism from $(B_1, \square_1)$ to $(B_2, \square_2)$, by Exercise 3.57, $f(X_1)$ is a $\square_2$–closed subset of B_2. By Exercise 3.56, it is a morphism from $(X_1, \square_1)$ to $(f(X_1), \square_2)$, therefore it an isomorphism from $(B_1, \square_1)$ to $(B_2, \square_2)$. □

Similar results hold for the restriction of an isomorphism of external binary operations, unary operations and nullary operations.

Exercise 3.59 (inverse of an isomorphism) – Let $\square_1$ be an internal binary operation on a set B_1 and let $\square_2$ be an internal binary operation on a set B_2. Prove that if f is an isomorphism from $(B_1, \square_1)$ to $(B_2, \square_2)$ then f^{-1} is an isomorphism from $(B_2, \square_2)$ to $(B_1, \square_1)$. □

Solution – For any a and b in B_2,

$$\begin{aligned} f^{-1}(a \square_2 b) &= f^{-1}(f(f^{-1}(a)) \square_2 f(f^{-1}(b))) && (f^{-1} \text{ is a right inverse}) \\ &= f^{-1}(f(f^{-1}(a) \square_1 f^{-1}(b))) && (f \text{ is an isomorphism}) \\ &= f^{-1}(a) \square_1 f^{-1}(b). && (f^{-1} \text{ is a left inverse}) \end{aligned}$$

□

Similar results hold for the inverse of an isomorphism of external binary operations, unary operations and nullary operations.

A *morphism of monoids* from $(A_1, \square_1, e_1)$ to $(A_2, \square_2, e_2)$ is a morphism for the binary operations $\square_1$ and $\square_2$, and a morphism for the nullary operations that select the elements e_1 and e_2. We denote by $\mathrm{M}(A_1, A_2)$ the set of these morphisms.

Definition 3.60 (linear mapping) – A *linear mapping* from the vector space $(V_1, +, \cdot, o)$ over $\mathbf{R}$ to the vector space $(V_2, +, \cdot, o)$ over $\mathbf{R}$ is a morphism of vector spaces, that is, a morphism for the additions $+$ on V_1 and $+$ on V_2, and a morphism for the products $\cdot$ on V_1 and $\cdot$ on V_2 by a scalar in $\mathbf{R}$. We denote by $\mathrm{L}(V_1, V_2)$ the set of the linear mappings from V_1 to V_2. □

Exercise 3.61 (linear mapping property) – Prove that if l is a linear mapping from V_1 to V_2, then l is a morphism for the nullary operations o on V_1 and o on V_2. □

Let $(A_1, \leq)$ and $(A_2, \leq)$ be two lattices, then a morphism from A_1 to A_2 for the binary operations $\vee_\leq$ (on A_1) and $\vee_\leq$ (on A_2) is called a *joint morphism*, and a morphism from A_1 to A_2 for the binary operations $\wedge_\leq$ (on A_1) and $\wedge_\leq$ (on A_2) is called a *meet morphism* (Birkhoff, 1967).

From two finite lattices (or more generally from lattices which have a least element and greatest element) we can make up four classes of mappings which we call dilations, erosions, anti–dilations and anti–erosions.

Definition 3.62 (elementary morphological mappings) – Let $(A_1, \leq)$ and $(A_2, \leq)$ be two finite lattices, or more generally lattices which have a least element denoted by o, and greatest element denoted by i.

A morphism for the commutative and idempotent monoids $(A_1, \vee_\leq, o)$ and $(A_2, \vee_\leq, o)$ is called a *dilation* (Serra, 1988). We denote by $\mathrm{D}(A_1, A_2)$ the set of the dilations.

A morphism for the commutative and idempotent monoids $(A_1, \wedge_\leq, i)$ and $(A_2, \wedge_\leq, i)$ is called an *erosion* (Serra, 1988). We denote by $\mathrm{E}(A_1, A_2)$ the set of the erosions.

A morphism for the commutative and idempotent monoids $(A_1, \vee_\leq, o)$ and $(A_2, \wedge_\leq, i)$ is called an *anti–dilation* (Serra, 1987; Banon & Barrera, 1993). We denote by $\mathrm{D^a}(A_1, A_2)$ the set of the anti–dilations.

A morphism for the commutative and idempotent monoids $(A_1, \wedge_\leq, i)$ and $(A_2, \vee_\leq, o)$ is called an *anti–erosion* (Serra, 1987; Banon & Barrera, 1993). We denote by $\mathrm{E^a}(A_1, A_2)$ the set of the anti–erosions.

The dilations, erosions, anti–dilations and anti–erosions are called *elementary morphological mappings* or, simply, *elementary mappings*. □

If the posets $(A_1, \leq)$ and $(A_2, \leq)$ are finites lattices then we have the following results. We will omit the reference to the partial order when using the union and intersection symbols.

Proposition 3.63 (lattice versus increasing mapping) – Let $(A_1, \leq)$ and $(A_2, \leq)$ be two lattices. A mapping f from A_1 to A_2 is increasing iff, for any a and b in A_1,

$$f(a) \vee f(b) \leq f(a \vee b),$$

or equivalently, iff for any a and b in A_1,

$$f(a \wedge b) \leq f(a) \wedge f(b). \qquad \square$$

Proof – See Lemma 2.1, p. 260 of Heijmans & Ronse (1990) for the case where A_1 and A_2 are identical or Proposition 3.1, p. 33 of Banon & Barrera (1994) for the case where A_1 and A_2 are two Boolean lattices. □

Proposition 3.64 (finite lattice versus decreasing mapping) – Let $(A_1, \leq)$ and $(A_2, \leq)$ be two finite lattices. A mapping f from A_1 to A_2 is decreasing iff, for any a and b in A_1,

$$f(a \vee b) \leq f(a) \wedge f(b),$$

or equivalently, iff for any a and b in A_1,

$$f(a) \vee f(b) \leq f(a \wedge b). \qquad \square$$

Proof – The result follows from Proposition 3.63 by duality. □

As a consequence of these two propositions, we can state the following properties.

Proposition 3.65 (property of the elementary mappings) – Let $(A_1, \leq)$ and $(A_2, \leq)$ be two finite lattices. Let o_1 (resp., o_2) and i_1 (resp., i_2) be, respectively the least and greatest element of $(A_1, \leq)$ (resp., $(A_2, \leq)$). Let f a mapping from A_1 to A_2, then we have the following statements:

$$f \text{ is a dilation} \Rightarrow f \text{ is increasing and } f(o_1) = o_2$$

$$f \text{ is an erosion} \Rightarrow f \text{ is increasing and } f(i_1) = i_2$$

$$f \text{ is an anti–dilation} \Rightarrow f \text{ is decreasing and } f(o_1) = i_2$$

$$f \text{ is an anti–erosion} \Rightarrow f \text{ is decreasing and } f(i_1) = o_2. \qquad \square$$

Proof – The increase and decrease properties derive from the definition of the four classes of elementary mappings and from Propositions 3.63 and 3.64. The remaining properties derive from the definition of the four classes of elementary mappings. □

In Proposition 3.65, the converse statements are not true in general. Nevertheless, they are true when A_1 is a finite chain.

Proposition 3.66 (characterization of the elementary mappings whose domain is a finite chain) – Let $(A_1, \leq)$ a finite chain and let $(A_2, \leq)$ be a finite lattice. Let o_1 (resp., o_2) and i_1 (resp., i_2) be, respectively the least and greatest element of $(A_1, \leq)$ (resp., $(A_2, \leq)$). Let f a mapping from A_1 to A_2, then we have the following statements:

$$f \text{ is a dilation} \Leftrightarrow f \text{ is increasing and } f(o_1) = o_2$$

$$f \text{ is an erosion} \Leftrightarrow f \text{ is increasing and } f(i_1) = i_2$$

$$f \text{ is an anti–dilation} \Leftrightarrow f \text{ is decreasing and } f(o_1) = i_2$$

$$f \text{ is an anti–erosion} \Leftrightarrow f \text{ is decreasing and } f(i_1) = o_2. \qquad \square$$

Proof – Let us prove the first statement. Let us prove $\Rightarrow$. If f is a dilation then, by Proposition 3.65, f is increasing and $f(o_1) = o_2$. Let us prove $\Leftarrow$. For any a and b in A_1,

$$\begin{aligned} f(a \vee b) &\leq f(a) \vee f(b) && (a \vee b = a \text{ or } b, \text{ and } \vee \text{ property}) \\ &\leq f(a \vee b) && (f \text{ is increasing and Proposition 3.63}) \end{aligned}$$

In other words, by antisymmetry of $\leq$, for any a and b in A_1, $f(a \vee b) = f(a) \vee f(b)$. Together with the assumption that $f(o_1) = o_2$, this prove that f is a dilation.

Before ending this presentation on morphism, we state three propositions showing that the subsets of morphisms of some specific operations may be closed under some extended operations.

Proposition 3.67 (closure properties of the morphisms of internal binary operations) – Let $(A, \square)$ be a set equipped with an internal binary operation and let $(B, \square)$ be a commutative semigroup.

1. Then the set of morphisms $M_\square(A, B)$ is $\square_{BA}$–closed.

2. If $(B, \cdot)$ is a set equipped with an external binary operation with operand in C which is distributive over $\square$, then $M_\square(A, B)$ is $\cdot_{BA}$–closed.

3. If (B, e_B) is a set equipped with a nullary operation such that $e_B \square e_B = e_B$, then $M_\square(A, B)$ is e_{BA}–closed. $\square$

Proof – 1. For any f and g in $M_\square(A, B)$, and any x and y in A,

$$\begin{aligned} (f \square g)(x \square y) &= f(x \square y) \square g(x \square y) && (\text{extension definition}) \\ &= (f(x) \square f(y)) \square (g(x) \square g(y)) && (f \text{ and } g \text{ are morphisms}) \\ &= f(x) \square (f(y) \square (g(x) \square g(y))) && (\text{associativity of } \square \text{ on } B) \\ &= f(x) \square ((f(y) \square g(x)) \square g(y)) && (\text{associativity of } \square \text{ on } B) \\ &= f(x) \square ((g(x) \square f(y)) \square g(y)) && (\text{commutativity of } \square \text{ on } B) \\ &= f(x) \square (g(x) \square (f(y) \square g(y))) && (\text{associativity of } \square \text{ on } B) \\ &= (f(x) \square g(x)) \square (f(y) \square g(y)) && (\text{associativity of } \square \text{ on } B) \\ &= (f \square g)(x) \square (f \square g)(y). && (\text{extension definition}) \end{aligned}$$

In other words, $f \square g$ is in $M_\square(A, B)$.

2. For any f in $\mathrm{M}_{\square}(A, B)$, any α in C, and any x and y in A,

$$\begin{aligned}(\alpha{\cdot}f)(x \square y) &= \alpha{\cdot}f(x \square y) && \text{(extension definition)}\\ &= \alpha{\cdot}(f(x) \square f(y)) && \text{(f is morphism)}\\ &= \alpha{\cdot}f(x) \square \alpha{\cdot}f(y) && \text{(distributivity of $\cdot$over $\square$)}\\ &= (\alpha{\cdot}f)(x) \square (\alpha{\cdot}f)(y)\,. && \text{(extension definition)}\end{aligned}$$

In other words, $\alpha{\cdot}f$ is in $\mathrm{M}_{\square}(A, B)$.

3. Let e be the element of B^A selected by the extension to B^A of the nullary operation on B, for any x and y in A,

$$\begin{aligned}e(x \square y) &= e_B && \text{(extension definition)}\\ &= e_B \square e_B && \text{(hypothesis)}\\ &= e(x) \square e(y). && \text{(extension definition)}\end{aligned}$$

In other words, e is in $\mathrm{M}_{\square}(A, B)$. □

Proposition 3.68 (closure properties of the morphisms of external binary operations) – Let $(C, \cdot)$ be a set equipped with a commutative internal binary operation. Let $(A, \cdot)$ and $(B, \cdot)$ be two sets equipped with an external binary operation with operand in C, such that, for any b in B, and any α and β in C,

$$\alpha{\cdot}(\beta{\cdot}b) = (\alpha{\cdot}\beta){\cdot}b. \qquad \text{(mixt associativity)}$$

1. Then the set of morphisms $\mathrm{M}_{\cdot}(A, B)$ is $\cdot_{B^A}$–closed.

2. If $(B, \square)$ is a set equipped with an internal binary operation such that $\cdot$ on B is distributive over $\square$, then $\mathrm{M}_{\cdot}(A, B)$ is $\square_{B^A}$–closed.

3. If (B, e_B) is a set equipped with a nullary operation that selects e_B, and for α in C,

$$\alpha{\cdot}e_B = e_B,$$

then $\mathrm{M}_{\cdot}(A, B)$ is e_{B^A}–closed. □

Proof – 1. For any f in $\mathrm{M}_{\cdot}(A, B)$, any α and β in C, and any x in A,

$$\begin{aligned}(\alpha{\cdot}f)(\beta{\cdot}x) &= \alpha{\cdot}f(\beta{\cdot}x) && \text{(extension definition)}\\ &= \alpha{\cdot}(\beta{\cdot}f(x)) && \text{(f is morphism)}\\ &= (\alpha{\cdot}\beta){\cdot}f(x) && \text{(mixt associativity of $\cdot$ on C and B)}\\ &= (\beta{\cdot}\alpha){\cdot}f(x) && \text{(commutativity of $\cdot$ on C)}\\ &= \beta{\cdot}(\alpha{\cdot}f(x)) && \text{(mixt associativity of $\cdot$ on C and B)}\\ &= \beta{\cdot}(\alpha{\cdot}f)(x)\,. && \text{(extension definition)}\end{aligned}$$

In other words, $\alpha{\cdot}f$ is in $\mathrm{M}_{\cdot}(A, B)$.

2. For any f and g in $\mathrm{M}_{\cdot}(A, B)$, any α in C, and any x in A,

$$\begin{aligned}
(f \,\square\, g)(\alpha{\cdot}x) &= f(\alpha{\cdot}x) \,\square\, g(\alpha{\cdot}x) && \text{(extension definition)}\\
&= \alpha{\cdot}f(x) \,\square\, \alpha{\cdot}g(x) && \text{(f and g are morphisms)}\\
&= \alpha{\cdot}(f(x) \,\square\, g(x)) && \text{(distributivity of $\cdot$ on B over $\square$)}\\
&= \alpha{\cdot}(f \,\square\, g)(x). && \text{(extension definition)}
\end{aligned}$$

In other words, $f \,\square\, g$ is in $\mathrm{M}_{\cdot}(A, B)$.

3. Let e be the element of B^A selected by the extension to B^A of the nullary operation on B, for any α in C, and x in A,

$$\begin{aligned}
e(\alpha{\cdot}x) &= e_B && \text{(extension definition)}\\
&= \alpha{\cdot}e_B && \text{(e_B property)}\\
&= \alpha{\cdot}e(x). && \text{(extension definition)}
\end{aligned}$$

In other words, e is in $\mathrm{M}_{\cdot}(A, B)$. □

Proposition 3.69 (closure properties of the morphisms of nullary operations) – Let (A, e_A) and (B, e_B) be two sets equipped with a nullary operation.

1. Then the set of morphisms $\mathrm{M}_e(A, B)$ is e_{B^A}–closed.

2. If $(B, \square)$ is a set equipped with an internal binary operation such that $e_B \,\square\, e_B = e_B$, then $\mathrm{M}_e(A, B)$ is $\square_{B^A}$–closed.

3. If $(B, \cdot)$ is a set equipped with an external operation with operand in C, such that, for any α in C,

$$\alpha{\cdot}e_B = e_B,$$

then $\mathrm{M}_e(A, B)$ is $\cdot_{B^A}$–closed. □

Proof – 1. Let e be the element of B^A selected by the extension to B^A of the nullary operation on B, then $e(e_A) = e_B$ by extension definition. In other words, e is in $\mathrm{M}_e(A, B)$.

2. For any f and g in $\mathrm{M}_e(A, B)$,

$$\begin{aligned}
(f \,\square\, g)(e_A) &= f(e_A) \,\square\, g(e_A) && \text{(extension definition)}\\
&= e_B \,\square\, e_B && \text{(f and g are morphisms)}\\
&= e_B. && \text{(hypothesis)}
\end{aligned}$$

In other words, $f \,\square\, g$ is in $\mathrm{M}_e(A, B)$.

3. For any f in $\mathrm{M}_e(A, B)$, and any α in C,

$$\begin{aligned}
(\alpha{\cdot}f)(e_A) &= \alpha{\cdot}f(e_A) && \text{(extension definition)}\\
&= \alpha{\cdot}e_B && \text{(f is a morphism)}\\
&= e_B && \text{(e_B property)}
\end{aligned}$$

In other words, $\alpha{\cdot}f$ is in $\mathrm{M}_e(A, B)$. □

Extended binary relation and power lattice

Let A be a non–empty set. From a binary relation on B we can construct a binary relation on the set of mappings from A to B.

Let R be a binary relation on B. For any f and g in B^A, the *extension to B^A of the binary relation R on B* is the binary relation denoted by S_R and defined by

$$f\, S_R\, g \overset{\Delta}{\Leftrightarrow} (f(a)\, R\, g(a), \ \ \forall a \in A).$$

In the particular case of partial ordering relations, we have the following result.

Proposition 3.70 (partial order extension) – Let A and B be two non–empty sets. Then, the extension $\leq_{\leq}$ to B^A of a partial ordering relation $\leq$ on B is a partial ordering relation. □

Exercise 3.71 (partial order extension) – Prove Proposition 3.70. □

A consequence of the partial order extension is the following extension for lattices.

Proposition 3.72 (extension property for a lattice) – Let $(B, \leq)$ be a lattice. Then, $(B^A, \leq_{\leq})$ is a lattice and, for any f and g in B^A,

$$\sup\{f, g\} = f \vee_{B^A} g \quad \text{and} \quad \inf\{f, g\} = f \wedge_{B^A} g,$$

where $\vee_{B^A}$ and $\wedge_{B^A}$ are, respectively, the extensions to B^A of $\vee_{\leq}$ and $\wedge_{\leq}$ on B. □

Proof – For any f and g in B^A, any a in A and any b in B,

$$\begin{aligned} f(a) \leq b \text{ and } g(a) \leq b &\Leftrightarrow f(a) \vee_{\leq} g(a) \leq b && \text{(def. of } \vee_{\leq} \text{ and sup)} \\ &\Leftrightarrow (f \vee_{B^A} g)(a) \leq b, && \text{(definition of } \vee_{B^A}) \end{aligned}$$

by mapping definition, this implies that, for any distinct f, g and h in B^A, and any a in A,

$$f(a) \leq h(a) \text{ and } g(a) \leq h(a) \Leftrightarrow (f \vee_{B^A} g)(a) \leq h(a).$$

By definition of binary relation extension, the above statement is equivalent to, for any distinct f, g and h in B^A,

$$f \leq_{\leq} h \text{ and } g \leq_{\leq} h \Leftrightarrow (f \vee_{B^A} g) \leq_{\leq} h.$$

By Proposition 3.70, the binary relation $\leq_{\leq}$ in the above statement is a partial ordering, therefore, any distinct f and g in B^A have a supremum which is $f \vee_{B^A} g$.

The same reasoning applies to the infimum. Therefore $(B^A, \leq_{\leq})$ is a lattice. □

A consequence of the above proposition is that $\vee_{\leq_{\leq}}$ and $\wedge_{\leq_{\leq}}$ on B^A are, respectively, $\vee_{B^A}$ and $\wedge_{B^A}$.

The lattice $(B^A, \leq_{\leq})$ of the above proposition is called a *power lattice* or a *function lattice extended from the lattice* $(B, \leq)$.

Proposition 3.73 (extension property for a lattice having a least element and a greatest element) – Let $(B, \leq)$ be a lattice with the least element denoted by o and the greatest element denoted by i. Then, $(B^A, \leq_{\leq})$ has a least element and a greatest element, and these elements are, respectively, $a \mapsto o$ and $a \mapsto i$. Furthermore, the mappings $a \mapsto o$ and $a \mapsto i$, are, respectively, the unit elements of $\vee_{\leq_{\leq}}$ and $\wedge_{\leq_{\leq}}$ on B^A. □

Proof – By the least and greatest element definitions, for any f in B^A and any a in A,

$$o \leq f(a) \leq i;$$

therefore, by the binary relation extension, for any f in B^A,

$$(a \mapsto o) \leq_{\leq} f \leq_{\leq} (a \mapsto i),$$

that is, $a \mapsto o$ and $a \mapsto i$ are, respectively, the least element and the greatest element of $(B^A, \leq_{\leq})$.

This result and Proposition 3.48 imply the last statement. This last statement could also be obtained by observing that o and i are, respectively, the unit elements for $\vee_{\leq}$ and $\wedge_{\leq}$ on B, and by applying Proposition 2.48. □

We observe that the nullary operations on B^A that select, respectively, the least mapping and the greatest mapping are, the extensions to B^A of the nullary operations on B which select respectively the least element and greatest element of $(B, \leq)$.

Chapter 4

Filtering

Image simulation and restoration can be achieved through filtering. A spatially invariant filter is characterized by a measure. Its properties depend on the properties of its associated measure. So we begin to study the measures.

At the contrary of the previous chapter, in this one we will consider some properties of the image domain.

4.1 Measure

Measures can be made on images. We define a measure on a digital image as a measure on a mapping.

Definition 4.1 (measure) – An *image measure*, or, simply, *measure* is a mapping from an image set to a gray–scale. □

In this section, we denote the image domain by W.

If K_1 and K_2 are any gray–scales (continuous or discrete), we denote by $\mathrm{Me}({K_1}^W, K_2)$, or sometimes simply by Me, the set of measures from ${K_1}^W$ to K_2.

Figure 4.1 shows an image measure.

The mapping from $[0, 1]^W$ to $[0, \#W]$ (where the intervals are intervals of natural numbers), given by $g \mapsto \#g^{-1}(\{1\})$, is an example of useful measure. This measure gives for each binary image with domain W, its number of white pixels.

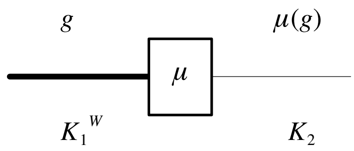


Fig. 4.1 – A measure.

For each i in a discrete gray–scale K_1, the mapping from $K_1^{\ W}$ to $\{0, 1/\#W, 2/\#W, ..., 1\}$, given by $g \mapsto H(g)(i)$, (where $\mathrm{H}(g)$ is the histogram of g), is another example of measure.

The mapping from $\mathbf{R}^W$ to $\mathbf{R}$ given by $g \mapsto \mathrm{m}(g)$, where $\mathrm{m}(g)$ is the mean of g as defined in Section 2.2 is one more example of measure.

When we compose a measure with an appropriate lut we get another measure. If the lut is a morphism of an external operation as defined in Proposition 4.36, then the composition reduces to an external operation.

Exercise 4.2 (composition of a measure and a lut which is a morphism of an external operation) – Let $(K_2, \cdot, e_2')$ be a commutative monoid and let $\mathrm{M}_{\cdot}(K_2, K_2)$ be the set of luts from K_2 to K_2 which are morphisms for the operation $\cdot$ viewed as an external binary operation (with operand in K_2). Let μ be a measure in $\mathrm{Me}(K_1^{\ W}, K_2)$, using Proposition 4.36, prove that for any l in $\mathrm{M}_{\cdot}(K_2, K_2)$,

$$l \circ \mu \; = \; \alpha_l \cdot \mu. \qquad \square$$

Solution – For any l in $\mathrm{M}_{\cdot}(K_2, K_2)$ and any g in $K_1^{\ W}$,

$$\begin{aligned}
(l \circ \mu)(g) &= l(\mu(g)) && \text{(composition definition)}\\
&= l_{\alpha_l}(\mu(g)) && \text{(Proposition 4.36)}\\
&= \mu(g) \cdot \alpha_l && (l_\alpha \text{ definition})\\
&= \alpha_l \cdot \mu(g) && \text{(commutativity)}\\
&= (\alpha_l \cdot \mu)(g). && \text{(extension definition)}
\end{aligned}$$

$\square$

We now present our last example of measure.

Let g be an image from W to K_1, for any u in W we denote by $\mu_u(g)$ the expression defined by

$$\mu_u(g) \triangleq g(u).$$

For each u in W, the above expression defines a mapping μ_u from $K_1^{\ W}$ to K_1. This measure gives the value of a pixel at position u, of the image g, for short we call it *read–pixel*.

The read–pixel measures have the following property.

Proposition 4.3 (read–pixel morphism property) – Let W be a non–empty set.

1. Let $(K_1, \square)$ be a set equipped with an internal binary operation and let $({K_1}^W, \square)$ be its extension to the set of images whose domain is W, then, for any u in W, the read–pixel measure μ_u is a morphism from $({K_1}^W, \square)$ to $(K_1, \square)$.

2. Let $(K_1, \cdot)$ be a set equipped with an external binary operation with operand in a set C, and let $({K_1}^W, \cdot)$ be its extension to the set of images whose domain is W, then, for any u in W, the read–pixel measure μ_u is a morphism from $({K_1}^W, \cdot)$ to $.(K_1, \cdot)$

3. Let (K_1, e_1) be a set equipped with a nullary operation and let $({K_1}^W, e)$ be its extension to the set of images whose domain is W, then, for any u in W, the read–pixel measure μ_u is a morphism from $({K_1}^W, e)$ to (K_1, e_1). □

Proof – 1. For any u in W, and any images g and h in ${K_1}^W$,

$$\begin{aligned} \mu_u(g \square h) &= (g \square h)(u) && (\mu_l \text{ definition}) \\ &= g(u) \square h(u) && (\text{extension def.}) \\ &= \mu_u(g) \square \mu_u(h). && (\mu_l \text{ definition}) \end{aligned}$$

2. For any u in W, any image g in ${K_1}^W$, and any α in C,

$$\begin{aligned} \mu_u(\alpha \cdot g) &= (\alpha \cdot g)(u) && (\mu_l \text{ definition}) \\ &= \alpha \cdot g(u) && (\text{extension def.}) \\ &= \alpha \cdot \mu_u(g). && (\mu_l \text{ definition}) \end{aligned}$$

3. For any u in W,

$$\begin{aligned} \mu_u(e) &= e(u) && (\mu_l \text{ definition}) \\ &= e_1. && (\text{extension def.}) \end{aligned}$$

□

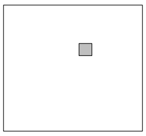
Before characterizing the measures that are morphisms, we introduce the notion of pulse image and we give some of its properties. We also give a result concerning an image decomposition.

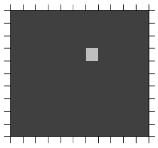
Definition 4.4 (pulse image) – Let e_1 be a given gray level in K_1, for any u in W and s in K_1, the *pulse image* or *pulse mapping* $g_{u,s}$ with a pulse of intensity s at u and with background e_1 is the mapping from W to K_1 defined by, for any v in W,

$$g_{u,s}(v) \triangleq \begin{cases} s & \text{if } v = u \\ e_1 & \text{otherwise.} \end{cases}$$ □

In other words, $g_{u,s}$ is the image of a "point" at position u, the value of the pixel at position u being s and the value of the remaining pixels being e_1. Figure 4.2 shows an example of a pulse image. The pair (u, s) defines a pixel in $W \times K_1$ which is shown on the upper part of the figure.

For each u in W, the mapping from K_1 to ${K_1}^W$, given by $s \mapsto g_{u,s}$ is a morphism as shown in the next proposition.

$(u, s) =$ 

 $g_{u,s}$

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&3&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

Fig. 4.2 – A pulse image.

Proposition 4.5 (pulse image property) – Let W be a non–empty set and let (K_1, e_1) be a set equipped with a nullary operation e_1 which selects the element e_1.

1. Let $(K_1, \square)$ be a set equipped with an internal binary operation such that $e_1 \square e_1 = e_1$, and let $(K_1{}^W, \square)$ be its extension to the set of images whose domain is W, then, for any u in W, and s and t in K_1,

$$g_{u,(s \square t)} = g_{u,s} \square g_{u,t}.$$

2. Let $(K_1, \cdot)$ be a set equipped with an external binary operation with operand in a set C, such that, for any α in C,

$$\alpha \cdot e_1 = e_1,$$

and let $(K_1{}^W, \cdot)$ be the extension of $(K_1, \cdot)$ to the set of images whose domain is W, then, for any u in W, s in K_1, and any α in C,

$$g_{u,\alpha \cdot s} = \alpha \cdot g_{u,s}.$$

3. Let $(K_1{}^W, e)$ be the extension of (K_1, e_1) to the set of images whose domain is W, then, for any u in W,

$$g_{u,e_1} = e. \qquad \square$$

Proof – 1. First, for any u in W, and s and t in K_1,

$$\begin{aligned} g_{u,(s \square t)}(u) &= s \square t && (g_{u,s} \text{ definition})\\ &= g_{u,s}(u) \square g_{u,t}(u) && (g_{u,s} \text{ definition})\\ &= (g_{u,s} \square g_{u,t})(u); && (\text{extension def.}) \end{aligned}$$

second, for any u and v in W, such that $u \neq v$, and s and t in K_1,

$$\begin{aligned} g_{u,(s \square t)}(v) &= e_1 && (g_{u,s} \text{ definition})\\ &= e_1 \square e_1 && (\square \text{ property})\\ &= g_{u,s}(v) \square g_{u,t}(v) && (g_{u,s} \text{ definition})\\ &= (g_{u,s} \square g_{u,t})(v); && (\text{extension def.}) \end{aligned}$$

that is, for any u in W, and s and t in K_1, $g_{u,(s \square t)} = g_{u,s} \square g_{u,t}$.

2. First, for any u in W, s in K_1, and α in C,

$$\begin{aligned} g_{u,\alpha\cdot s}(u) &= \alpha\cdot s && (g_{u,s}\text{ definition})\\ &= \alpha\cdot g_{u,s}(u) && (g_{u,s}\text{ definition})\\ &= (\alpha\cdot g_{u,s})(u); && (\text{extension def.}) \end{aligned}$$

second, for any u and v in W, such that $u \neq v$, s in K_1, and α in C,

$$\begin{aligned} g_{u,\alpha\cdot s}(v) &= e_1 && (g_{u,s}\text{ definition})\\ &= \alpha\cdot e_1 && (\cdot\text{ property})\\ &= \alpha\cdot g_{u,s}(v) && (g_{u,s}\text{ definition})\\ &= (\alpha\cdot g_{u,s})(v); && (\text{extension def.}) \end{aligned}$$

that is, for any u in W, s in K_1, and α in C, $g_{u,\alpha\cdot s} = \alpha\cdot g_{u,s}$.

3. For any u in W,

$$\begin{aligned} g_{u,e_1}(u) &= e_1 && (g_{u,s}\text{ definition})\\ &= e(u), && (\text{extension def.}) \end{aligned}$$

that is, for any u in W, $g_{u,e_1} = e$. □

Concerning an image decomposition we have the following important result.

Proposition 4.6 (image decomposition) – Let $(K_1, \Box, e_1)$ be a commutative monoid, then for any g in $K_1{}^W$,

$$\mathop{\Box}_{u \in W} g_{u,g(u)} = g. \qquad \Box$$

Proof – Let g be in $K_1{}^W$, for any v in W,

$$\begin{aligned} (\mathop{\Box}_{u \in W} g_{u,g(u)})(v) &= \mathop{\Box}_{u \in W} g_{u,g(u)}(v) && (\text{extension definition})\\ &= g_{v,g(v)} \Box (\mathop{\Box}_{\substack{u \in W\\ u \neq v}} g_{u,g(u)}(v)) && (\text{commutative monoid property})\\ &= g_{v,g(v)} \Box (\mathop{\Box}_{\substack{u \in W\\ u \neq v}} e_1) && (g_{u,s}\text{ definition})\\ &= g_{v,g(v)} \Box e_1 && (\text{unit element})\\ &= g_{v,g(v)} && (\text{unit element})\\ &= g(v), && (g_{u,s}\text{ definition}) \end{aligned}$$

that is, by mapping equality definition, $\mathop{\Box}_{u \in W} g_{u,g(u)} = g$. □

In order to exemplify, let us consider an image g from the domain $\mathbf{2} \times \mathbf{2}$ to the gray–scale $K = \{0, 1, 2, 3, 4\}$. With the usual order $\leq$ on the natural numbers, $(K, \leq)$ is a chain with 0 has least element and, from Propositions 3.43 and 3.48, $(K, \vee_{\leq}, 0)$, is a commutative and idempotent monoid, where 0 is the unit elements of $\vee_{\leq}$. By denoting simply by $\vee$ the extension to $K^{\mathbf{2}\times\mathbf{2}}$ of $\vee_{\leq}$ and applying Proposition 4.6, the decomposition of $g = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ is

$$g = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

To characterize the measures that are morphisms we will need to consider the mappings from W to ${K_2}^{K_1}$. We have encountered such mappings in Chapter 3 to characterize the pointwise operator, but this is merely a coincidence.

If l is such a mapping, we observe that, for any u in W, the value $l(u)$ is a lut from K_1 to K_2, for this reason we call l a family of luts.

We denote by $\mathrm{Lu}(W, {K_2}^{K_1})$, or sometimes simply Lu, the set of families of luts from W to ${K_2}^{K_1}$.

We now introduce some notation for extended operations on the measures and on the families of luts.

Let $\square$ be an internal binary operation on K_2, let $\cdot$ be an external operation on K_2 with operand in a set C, and let e_2 be a nullary operation on K_2.

For the measures, we denote by $\square$ the extension to $\mathrm{Me}({K_1}^W, K_2)$ of $\square$ on K_2, by $\cdot$ the extension to $\mathrm{Me}({K_1}^W, K_2)$ of $\cdot$ on K_2, and by ε the extension to $\mathrm{Me}({K_1}^W, K_2)$ of e_2 on K_2.

For the families of luts, we denote by $\square$ the extension to $\mathrm{Lu}(W, {K_2}^{K_1})$ of the extension to ${K_2}^{K_1}$ of $\square$ on K_2, by $\cdot$ the extension to $\mathrm{Lu}(W, {K_2}^{K_1})$ of the extension to ${K_2}^{K_1}$ of $\cdot$ on K_2, and by e the extension to $\mathrm{Lu}(W, {K_2}^{K_1})$ of the extension to ${K_2}^{K_1}$ of e_2 on K_2.

We are now ready to characterize the measures that are morphisms.

Let μ be in $\mathrm{Me}({K_1}^W, K_2)$, let u be an element of W and let s be an element of K_1, we denote by $l_\mu(u)(s)$ the expression defined by

$$l_\mu(u)(s) \triangleq \mu(g_{u,s}).$$

Let l be in $\mathrm{Lu}(W, {K_2}^{K_1})$, we denote by μ_l the expression defined by

$$\mu_l \triangleq \mathop{\square}_{u \in W} l(u) \circ \mu_u.$$

Proposition 4.7 (characterization of the measures which are morphisms) – Let W be a finite set.

1. Let $(K_1, \square, e_1)$ and $(K_2, \square, e_2)$ be two commutative monoids, then the mapping $\mu \mapsto l_\mu$ from $\mathrm{M}_{\square,e}({K_1}^W, K_2)$ to the set $\mathrm{Lu}(W, \mathrm{M}_{\square,e}(K_1, K_2))$ of mappings from W to $\mathrm{M}_{\square,e}(K_1, K_2)$ is a bijection. Its inverse is $l \mapsto \mu_l$.

2. Let $(C, \cdot)$ be a set equipped with a commutative internal binary operation. Let $(K_1, \square, \cdot, e_1)$ set equipped with an internal binary operation $\square$, an external binary operation $\cdot$ and a nullary operation e_1, such that $(K_1, \square, e_1)$ is a commutative monoid. If $(K_2, \square, \cdot, e_2)$ is such that $(K_2, \square, e_2)$ is a commutative monoid and $\cdot$is an external binary operation (with operand in C) satisfying for any s and t in K_2, and any α and β in C,

$$\alpha \cdot e_2 = e_2, \quad \alpha \cdot (\beta \cdot t) = (\alpha . \beta) \cdot t \quad \text{and} \quad \alpha \cdot (s \square t) = \alpha \cdot s \square \alpha \cdot t,$$

then the mapping $\mu \mapsto l_\mu$ from $\mathrm{M}_{\square, \cdot, e}({K_1}^W, K_2)$ to the set $\mathrm{Lu}(W, \mathrm{M}_{\square, \cdot, e}(K_1, K_2))$ of mappings from W to $\mathrm{M}_{\square, \cdot, e}(K_1, K_2)$ is a bijection. Its inverse is $l \mapsto \mu_l$. □

Proof – 1.1. Let μ be a measure from ${K_1}^W$ to K_2, then by definition l_μ is a mapping from W to ${K_2}^{K_1}$. Furthermore, let μ be in $\mathrm{M}_{\square, e}({K_1}^W, K_2)$. First, for any u in W, and s and t in K_1,

$$\begin{aligned} l_\mu(u)(s \square t) &= \mu(g_{u, (s \square t)}) && (l_\mu \text{ definition}) \\ &= \mu(g_{u,s} \square g_{u,t}) && (\text{Proposition 4.5}) \\ &= \mu(g_{u,s}) \square \mu(g_{u,t}) && (\mu \text{ is a morphism}) \\ &= l_\mu(u)(s) \square l_\mu(u)(t); && (l_\mu \text{ definition}) \end{aligned}$$

second, for any u in W,

$$\begin{aligned} l_\mu(u)(e_1) &= \mu(g_{u, e_1}) && (l_\mu \text{ definition}) \\ &= \mu(e) && (\text{Proposition 4.5}) \\ &= e_2; && (\mu \text{ is a morphism}) \end{aligned}$$

that is, for any u in W, the lut $l_\mu(u)$ is in $\mathrm{M}_{\square, e}(K_1, K_2)$.

1.2. Let l be a mapping from W to ${K_2}^{K_1}$, then by definition, μ_l is a measure from ${K_1}^W$ to K_2. Furthermore, for any u in W, let $l(u)$ be in $\mathrm{M}_{\square, e}(K_1, K_2)$. From Proposition 4.3, the read–pixel measure μ_u is a morphism from $({K_1}^W, \square, e)$ to $(K_1, \square, e_1)$, then from Proposition 4.40, $l(u) \circ \mu_u$ is a morphism from $({K_1}^W, \square, e)$ to $(K_2, \square, e_2)$. Therefore, by Propositions 3.67 and 3.69, W being finite, $\underset{u \in W}{\square} l(u) \circ \mu_u$ is a morphism from $({K_1}^W, \square, e)$ to $(K_2, \square, e_2)$, that is, μ_l is in $\mathrm{M}_{\square, e}({K_1}^W, K_2)$.

1.3. Let μ be a measure in $\mathrm{M}_{\square, e}({K_1}^W, K_2)$, for any g in ${K_1}^W$,

$$\begin{aligned} \mu_{l_\mu}(g) &= (\underset{u \in W}{\square} l_\mu(u) \circ \mu_u)(g) && (\mu_l \text{ definition}) \\ &= \underset{u \in W}{\square} (l_\mu(u) \circ \mu_u)(g) && (\text{extension definition}) \\ &= \underset{u \in W}{\square} l_\mu(u)(\mu_u(g)) && (\text{composition definition}) \end{aligned}$$

$$= \square_{u \in W} l_\mu(u)(g(u)) \qquad (\mu_u \text{ definition})$$

$$= \square_{u \in W} \mu(g_{u,g(u)}) \qquad (l_\mu \text{ definition})$$

$$= \mu(\square_{u \in W} g_{u,g(u)}) \qquad (\mu \text{ is a morphism})$$

$$= \mu(g), \qquad (\text{Proposition 4.6})$$

that is, $\mu \mapsto l_\mu$ is an injection.

1.4. Let l be a mapping from W to $M_{\square,e}(K_1, K_2)$, for any v in W and s in K_1,

$$l_{\mu_l}(v)(s) = \mu_l(g_{v,s}) \qquad (l_\mu \text{ definition})$$

$$= (\square_{u \in W} l(u) \circ \mu_u)(g_{v,s}) \qquad (\mu_u \text{ definition})$$

$$= \square_{u \in W} (l(u) \circ \mu_u)(g_{v,s}) \qquad (\text{extension definition})$$

$$= \square_{u \in W} l(u)(\mu_u(g_{v,s})) \qquad (\text{composition def.})$$

$$= \square_{u \in W} l(u)(g_{v,s}(u)) \qquad (\mu_u \text{ definition})$$

$$= l(v)(g_{v,s}(v)) \,\square\, (\square_{\substack{u \in W \\ u \neq v}} l(u)(g_{u,s}(u)))$$

$$((K_2, \square) \text{ is a commutative semigroup})$$

$$= l(v)(g_{v,s}(v)) \,\square\, (\square_{\substack{u \in W \\ u \neq v}} l(u)(e_1)) \qquad (g_{u,s} \text{ def.})$$

$$= l(v)(g_{v,s}(v)) \,\square\, (\square_{\substack{u \in W \\ u \neq v}} e_2) \qquad (l(u) \text{ is a morphism})$$

$$= l(v)(g_{v,s}(v)) \,\square\, e_2 \qquad (\text{unit element for } (K_2, \square))$$

$$= l(v)(g_{v,s}(v)) \qquad (\text{unit element for } (K_2, \square))$$

$$= l(v)(s), \qquad (g_{u,s} \text{ def.})$$

that is, $\mu \mapsto l_\mu$ is a surjection.

This prove that the mapping $\mu \mapsto l_\mu$ from $M_{\square,e}({K_1}^W, K_2)$ to the set of mappings from W to $M_{\square,e}(K_1, K_2)$ is a bijection.

2.1. Let μ be in $\mathrm{M}_{\square,\cdot,e}(K_1{}^W, K_2)$. For any u in W, s in K_1, and α in C,

$$\begin{aligned} l_\mu(u)(\alpha\cdot s) &= \mu(g_{u,(\alpha.s)}) && (l_\mu \text{ definition}) \\ &= \mu(\alpha\cdot g_{u,s}) && (\text{Proposition 4.5}) \\ &= \alpha\cdot\mu(g_{u,s}) && (\mu \text{ is a morphism}) \\ &= \alpha\cdot l_\mu(u)(s); && (l_\mu \text{ definition}) \end{aligned}$$

that is, together with 1.1, for any u in W, the lut $l_\mu(u)$ is in $\mathrm{M}_{\square,\cdot,e}(K_1, K_2)$.

2.2. For any u in W, let $l(u)$ be in $\mathrm{M}_{\square,\cdot,e}(K_1, K_2)$. From Proposition 4.3, the read–pixel measure μ_u is a morphism from $(K_1{}^W, \square, \cdot, e)$ to $(K_1, \square, \cdot, e_1)$, then from Proposition 4.40, $l(u) \circ \mu_u$ is a morphism from $(K_1{}^W, \square, \cdot, e)$ to $(K_2, \square, \cdot, e_2)$. Therefore, by Propositions 3.67 to 3.69, W being finite, $\underset{u \in W}{\square}\ l(u) \circ \mu_u$ is a morphism from $(K_1{}^W, \square, \cdot, e)$ to $(K_2, \square, \cdot, e_2)$, that is, μ_l is in $\mathrm{M}_{\square,\cdot,e}(K_1{}^W, K_2)$.

Therefore, from Part 1 and by Exercise 1.26, the mapping $\mu \mapsto l_\mu$ from $\mathrm{M}_{\square,\cdot,e}(K_1{}^W, K_2)$ to the set of mappings from W to $\mathrm{M}_{\square,\cdot,e}(K_1, K_2)$ is a bijection. □

Figure 4.3 shows the bijection $\mu \mapsto l_\mu$ of Proposition 4.7 for the morphisms of monoids.

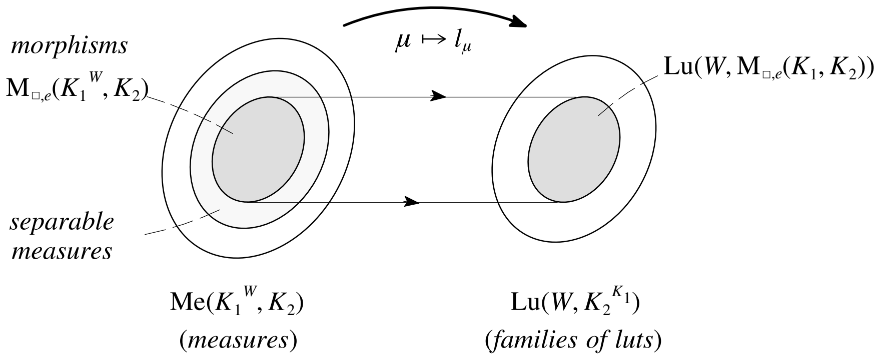


Fig. 4.3 – Characterization of the measures which are morphisms of monoids.

In Figure 4.3 we have represented the set of separable measures which definition is given below.

A measure μ from $K_1{}^W$ to K_2 is called *separable with respect to an internal binary operation* $\square$ *on* $\mathrm{Me}(K_1{}^W, K_2)$, iff there exists a family l of luts from W to $K_2{}^{K_1}$ such that

$$\mu = \underset{u \in W}{\square}\ l(u) \circ \mu_u. \tag{4.1}$$

In other words, the set of separable measures with respect to an internal binary operation is the image of the set of families of luts through the mapping $l \mapsto \mu_l$.

Let us verify, in the discrete case, that there exist measures which are **not** separable.

The number of measures from ${K_1}^W$ to K_2 is $(\#K_2)^{((\#K_1)^{\#W})}$ and the number of separable measures is bounded above by $((\#K_2)^{\#K_1})^{\#W}$ (the number of lut families). If we choose K_1, K_2 and W such that $\#K_1 = \#K_2 = 2$ and $\#W = 4$, then we find that the number of measures $2^{(2^4)} = 2^{16}$ is greater than the upper bound $(2^2)^4 = 2^8$ for the number of separable measures.

A measure satisfying expression (4.1) is called separable since its value at g is a combination, through an associative binary operation, of each value $g(u)$, u running over W. Figure 4.4 shows a separable measure μ.

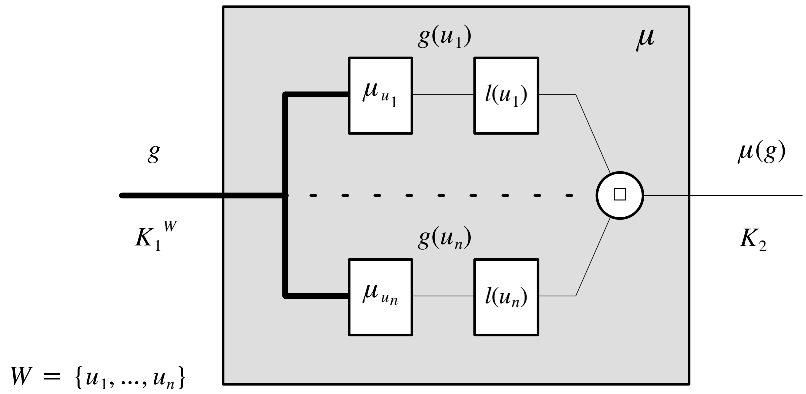


Fig. 4.4 – A separable measure.

It could be interesting to prove that the mapping $l \mapsto \mu_l$ from the set of families of luts, may not be injective.

An important consequence of Proposition 4.7, is that any measure which is a morphism of commutative monoids is a separable measure.

The idea of the mapping $l \mapsto \mu_l$. can be found in Heijmans (1994, Prop. 5.3, p. 126).

Let V be a subset of W, and let μ be a measure from ${K_1}^W$ to K_2 which is a morphism of commutative monoids (as in Proposition 4.7). A measure from ${K_1}^V$ to K_2, denoted μ_V, is called a *submeasure of* μ iff

$$\mu_V \triangleq \mathop{\square}_{u \in V} l_\mu(u) \circ (\mu_u / V).$$

We observe that $\mu_W = \mu$.

The notion of submeasure will be useful in the next section to deal with border effect in image processing.

We now study some properties of the above characterization.

Proposition 4.8 (morphism between families of luts and measures) – Let W be a finite set and let $(K_2, \square, e_2)$ be a commutative monoid.

1. The mapping $l \mapsto \mu_l$ from the set $\mathrm{Lu}(W, {K_2}^{K_1})$ of families of luts from K_1 to K_2 to the set $\mathrm{Me}({K_1}^W, K_2)$ of measures from ${K_1}^W$ to K_2, is a morphism of the commutative monoids $(\mathrm{Lu}(W, {K_2}^{K_1}), \square, e)$ and $(\mathrm{Me}({K_1}^W, K_2), \square, \varepsilon)$. That is, we have, for any families of luts l and m in $\mathrm{Lu}(W, {K_2}^{K_1})$,

$$\mu_{l \,\square\, m} = \mu_l \,\square\, \mu_m \quad \text{and} \quad \mu_e = \varepsilon.$$

2. If $(K_2, \cdot)$ is a set equipped with an external binary operation, with operand in a set C, which is distributive over $\square$, then the mapping $l \mapsto \mu_l$ from $\mathrm{Lu}(W, {K_2}^{K_1})$ to $\mathrm{Me}({K_1}^W, K_2)$ is a morphism from $(\mathrm{Lu}(W, {K_2}^{K_1}), \cdot)$ to $(\mathrm{Me}({K_1}^W, K_2), \cdot)$. That is, we have, for any family of luts l in $\mathrm{Lu}(W, {K_2}^{K_1})$ and α in C,

$$\mu_{\alpha \cdot l} = \alpha \cdot \mu_l \, . \qquad \square$$

Proof – By Proposition 3.51, $(\mathrm{Lu}(W, {K_2}^{K_1}), \square, e)$ and $(\mathrm{Me}({K_1}^W, K_2), \square, \varepsilon)$ are two commutative monoids.

1.1. For any mapping l and m in $\mathrm{Lu}(W, {K_2}^{K_1})$, any g in ${K_1}^W$,

$$\begin{aligned}
\mu_{l \,\square\, m}(g) &= \Big(\mathop{\square}_{u \in W} (l \,\square\, m)(u) \circ \mu_u\Big)(g) && (\mu_l \text{ definition})\\
&= \mathop{\square}_{u \in W} ((l \,\square\, m)(u) \circ \mu_u)(g) && (\text{extension definition})\\
&= \mathop{\square}_{u \in W} (l \,\square\, m)(u)(\mu_u(g)) && (\text{composition definition})\\
&= \mathop{\square}_{u \in W} (l \,\square\, m)(u)(g(u)) && (\mu_u \text{ definition})\\
&= \mathop{\square}_{u \in W} (l(u) \,\square\, m(u))(g(u)) && (\text{extension definition})\\
&= \mathop{\square}_{u \in W} l(u)(g(u)) \,\square\, m(u)(g(u)) && (\text{extension definition})\\
&= \mathop{\square}_{u \in W} l(u)(g(u)) \,\square\, \mathop{\square}_{u \in W} m(u)(g(u)) && (\text{commutativity and associativity})\\
&= \mathop{\square}_{u \in W} l(u)(\mu_u(g)) \,\square\, \mathop{\square}_{u \in W} m(u)(\mu_u(g)) && (\mu_u \text{ definition})\\
&= \mathop{\square}_{u \in W} (l(u) \circ \mu_u)(g) \,\square\, \mathop{\square}_{u \in W} (m(u) \circ \mu_u)(g) && (\text{composition definition})
\end{aligned}$$

$$= \Big(\bigsqcup_{u \in W} l(u) \circ \mu_u\Big)(g) \sqcup \Big(\bigsqcup_{u \in W} m(u) \circ \mu_u\Big)(g) \qquad \text{(extension definition)}$$

$$= \mu_l(g) \sqcup \mu_m(g) \qquad (\mu_l \text{ definition})$$

$$= (\mu_l \sqcup \mu_m)(g), \qquad \text{(extension definition)}$$

that is, for any mapping l and m in $\mathrm{Lu}(W, K_2{}^{K_1})$, $\mu_{l \sqcup m} = \mu_l \sqcup \mu_m$.

1.2. For any g in $K_1{}^W$,

$$\mu_e(g) = \Big(\bigsqcup_{u \in W} e(u) \circ \mu_u\Big)(g) \qquad (\mu_l \text{ definition})$$

$$= \bigsqcup_{u \in W} (e(u) \circ \mu_u)(g) \qquad \text{(extension definition)}$$

$$= \bigsqcup_{u \in W} e(u)(\mu_u(g)) \qquad \text{(composition definition)}$$

$$= \bigsqcup_{u \in W} e(u)(g(u)) \qquad (\mu_u \text{ definition})$$

$$= \bigsqcup_{u \in W} e_2 \qquad (e \text{ definition})$$

$$= e_2 \qquad \text{(unit element)}$$

$$= \varepsilon(g), \qquad (\varepsilon \text{ definition})$$

that is, $\mu_e = \varepsilon$.

2. For any mapping l in $\mathrm{Lu}(W, K_2{}^{K_1})$, any α in C, and any g in $K_1{}^W$,

$$\mu_{\alpha \cdot l}(g) = \Big(\bigsqcup_{u \in W} (\alpha \cdot l)(u) \circ \mu_u\Big)(g) \qquad (\mu_l \text{ definition})$$

$$= \bigsqcup_{u \in W} ((\alpha \cdot l)(u) \circ \mu_u)(g) \qquad \text{(extension definition)}$$

$$= \bigsqcup_{u \in W} (\alpha \cdot l)(u)(\mu_u(g)) \qquad \text{(composition definition)}$$

$$= \bigsqcup_{u \in W} (\alpha \cdot l)(u)(g(u)) \qquad (\mu_u \text{ definition})$$

$$= \bigsqcup_{u \in W} (\alpha \cdot l(u))(g(u)) \qquad \text{(extension definition)}$$

$$= \bigsqcup_{u \in W} \alpha \cdot l(u)(g(u)) \qquad \text{(extension definition)}$$

$$= \alpha\cdot(\mathop{\square}_{u\,\in\,W} l(u)(g(u))) \qquad \text{(distributivity)}$$

$$= \alpha\cdot(\mathop{\square}_{u\,\in\,W} l(u)(\mu_u(g))) \qquad (\mu_u \text{ definition})$$

$$= \alpha\cdot(\mathop{\square}_{u\,\in\,W} (l(u)\circ\mu_u)(g)) \qquad \text{(composition definition)}$$

$$= \alpha\cdot(\mathop{\square}_{u\,\in\,W} l(u)\circ\mu_u)(g) \qquad \text{(extension definition)}$$

$$= \alpha\cdot\mu_l(g) \qquad (\mu_l \text{ definition})$$

$$= (\alpha\cdot\mu_l)(g), \qquad \text{(extension definition)}$$

that is, for any mapping l in $\mathrm{Lu}(W, {K_2}^{K_1})$ and α in C, $\mu_{\alpha.l} = \alpha\cdot\mu_l$. □

From the above two propositions we can state the following result.

Proposition 4.9 (properties of the characterization of the measures which are morphisms) – Let W be a finite set.

1. Let $(K_1, \square, e_1)$ and $(K_2, \square, e_2)$ be two commutative monoids, then the mapping $\mu \mapsto l_\mu$ from $\mathrm{M}_{\square,e}({K_1}^W, K_2)$ to the set $\mathrm{Lu}(W, \mathrm{M}_{\square,e}(K_1, K_2))$ of mappings from W to $\mathrm{M}_{\square,e}(K_1, K_2)$ is an isomorphism from the $\square_{\mathrm{Me}}$–closed and $\varepsilon_{\mathrm{Me}}$–closed set $(\mathrm{M}_{\square,e}({K_1}^W, K_2), \square, \varepsilon)$ to the $\square_{\mathrm{Lu}}$–closed and e_{Lu}–closed set $(\mathrm{Lu}(W, \mathrm{M}_{\square,e}(K_1, K_2)), \square, e)$. Its inverse is $l \mapsto \mu_l$.

In particular, we have, for any measures μ and ν in $\mathrm{M}_{\square,e}({K_1}^W, K_2)$,

$$l_{\mu\,\square\,\nu} = l_\mu \mathbin{\square} l_\nu \quad\text{and}\quad l_\varepsilon = e,$$

and, for any l and m in $\mathrm{Lu}(W, \mathrm{M}_{\square,e}(K_1, K_2))$,

$$\mu_{l\,\square\,m} = \mu_l \mathbin{\square} \mu_m \quad\text{and}\quad \mu_e = \varepsilon.$$

2. Let $(C, \cdot)$ be a set equipped with a commutative internal binary operation. Let $(K_1, \square, \cdot, e_1)$ be a set equipped with an internal binary operation $\square$, an external binary operation $\cdot$ (with operand in C) and a nullary operation e_1, such that $(K_1, \square, e_1)$ is a commutative monoid. If $(K_2, \square, \cdot, e_2)$ is such that $(K_2, \square, e_2)$ is a commutative monoid and $\cdot$is an external binary operation (with operand in C) satisfying for any s and t in K_2, and any α and β in C,

$$\alpha\cdot e_2 = e_2, \quad \alpha\cdot(\beta\cdot t) = (\alpha\cdot\beta)\cdot t \quad\text{and}\quad \alpha\cdot(s \mathbin{\square} t) = \alpha\cdot s \mathbin{\square} \alpha\cdot t,$$

then the mapping $\mu \mapsto l_\mu$ from the $\square_{\mathrm{Me}}$–closed, $\cdot_{\mathrm{Me}}$–closed, and $\varepsilon_{\mathrm{Me}}$–closed set $\mathrm{M}_{\square,\cdot,e}({K_1}^W, K_2)$ to the $\square_{\mathrm{Lu}}$–closed, $\cdot_{\mathrm{Lu}}$–closed and e_{Lu}–closed set $\mathrm{Lu}(W, \mathrm{M}_{\square,\cdot,e}(K_1, K_2))$ of mappings from W to $\mathrm{M}_{\square,\cdot,e}(K_1, K_2)$ is an isomorphism from $(\mathrm{M}_{\square,\cdot,e}({K_1}^W, K_2), \square, \cdot, \varepsilon)$ to $(\mathrm{Lu}(W, \mathrm{M}_{\square,\cdot,e}(K_1, K_2)), \square, \cdot, e)$. Its inverse is $l \mapsto \mu_l$.

In particular, we have, for any measures μ in $\mathrm{M}_{\square,\cdot,e}({K_1}^W, K_2)$ and any α in C,

$$l_{\alpha\cdot\mu} = \alpha\cdot l_\mu,$$

and for any l in $\mathrm{Lu}(W, \mathrm{M}_{\square,\cdot,e}(K_1, K_2))$ and α in C,

$$\mu_{\alpha\cdot l} = \alpha\cdot\mu_l\,. \qquad \square$$

Proof – 1.1. By Propositions 3.67 and 3.69, the sets $\mathrm{M}_\square({K_1}^W, K_2)$ and $\mathrm{M}_e({K_1}^W, K_2)$ are $\square_{\mathrm{Me}}$–closed and $\varepsilon_{\mathrm{Me}}$–closed. Consequently, by Proposition 3.55 their intersection $(\mathrm{M}_{\square,e}({K_1}^W, K_2), \square, \varepsilon)$ is also $\square_{\mathrm{Me}}$–closed and $\varepsilon_{\mathrm{Me}}$–closed.

Let denote by $\mathrm{Lu}(W, \mathrm{M}_\square(K_1, K_2))$ the set of families of luts which are morphisms for $\square$ on K_1 and K_2, and let denote by $\mathrm{Lu}(W, \mathrm{M}_e(K_1, K_2))$ the set of families of luts which are morphisms for the nullary operations on K_1 and K_2. Since $\mathrm{M}_\square(K_1, K_2)$ and $\mathrm{M}_e(K_1, K_2)$ are $\square_{{K_2}^{K_1}}$–closed and $e_{{K_2}^{K_1}}$–closed, by Proposition 4.41, $\mathrm{Lu}(W, \mathrm{M}_\square(K_1, K_2))$, and $\mathrm{Lu}(W, \mathrm{M}_e(K_1, K_2))$ are $\square_{\mathrm{Lu}}$–closed and e_{Lu}–closed. Consequently, by Proposition 3.55 their intersection $(\mathrm{Lu}(W, \mathrm{M}_{\square,e}(K_1, K_2)), \square, e)$ is also $\square_{\mathrm{Lu}}$–closed and e_{Lu}–closed.

1.2. By Proposition 4.7, $\mu \mapsto l_\mu$ from $\mathrm{M}_{\square,e}({K_1}^W, K_2)$ to $\mathrm{Lu}(W, \mathrm{M}_{\square,e}(K_1, K_2))$ is a bijection, and by Proposition 4.8, $l \mapsto \mu_l$ is a morphism of the commutative monoids $(\mathrm{Lu}(W, {K_2}^{K_1}), \square, e)$ and $(\mathrm{Me}({K_1}^W, K_2), \square, \varepsilon)$, therefore, $l \mapsto \mu_l$ and its inverse $\mu \mapsto l_\mu$ are isomorphisms.

2.1. By Propositions 3.67 to 3.69, the sets $\mathrm{M}_\square({K_1}^W, K_2)$, $\mathrm{M}_\cdot({K_1}^W, K_2)$ and $\mathrm{M}_e({K_1}^W, K_2)$ are $\square_{\mathrm{Me}}$–closed, $\cdot_{\mathrm{Me}}$–closed, and $\varepsilon_{\mathrm{Me}}$–closed. Consequently, by Proposition 3.55 their intersection $(\mathrm{M}_{\square,\cdot,e}({K_1}^W, K_2), \square, \cdot, \varepsilon)$ is also $\square_{\mathrm{Me}}$–closed, $\cdot_{\mathrm{Me}}$–closed, and $\varepsilon_{\mathrm{Me}}$–closed.

Let denote by $\mathrm{Lu}(W, \mathrm{M}_\cdot(K_1, K_2))$ the set of families of luts which are morphisms for $\cdot$ on K_1 and K_2. Since $\mathrm{M}_\square(K_1, K_2)$, $\mathrm{M}_\cdot(K_1, K_2)$ and $\mathrm{M}_e(K_1, K_2)$ are $\square_{{K_2}^{K_1}}$–closed, $\cdot_{{K_2}^{K_1}}$ and $e_{{K_2}^{K_1}}$–closed, by Proposition 4.41, $\mathrm{Lu}(W, \mathrm{M}_\square(K_1, K_2))$, $\mathrm{Lu}(W, \mathrm{M}_\cdot(K_1, K_2))$ and $\mathrm{Lu}(W, \mathrm{M}_e(K_1, K_2))$ are $\square_{\mathrm{Lu}}$–closed, $\cdot_{\mathrm{Lu}}$–closed and e_{Lu}–closed. Consequently, by Proposition 3.55 their intersection $(\mathrm{Lu}(W, \mathrm{M}_{\square,\cdot,e}(K_1, K_2)), \square, \cdot, e)$ is also $\square_{{K_2}^{K_1}}$–closed, $\cdot_{{K_2}^{K_1}}$ and $e_{{K_2}^{K_1}}$–closed.

2.2. By Proposition 4.7, the mapping $\mu \mapsto l_\mu$ from $\mathrm{M}_{\square,\cdot,e}({K_1}^W, K_2)$ to the set $\mathrm{Lu}(W, \mathrm{M}_{\square,\cdot,e}(K_1, K_2))$ is a bijection, and by Proposition 4.8, $l \mapsto \mu_l$ is a morphism of $(\mathrm{Lu}(W, {K_2}^{K_1}), \square, \cdot, e)$ and $(\mathrm{Me}({K_1}^W, K_2), \square, \cdot, \varepsilon)$, therefore, $l \mapsto \mu_l$ and its inverse $\mu \mapsto l_\mu$ are isomorphisms. $\square$

We now specialize Proposition 4.9 to the cases of continuous and discrete gray–scales.

Let us consider first the continuous case.

For the measures, the addition $+$ on $\mathbf{R}$ can be extended to the set $\mathrm{Me}(\mathbf{R}^W, \mathbf{R})$ (see Section 2.4), resulting in an internal binary operation on the set of measures, that we call addition and we denote by $+$. In the same way, the multiplication on $\mathbf{R}$ can be extended to the set $\mathrm{Me}(\mathbf{R}^W, \mathbf{R})$, resulting in an external binary operation on the set of measures, with operand in $\mathbf{R}$, that we call multiplication by a scalar and we denote by $\cdot$.

The nullary operation 0 on $\mathbf{R}$ (that selects 0), can be extended to the set $\mathrm{Me}(\mathbf{R}^W, \mathbf{R})$, resulting in a nullary operation that selects the constant measure $g \mapsto 0$. We denote this measure by o.

As in the case of the real images of Chapter 2, the algebraic structure $(\mathrm{Me}(\mathbf{R}^W, \mathbf{R}), +, \cdot, o)$ consisting of the set of measures equipped with the above three operations is a linear vector space.

Furthermore, Propositions 3.67 and 3.68 apply to the set of linear measures, in other words, $\mathrm{M}_+(\mathbf{R}^W, \mathbf{R})$ and $\mathrm{M}_\cdot(\mathbf{R}^W, \mathbf{R})$ are $+_{\mathrm{Me}}$–closed and $\cdot_{\mathrm{Me}}$–closed. Consequently, by Propositions 3.55 and 3.53, $(\mathrm{L}(\mathbf{R}^W, \mathbf{R}), +, \cdot, o)$ is a sub space of $(\mathrm{Me}(\mathbf{R}^W, \mathbf{R}), +, \cdot, o)$.

For the families of luts, the addition $+$ on $\mathbf{R}^\mathbf{R}$ (see Section 3.1) can be extended to the set $\mathrm{Lu}(W, \mathbf{R}^\mathbf{R})$ (see Section 2.4), resulting in an internal binary operation on the set of families of luts, that we call addition and we denote by $+$. In the same way, the multiplication on $\mathbf{R}^\mathbf{R}$ can be extended to the set $\mathrm{Lu}(W, \mathbf{R}^\mathbf{R})$, resulting in an external binary operation on the set of families of luts, with operand in $\mathbf{R}$, that we call multiplication by a scalar and we denote by $\cdot$.

The nullary operation o on $\mathbf{R}^\mathbf{R}$ (that selects the constant lut $o : s \mapsto 0$), can be extended to the set $\mathrm{Lu}(W, \mathbf{R}^\mathbf{R})$, resulting in a nullary operation that selects the constant family $x \mapsto o$. We denote this family of luts by o too.

As in the case of the real images of Chapter 2, the algebraic structure $(\mathrm{Lu}(W, \mathbf{R}^\mathbf{R}), +, \cdot, o)$ consisting of the set of families of luts, equipped with the above three operations is a linear vector space.

Let denote by $\mathrm{Lu}(W, \mathrm{M}_+(\mathbf{R}, \mathbf{R}))$ the set of families of luts which are morphisms for the addition on $\mathbf{R}$, let denote by $\mathrm{Lu}(W, \mathrm{M}_\cdot(\mathbf{R}, \mathbf{R}))$ the set of families of real luts which are morphisms for the multiplication on $\mathbf{R}$, and let denote by $\mathrm{Lu}(W, \mathrm{M}_0(\mathbf{R}, \mathbf{R}))$ the set of families of real luts which are morphisms for the nullary operation on $\mathbf{R}$ which selects 0.

Since $\mathrm{M}_+(\mathbf{R}, \mathbf{R})$ and $\mathrm{M}_\cdot(\mathbf{R}, \mathbf{R})$ are $+_{\mathbf{R}^\mathbf{R}}$–closed, $\cdot_{\mathbf{R}^\mathbf{R}}$–closed and $o_{\mathbf{R}^\mathbf{R}}$–closed, by Proposition 4.41, $\mathrm{Lu}(W, \mathrm{M}_+(\mathbf{R}, \mathbf{R}))$ and $\mathrm{Lu}(W, \mathrm{M}_\cdot(\mathbf{R}, \mathbf{R}))$ are $+_{\mathrm{Lu}}$–closed and $\cdot_{\mathrm{Lu}}$–closed. Consequently, by Propositions 3.55 and 3.53, the set $(\mathrm{Lu}(W, \mathrm{L}(\mathbf{R}, \mathbf{R})), +, \cdot, o)$ of families of linear luts is a sub space of $(\mathrm{Lu}(W, \mathbf{R}^\mathbf{R}), +, \cdot, o)$.

By applying Proposition 4.9 to the continuous case, we can state the following result.

Proposition 4.10 (characterization of the linear measures) – Let W be a finite set, and let $(\mathbf{R}, +, \cdot, o)$ be the linear vector space of the real numbers. The mapping $\mu \mapsto l_\mu$ from the set $\mathrm{L}(\mathbf{R}^W, \mathbf{R})$ of linear measures to the set $\mathrm{Lu}(W, \mathrm{L}(\mathbf{R}, \mathbf{R}))$ of families of luts which are linear is an isomorphism of linear vector spaces. Its inverse is $l \mapsto \mu_l$. In particular, we have, for any measures μ and ν in $\mathrm{L}(\mathbf{R}^W, \mathbf{R})$, and α in $\mathbf{R}$,

$$l_{\mu+\nu} = l_\mu + l_\nu\,, \quad l_{\alpha.\mu} = \alpha \cdot l_\mu \quad \text{and} \quad l_o = o,$$

and, for any l and m in $\mathrm{Lu}(W, \mathrm{L}(\mathbf{R}, \mathbf{R}))$, and α in $\mathbf{R}$,

$$\mu_{l+m} = \mu_l + \mu_m\,, \quad \mu_{\alpha.l} = \alpha \cdot \mu_l \quad \text{and} \quad \mu_o = o.$$ □

Proposition 4.10 shows that any linear measure μ is a linear combination of the read–pixel measures μ_u when u runs over W. Furthermore, since the set of these read–pixel measures is linearly independent, these measures form a base for the linear vector space of the linear measures.

Exercise 4.11 (coordinates of a linear measure) – Let u be in W. Prove that the coordinate of a measure μ in $\mathrm{L}(\mathbf{R}^W, \mathbf{R})$ with respect to the read–pixel measure μ_u is $\mu(g_{u,1})$. □

Solution – For any μ in $\mathrm{L}(\mathbf{R}^W, \mathbf{R})$,

$$\begin{aligned}
\mu &= \sum_{u \in W} l_\mu(u) \circ \mu_u && \text{(Proposition 4.7)}\\
&= \sum_{u \in W} \alpha_{l_\mu(u)} \cdot \mu_u && \text{(Exercise 4.2)}\\
&= \sum_{u \in W} l_\mu(u)(1) \cdot \mu_u && (\alpha_l \text{ definition})\\
&= \sum_{u \in W} \mu(g_{u,1}) \cdot \mu_u. && (l_\mu \text{ definition})
\end{aligned}$$

□

Let us consider last the discrete case.

For the measures, assuming that $(K_1, \leq)$ and $(K_2, \leq)$ are two subposets of the poset of $\mathbf{Z}$ of natural numbers under the usual partial order, where $K_1 = [0, k_1] \subset \mathbf{Z}$ and $K_2 = [0, k_2] \subset \mathbf{Z}$, then, by Propositions 3.72, the poset $(\mathrm{Me}({K_1}^W, K_2), \leq)$ is a lattice and we can introduce the following four operations on the set $\mathrm{Me}({K_1}^W, K_2)$.

The *measure union*, denoted by $\vee$, is the union on $(\mathrm{Me}({K_1}^W, K_2), \leq)$, or equivalently, the extension to $\mathrm{Me}({K_1}^W, K_2)$ of the union $\vee$ on K_2.

The *measure intersection*, denoted by $\wedge$, is the intersection on $(\mathrm{Me}({K_1}^W, K_2), \leq)$, or equivalently, the extension to $\mathrm{Me}({K_1}^W, K_2)$ of the intersection $\wedge$ on K_2.

The *least measure selection*, denoted by o, is the extension to $\mathrm{Me}({K_1}^W, K_2)$ of the nullary operation on K_2 which selects the value 0. This operation selects the *least measure*, that is, the constant measure, $g \mapsto 0$. We denote by o this measure.

The *greatest measure selection*, denoted by ι, is the extension to $\mathrm{Me}({K_1}^W, K_2)$ of the nullary operation on K_2 which selects the value k_2. This operation selects the *greatest measure*, that is, the constant measure $g \mapsto k_2$. We denote by ι this measure.

In the lattice $(\mathrm{Me}({K_1}^W, K_2), \vee, \wedge)$, we can identify the two idempotent and commutative monoids $(\mathrm{Me}({K_1}^W, K_2), \vee, o)$ and $(\mathrm{Me}({K_1}^W, K_2), \wedge, \iota)$.

Furthermore, parts of Propositions 3.67 and 3.69 apply to the set of measures which are dilations, in other words, the sets $\mathrm{M}_\vee({K_1}^W, K_2)$ and $\mathrm{M}_{o0}({K_1}^W, K_2)$ are $\vee_{\mathrm{Me}}$–closed and o_{Me}–closed. Consequently, by Propositions 3.55 and 3.52, $(\mathrm{D}({K_1}^W, K_2), \vee, o)$ is a submonoid of $(\mathrm{Me}({K_1}^W, K_2), \vee, o)$.

Using the same arguments, $(\mathrm{E}^{\mathrm{a}}({K_1}^W, K_2), \vee, o)$ is a submonoid of $(\mathrm{Me}({K_1}^W, K_2), \vee, o)$, and $(\mathrm{E}({K_1}^W, K_2), \wedge, \iota)$ and $(\mathrm{D}^{\mathrm{a}}({K_1}^W, K_2), \wedge, \iota)$ are submonoids of $(\mathrm{Me}({K_1}^W, K_2), \wedge, \iota)$.

We must observe that $\mathrm{D}({K_1}^W, K_2)$ may not be $\wedge_{\mathrm{Me}}$–closed as shown is the next exercise.

Exercise 4.12 (counter example) – Let $W = \{1, 2\}$, $K = \{0, 1\}$, and l and m be two families of luts from K to K^W defined by, for any s in K,

$$l(1)(s) = s \quad \text{and} \quad l(2)(s) = 0$$

$$m(1)(s) = 0 \quad \text{and} \quad m(2)(s) = s.$$

Since $l(1)$, $l(2)$, $m(1)$ and $m(2)$ are dilations, μ_l and μ_m are dilations too (see Proposition 4.9).

Let g_1 and g_2 be two images from W to K defined by,

$$g_1(1) = 0 \quad \text{and} \quad g_1(2) = 1$$

$$g_2(1) = 1 \quad \text{and} \quad g_2(2) = 0.$$

Prove that $\mu_l \wedge \mu_m$ is not a dilation by proving that

$$(\mu_l \wedge \mu_m)(g_1 \vee g_2) \neq (\mu_l \wedge \mu_m)(g_1) \vee (\mu_l \wedge \mu_m)(g_2). \qquad \square$$

Solution – 1. For any image g from W to K,

$$\begin{aligned}
(\mu_l \wedge \mu_m)(g) &= \mu_l(g) \wedge \mu_m(g) && \text{(extension definition)}\\
&= (l(1)(g(1)) \vee l(2)(g(2))) \wedge (m(1)(g(1)) \vee m(2)(g(2))) && (\mu_l \text{ def. for dil.})\\
&= (g(1) \vee 0) \wedge (0 \vee g(2)) && (l \text{ and } m \text{ definitions})\\
&= g(1) \wedge g(2). && \text{(union and intersection properties)}
\end{aligned}$$

That is, for any g from W to K, $(\mu_l \wedge \mu_m)(g) = g(1) \wedge g(2)$.

2. On one hand,

$$\begin{aligned}(\mu_l \wedge \mu_m)(g_1 \vee g_2) &= (g_1 \vee g_2)(1) \wedge (g_1 \vee g_2)(2) && \text{(Part 1)}\\ &= (g_1(1) \vee g_2(1)) \wedge (g_1(2) \vee g_2(2)) && \text{(extension definition)}\\ &= (0 \vee 1) \wedge (1 \vee 0) && (g_1 \text{ and } g_2 \text{ definitions})\\ &= 1. && \text{(union and intersection properties)}\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mu_l \wedge \mu_m)(g_1) \vee (\mu_l \wedge \mu_m)(g_2) &= (g_1(1) \wedge g_1(2)) \vee (g_2(1) \wedge g_2(2)) && \text{(Part 1)}\\ &= (0 \wedge 1) \vee (1 \wedge 0) && (g_1 \text{ and } g_2 \text{ definitions})\\ &= 0. && \text{(union and intersection properties)}\end{aligned}$$

□

For the families of luts, assuming that $(K_1, \leq)$ and $(K_2, \leq)$ are two subposets of the poset of $\mathbf{Z}$ of natural numbers under the usual partial order, where $K_1 = [0, k_1] \subset \mathbf{Z}$ and $K_2 = [0, k_2] \subset \mathbf{Z}$, then, by Propositions 3.72, the poset $(\mathrm{Lu}(W, K_2{}^{K_1}), \leq)$ is a lattice and we can introduce the following four operations on the set $\mathrm{Lu}(W, K_2{}^{K_1})$.

The *union of families of luts*, denoted by $\vee$, is the union on $(\mathrm{Lu}(W, K_2{}^{K_1}), \leq)$, or equivalently, the extension to $\mathrm{Lu}(W, K_2{}^{K_1})$ of the union $\vee$ on $K_2{}^{K_1}$.

The *intersection of families of luts*, denoted by $\wedge$, is the intersection on $(\mathrm{Lu}(W, K_2{}^{K_1}), \leq)$, or equivalently, the extension to $\mathrm{Lu}(W, K_2{}^{K_1})$ of the intersection $\wedge$ on $K_2{}^{K_1}$.

The *least family of luts selection*, denoted by o, is the extension to $\mathrm{Lu}(W, K_2{}^{K_1})$ of the nullary operation on $K_2{}^{K_1}$ which selects the constant lut $s \mapsto 0$. This operation selects the *least family of luts*, that is, the constant family, $x \mapsto (g \mapsto 0)$. We denote by o this family.

The *greatest family of luts selection*, denoted by i, is the extension to $\mathrm{Lu}(W, K_2{}^{K_1})$ of the nullary operation on $K_2{}^{K_1}$ which selects the lut $s \mapsto k_2$. This operation selects the *greatest family of luts*, that is, the constant family $x \mapsto (g \mapsto k_2)$. We denote by i this family.

In the lattice $(\mathrm{Lu}(W, K_2{}^{K_1}), \vee, \wedge)$, we can identify the two idempotent and commutative monoids $(\mathrm{Lu}(W, K_2{}^{K_1}), \vee, o)$ and $(\mathrm{Lu}(W, K_2{}^{K_1}), \wedge, i)$.

Let denote by $\mathrm{Lu}(W, \mathrm{M}_\vee(K_1, K_2))$ the set of families of luts which are morphisms for the unions on K_1 and K_2, and let denote by $\mathrm{Lu}(W, \mathrm{M}_0(K_1, K_2))$ the set of families of luts which are morphisms for the nullary operations on K_1 and K_2 which select 0.

Since $\mathrm{M}_\vee(K_1, K_2)$ and $\mathrm{M}_0(K_1, K_2)$ are $\vee_{K_2{}^{K_1}}$–closed and $o_{K_2{}^{K_1}}$–closed, by Proposition 4.41, $\mathrm{Lu}(W, \mathrm{M}_\vee(K_1, K_2))$, and $\mathrm{Lu}(W, \mathrm{M}_0(K_1, K_2))$ are $\vee_{\mathrm{Lu}}$–closed and o_{Lu}–closed. Consequently, by Propositions 3.55 and 3.52, the set $\mathrm{Lu}(W, \mathrm{D}(K_1, K_2))$ of families of luts which are dilations is a submonoid of $(\mathrm{Lu}(W, K_2{}^{K_1}), \vee, o)$.

Using the same arguments, $\mathrm{Lu}(W, \mathrm{E}^{\mathrm{a}}(K_1, K_2))$ is a submonoid of $(\mathrm{Lu}(W, K_2{}^{K_1}), \vee, o)$, and $\mathrm{Lu}(W, \mathrm{E}(K_1, K_2))$ and $\mathrm{Lu}(W, \mathrm{D}^{\mathrm{a}}(K_1, K_2))$ are submonoids of $(\mathrm{Lu}(W, K_2{}^{K_1}), \wedge, i)$.

By applying Part 1 of Proposition 4.9 to the discrete case, we can state the following result.

Proposition 4.13 (characterization of the elementary morphological measures) – Let W be a finite set, and let $(K_1, \leq)$ and $(K_2, \leq)$ be, respectively, the intervals $[0, k_1]$ and $[0, k_2]$ of natural numbers equipped with the usual order relation.

1. The mapping $\mu \mapsto l_\mu$ from the set $\mathrm{D}({K_1}^W, K_2)$ (resp. $\mathrm{E^a}({K_1}^W, K_2)$) of measures which are dilations (resp. anti–erosions) to the set $\mathrm{Lu}(W, \mathrm{D}(K_1, K_2))$ of families of luts which are dilations is an isomorphism of monoids. Its inverse is $l \mapsto \mu_l$. In particular, we have, for any measures μ and ν in $\mathrm{D}({K_1}^W, K_2)$ (resp. $\mathrm{E^a}({K_1}^W, K_2)$),

$$l_{\mu \vee \nu} = l_\mu \vee l_\nu \quad \text{and} \quad l_o = o,$$

and, for any l and m in $\mathrm{Lu}(W, \mathrm{D}(K_1, K_2))$ (resp. $\mathrm{Lu}(W, \mathrm{E^a}(K_1, K_2))$),

$$\mu_{l \vee m} = \mu_l \vee \mu_m \quad \text{and} \quad \mu_o = o.$$

2. The mapping $\mu \mapsto l_\mu$ from the set $\mathrm{E}({K_1}^W, K_2)$ (resp. $\mathrm{D^a}({K_1}^W, K_2)$) of measures which are erosions (resp. anti–dilations) to the set $\mathrm{Lu}(W, \mathrm{E}(K_1, K_2))$ of families of luts which are erosions is an isomorphism of monoids. Its inverse is $l \mapsto \mu_l$. In particular, we have, for any measures μ and ν in $\mathrm{E}({K_1}^W, K_2)$ (resp. $\mathrm{D^a}({K_1}^W, K_2)$),

$$l_{\mu \wedge \nu} = l_\mu \wedge l_\nu \quad \text{and} \quad l_\iota = i,$$

and, for any l and m in $\mathrm{Lu}(W, \mathrm{E}(K_1, K_2))$ (resp. $\mathrm{Lu}(W, \mathrm{D^a}(K_1, K_2))$),

$$\mu_{l \wedge m} = \mu_l \wedge \mu_m \quad \text{and} \quad \mu_i = \iota. \qquad \square$$

Exercise 4.14 (dilation inequality) – Prove that for any l and m in $\mathrm{Lu}(W, \mathrm{D}(K_1, K_2))$,

$$\mu_{l \wedge m} \leq \mu_l \wedge \mu_m. \qquad \square$$

4.2 Spatially invariant window operator

Let K_1 and K_2 be two gray–scales (continuous or discrete), and let ${K_1}^{E_1}$ and ${K_2}^{E_2}$ be two sets of images whose domains are E_1 and E_2. In this section, we consider the image operators from ${K_1}^{E_1}$ to ${K_2}^{E_2}$. By assumption, the output images may have a different domain and a different gray–scale compared to the domain and gray–scale of the input images.

We denote by $\mathrm{Op}({K_1}^{E_1}, {K_2}^{E_2})$, or sometimes simply by Op, the set of operators from ${K_1}^{E_1}$ to ${K_2}^{E_2}$, and we usually denote an image operator by a capital Greek letter to distinguish from an image measure.

The set Op is usually very big and it is worthwhile to work with some smaller subsets (or classes) of operators. We now introduce two useful classes of image operators which could be seen as mutually dual. The need for two distinct classes comes from the fact that we are considering finite image domains. In the infinite domain case the two classes become the same class.

Definition 4.15 (window operator of type 1) – Let E_1 and E_2 be two non–empty sets. Let B be a mapping from E_2 to $\boldsymbol{P}(E_1)$, the collection of all parts of E_1. An operator Ψ from ${K_1}^{E_1}$ to ${K_2}^{E_2}$ is a *window operator of type 1* with respect to B, iff, for any y in E_2, and any f and g in ${K_1}^{E_1}$,

$$f/B(y) = g/B(y) \;\Rightarrow\; \Psi(f)(y) = \Psi(g)(y). \qquad \square$$

If X is a subset of the domain of an image f, then f/X is called the *subimage* of f restricted to X.

In other words, Ψ is a window operator of type 1 iff the pixel value $\Psi(f)(y)$ depends only on the subimage $f/B(y)$, or again, the input pixel values $f(x)$ outside $B(y)$ have no influence on the output pixel value $\Psi(f)(y)$.

Definition 4.16 (window operator of type 2) – Let E_1 and E_2 be two non–empty sets. Let A be a mapping from E_1 to $\boldsymbol{P}(E_2)$, the collection of all parts of E_2. An operator Φ from ${K_1}^{E_1}$ to ${K_2}^{E_2}$ is a *window operator of type 2* with respect to A, iff, for any x in E_1, and any f and g in ${K_1}^{E_1}$,

$$f/(E_1 - \{x\}) = g/(E_1 - \{x\}) \;\Rightarrow\; \Phi(f)/(E_2 - A(x)) = \Phi(g)/(E_2 - A(x)). \qquad \square$$

In other words, Φ is a window operator of type 2 iff the subimage $\Phi(f)/(E_2 - A(x))$ depends only on the subimage $f/(E_1 - \{x\})$, or again, the input pixel value $f(x)$ has no influence on the output subimage $\Phi(f)/(E_2 - A(x))$.

As we will see now, a window operator of type 1 can be characterized in terms of a family of measures. Nevertheless, in the general case, we do not know any characterization for the window operators of type 2. To get a characterization for type 2 we will have to restrict ourselves to the case of the window operators which are morphisms (see last part of Section 4.3).

Let Ψ be a window operator of type 1 (w.r.t. B) from ${K_1}^{E_1}$ to ${K_2}^{E_2}$. For any y in E_2, and g in ${K_1}^{B(y)}$, we denote by $\psi_\Psi(y)(g)$ the expression defined by

$$\psi_\Psi(y)(g) \triangleq \Psi(f)(y),$$

where f is any image of ${K_1}^{E_1}$ such that $f/B(y) = g$.

Let ψ be a mapping from E_2 to $M_B \triangleq \bigcup_{y \in E_2} \mathrm{Me}({K_1}^{B(y)}, K_2)$, such that, for any y in E_2, $\psi(y)$ is a mapping from ${K_1}^{B(y)}$ to K_2. For any f in ${K_1}^{E_1}$, and y in E_2, we denote by $\Psi_\psi(f)(y)$ the expression defined by

$$\Psi_\psi(f)(y) \triangleq \psi(y)(f/B(y)).$$

Proposition 4.17 (characterization of the window operators of type 1 in terms of families of measures) – The mapping $\Psi \mapsto \psi_\Psi$ from the set of window operators of type 1 (w.r.t. B) from ${K_1}^{E_1}$ to ${K_2}^{E_2}$, to the set of mappings from E_2 to M_B, such that, for any y in E_2, $\psi(y)$ is a mapping from ${K_1}^{B(y)}$ to K_2, is a bijection. Its inverse is $\psi \mapsto \Psi_\psi$. $\square$

Proof – 1. By definition, ψ_Ψ is a mapping from E_2 to M_B, such that, for any y in E_2, $\psi_\Psi(y)$ is a mapping from ${K_1}^{B(y)}$ to K_2.

2. By definition, Ψ_ψ is an operator from ${K_1}^{E_1}$ to ${K_2}^{E_2}$, furthermore, for any y in E_2, and f and g in ${K_1}^E$ such that $f/B(y) = g/B(y)$,

$$\begin{aligned} \Psi_\psi(f)(y) &= \psi(y)(f/B(y)) && (\Psi_\psi \text{ definition}) \\ &= \psi(y)(g/B(y)) && (\text{hypothesis}) \\ &= \Psi_\psi(g)(y), && (\Psi_\psi \text{ definition}) \end{aligned}$$

that is, Ψ_ψ is a window operator w.r.t. B.

3. Let Ψ be a window operator (w.r.t. B) from ${K_1}^{E_1}$ to ${K_2}^{E_2}$, for any y in E_2, and f in ${K_1}^{E_1}$,

$$\begin{aligned} \Psi_{\psi_\Psi}(f)(y) &= \psi_\Psi(y)(f/B(y)) && (\Psi_\psi \text{ definition}) \\ &= \Psi(f)(y), && (\psi_\Psi \text{ definition}) \end{aligned}$$

that is, $\Psi \mapsto \psi_\Psi$ is an injection.

4. Let ψ be a mapping from E_2 to M_B, such that, for any y in E_2, $\psi(y)$ is a mapping from ${K_1}^{B(y)}$ to K_2, for any y in E, g in ${K_1}^{B(y)}$, and f in ${K_1}^E$ such that $f/B(y) = g$,

$$\begin{aligned} \psi_{\Psi_\psi}(y)(g) &= \Psi_\psi(f)(y) && (\psi_\Psi \text{ definition}) \\ &= \psi(y)(f/B(y)) && (\Psi_\psi \text{ definition}) \\ &= \psi(y)(g), && (\text{hypothesis}) \end{aligned}$$

that is, $\Psi \mapsto \psi_\Psi$ is a surjection. □

Figure 4.5 shows a window operator of type 1 characterized by a certain mapping ψ.

Most of the useful window operators are spatially invariant. In order to introduce the concept of spatially invariant window operator of type 1 and 2, we consider that the image domains E_1 and E_2 are subsets of $\mathbf{Z}^2$, and we will use the properties of the Abelian group $(\mathbf{Z}^2, +, o)$.

Definition 4.18 (spatially invariant window operator of type 1) – Let E_1 and E_2 be two non–empty subsets of $\mathbf{Z}^2$. A window operator of type 1 (w.r.t. B) Ψ from ${K_1}^{E_1}$ to ${K_2}^{E_2}$ is *spatially invariant with window* W iff there exists a subset W of $\mathbf{Z}^2$ such that $E_2 = E_1 \ominus W$ and $B(y) = W + y$ for any y in E_2, and for any f in ${K_1}^{E_1}$ and any y_1 and y_2 in E_2,

$$f \circ b_{y_1} = f \circ b_{y_2} \Rightarrow \Psi(f)(y_1) = \Psi(f)(y_2),$$

where, for any y in E_2, the mapping b_y from W to E_1 is defined by, for any u in W,

$$b_y(u) \triangleq u + y.$$

We denote by $\mathrm{IW}_1({K_1}^{E_1}, {K_2}^{E_2})$ the set of spatially invariant window operators of type 1. □

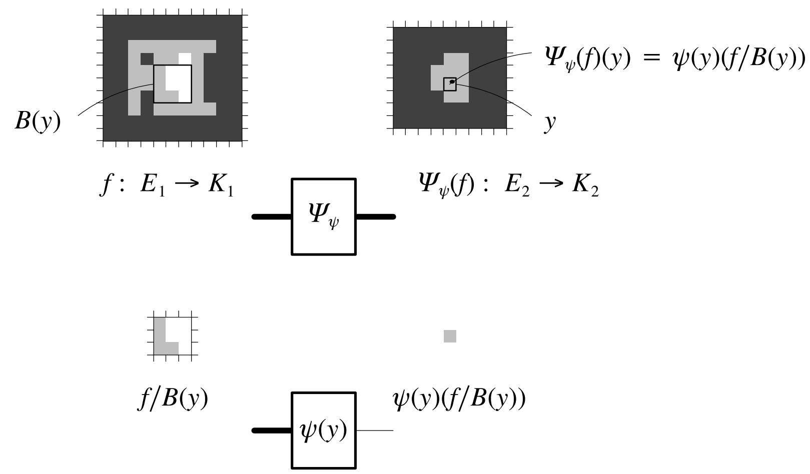


Fig. 4.5 – A window operator.

In other words, a window operator of type 1 is spatially invariant if the $B(y)$'s are translated version of a unique subset W and any two output pixel values $\Psi(f)(y_1)$ and $\Psi(f)(y_2)$ are equal if the two subimages $f/B(y_1)$ and $f/B(y_2)$ are equal up to a domain translation.

Definition 4.19 (spatially invariant window operator of type 2) – Let E_1 and E_2 be two non–empty subsets of $\mathbf{Z}^2$. A window operator of type 2 (w.r.t. A) Φ from ${K_1}^{E_1}$ to ${K_2}^{E_2}$ is *spatially invariant with window W* iff there exists a subset W of $\mathbf{Z}^2$ such that $E_2 = E_1 \oplus W$ and $A(x) = W + x$ for any x in E_1, and for any s_1 and s_2 in K_1, and any x_1 and x_2 in E_1,

$$(s_1 = s_2) \Rightarrow \Phi(f_{x_1, s_1}) \circ a_{x_1} = \Phi(f_{x_2, s_2}) \circ a_{x_2},$$

where, for any x in E_1, the mapping a_x from W to E_2 is defined by, for any u in W,

$$a_x(u) \triangleq u + x.$$

We denote by $\mathrm{IW}_2({K_1}^{E_1}, {K_2}^{E_2})$ the set of spatially invariant window operators of type 2. □

In other words, a window operator of type 2 is spatially invariant if the $A(x)$'s are translated version of a unique subset W and any two output subimages $\Phi(f)/A(x_1)$ and $\Phi(f)/A(x_2)$ are equal up to a domain translation if the two pulse images f_{x_1, s_1} and f_{x_2, s_2} have the same pulse intensity.

In the above definitions, $\oplus$ and $\ominus$ are, respectively, the Minkowski addition and subtraction (Banon & Barrera, 1998).

We can illustrate the two domain conditions stated in Definitions 4.18 and 4.19. Let m_1 and n_1 be two natural numbers and let v and w be two natural numbers smaller than, respectively, m_1 and n_1.

If $E_1 = \mathbf{m}_1 \times \mathbf{n}_1$ and W is the rectangular window defined by

$$W \triangleq [0, v] \times [0, w] \subset \mathbf{Z}^2,$$

then $E_2 = E_1 \ominus W$ is the Cartesian product $\mathbf{m}_2 \times \mathbf{n}_2$ with $m_2 = m_1 - v$ and $n_2 = n_1 - w$. The left part of Figure 4.6 shows a 3 by 3 window W and the two sets E_1 and E_2 satisfying the relationship $E_2 = E_1 \ominus W$.

Under the same assumption, $E_2 = E_1 \oplus W$ is the Cartesian product $\mathbf{m}_2 \times \mathbf{n}_2$ with $m_2 = m_1 + v$ and $n_2 = n_1 + w$. The right part of Figure 4.6 shows a 3 by 3 window W and the two sets E_1 and E_2 satisfying the relationship $E_2 = E_1 \oplus W$.

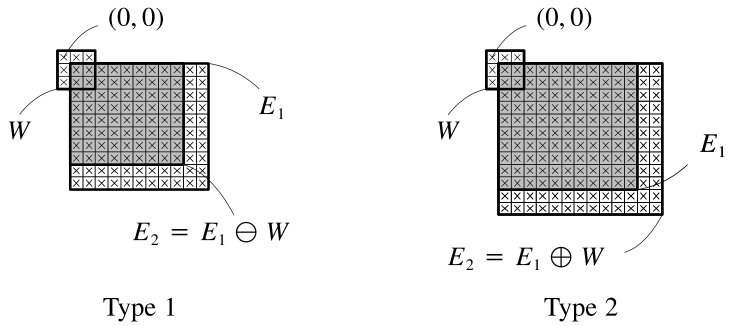


Fig. 4.6 – Window and image domains.

In the case of spatial invariance, we can simplify the characterization of the window operators of type 1.

Let Ψ be a spatially invariant window operator of type 1 (with window W) from ${K_1}^{E_1}$ to ${K_2}^{E_2}$. For any g in ${K_1}^W$, we denote by $\mu_\Psi(g)$ the expression defined by

$$\mu_\Psi(g) \triangleq \Psi(f)(y),$$

where y is any element of E_2, and f is any image of ${K_1}^{E_1}$ such that $f \circ b_y = g$, where b_y is the mapping of Definition 4.18.

Let μ be a measure from ${K_1}^W$ to K_2. For any f in ${K_1}^{E_1}$, and y in E_2, we denote by $\Psi_\mu(f)(y)$ the expression defined by

$$\Psi_\mu(f)(y) \triangleq \mu(f \circ b_y),$$

where b_y is the mapping of Definition 4.18.

Proposition 4.20 (characterization of the spatially invariant window operators of type 1 – characterization in terms of measures) – Let E_1, E_2 and W be three non–empty subsets of $\mathbf{Z}^2$ such that $E_2 = E_1 \ominus W$. The mapping $\Psi \mapsto \mu_\Psi$ from the set of spatially invariant window operators of type 1 (with window W) from ${K_1}^{E_1}$ to ${K_2}^{E_2}$, to the set of measures from ${K_1}^W$ to K_2 is a bijection. Its inverse is $\mu \mapsto \Psi_\mu$. □

Proof – 1. By definition, μ_Ψ is a mapping from ${K_1}^W$ to K_2.

2. By definition, Ψ_μ is an operator from ${K_1}^{E_1}$ to ${K_2}^{E_2}$. Let us show that is a spatially invariant window operator of type 1. First let us show that it is a window operator of type 1. For any y in E_2, and f and g in ${K_1}^{E_1}$ such that $f/(W + y) = g/(W + y)$, we have, for any u in W,

$$\begin{aligned}
(f \circ b_y)(u) &= f(b_y(u)) && \text{(composition definition)}\\
&= f(u + y) && (b_y \text{ definition})\\
&= g(u + y) && \text{(hypothesis)}\\
&= g(b_y(u)) && (b_y \text{ definition})\\
&= (g \circ b_y)(u), && \text{(composition definition)}
\end{aligned}$$

that is, $f/(W + y) = g/(W + y)$ implies $f \circ b_y = g \circ b_y$.

Hence, under the same assumptions we have,

$$\begin{aligned}
\Psi_\mu(f)(y) &= \mu(f \circ b_y) && (\Psi_\mu \text{ definition})\\
&= \mu(g \circ b_y) && \text{(hypothesis)}\\
&= \Psi_\mu(g)(y), && (\Psi_\mu \text{ definition})
\end{aligned}$$

that is, Ψ_μ is a window operator of type 1 w.r.t. $y \mapsto W + y$.

Let us show that Ψ_μ is spatially invariant. For any y_1 and y_2 in E_2 and f in ${K_1}^{E_1}$ such that $f \circ b_{y_1} = f \circ b_{y_2}$, we have,

$$\begin{aligned}
\Psi_\mu(f)(y_1) &= \mu(f \circ b_{y_1}) && (\Psi_\mu \text{ definition})\\
&= \mu(f \circ b_{y_2}) && \text{(hypothesis)}\\
&= \Psi_\mu(f)(y_2), && (\Psi_\mu \text{ definition})
\end{aligned}$$

that is, Ψ_μ is a spatially invariant window operator of type 1, with window W.

The rest of the proof is similar to Parts 3 and 4 of the proof of Proposition 4.17. □

Figure 4.7 shows two ways of representing a spatially invariant window operator of type 1.

Before ending this section, we will state two propositions that will be useful next section for the characterization of the spatially invariant operators of type 1 which are morphisms. One proposition is about morphism properties of the construction $\mu \mapsto \Psi_\mu$ and the second one is about closure properties of the set of spatially invariant window operators of type 1.

Our notations for extended operations on the set of operators are the same as in the previous chapter (see Section 3.2) except that the nullary operation is denoted by a capital Greek letter instead of a small Greek letter.

Fig. 4.7 – A spatially invariant window operator of type 1.

Proposition 4.21 (morphism from measures to operators) – Let K_1 be a non–empty set, and let E_1, E_2 and W be three non–empty subsets of $\mathbf{Z}^2$ such that $E_2 = E_1 \ominus W$.

1. If $(K_2, \square)$ is a set equipped with an internal binary operation, then the mapping $\mu \mapsto \Psi_\mu$ from the set $\mathrm{Me}({K_1}^W, K_2)$ to the set $\mathrm{Op}({K_1}^{E_1}, {K_2}^{E_2})$ is a morphism from $(\mathrm{Me}({K_1}^W, K_2), \square)$ to $(\mathrm{Op}({K_1}^{E_1}, {K_2}^{E_2}), \square)$. That is, we have, for any measures μ and ν from ${K_1}^W$ to K_2,

$$\Psi_{\mu \,\square\, \nu} = \Psi_\mu \,\square\, \Psi_\nu.$$

2. If $(K_2, \cdot)$ is a set equipped with an external binary operation with operand in a set C, then the above mapping is a morphism from $(\mathrm{Me}({K_1}^W, K_2), \cdot)$ to $(\mathrm{Op}({K_1}^{E_1}, {K_2}^{E_2}), \cdot)$. That is, we have, for any measures μ from ${K_1}^W$ to K_2, and any α in C,

$$\Psi_{\alpha \cdot \mu} = \alpha \cdot \Psi_\mu.$$

3. If (K_2, e_2) is a set equipped with a nullary operation, then the above mapping is a morphism from $(\mathrm{Me}({K_1}^W, K_2), \varepsilon)$ to $(\mathrm{Op}({K_1}^{E_1}, {K_2}^{E_2}), E)$. That is, we have,

$$\Psi_\varepsilon = E. \qquad \square$$

Proof – 1. For any mapping μ and ν from ${K_1}^W$ to K_2, any f in ${K_1}^{E_1}$, and any y in E_2,

$$\begin{aligned}
\Psi_{\mu \,\square\, \nu}(f)(y) &= (\mu \,\square\, \nu)(f \circ b_y) && (\Psi_\mu \text{ definition})\\
&= \mu(f \circ b_y) \,\square\, \nu(f \circ b_y) && (\text{extension definition})\\
&= \Psi_\mu(f)(y) \,\square\, \Psi_\nu(f)(y) && (\Psi_\mu \text{ definition})\\
&= (\Psi_\mu(f) \,\square\, \Psi_\nu(f))(y) && (\text{extension definition})\\
&= (\Psi_\mu \,\square\, \Psi_\nu)(f)(y), && (\text{extension definition})
\end{aligned}$$

that is, for any mapping μ and ν from ${K_1}^W$ to K_2, $\Psi_{\mu \,\square\, \nu} = \Psi_\mu \,\square\, \Psi_\nu$.

2. For any mapping μ from ${K_1}^W$ to K_2, any α in C, any f in ${K_1}^{E_1}$, and any y in E_2,

$$\begin{aligned}
\Psi_{\alpha.\mu}(f)(y) &= (\alpha.\mu)(f \circ b_y) && (\Psi_\mu \text{ definition})\\
&= \alpha.\mu(f \circ b_y) && (\text{extension definition})\\
&= \alpha.\Psi_\mu(f)(y) && (\Psi_\mu \text{ definition})\\
&= (\alpha.\Psi_\mu(f))(y) && (\text{extension definition})\\
&= (\alpha.\Psi_\mu)(f)(y), && (\text{extension definition})
\end{aligned}$$

that is, for any mapping μ from ${K_1}^W$ to K_2, and any α in C, $\Psi_{\alpha.\mu} = \alpha.\Psi_\mu$.

3. For any f in ${K_1}^{E_1}$, and any y in E_2,

$$\begin{aligned}
\Psi_\varepsilon(f)(y) &= \varepsilon(f \circ b_y) && (\Psi_\mu \text{ definition})\\
&= e_2 && (\text{extension def., } \varepsilon \text{ is a measure})\\
&= e(y) && (\text{extension def., } e \text{ is an image})\\
&= E(f)(y), && (\text{extension def., } E \text{ is an operator})
\end{aligned}$$

that is, $\Psi_\varepsilon = E$. □

Figure 4.8 illustrates graphically the equality $\Psi_{\mu \,\Box\, \nu} = \Psi_\mu \,\Box\, \Psi_\nu$ of Proposition 4.21. We observe that the right handside implementation is simpler than the left handside one.

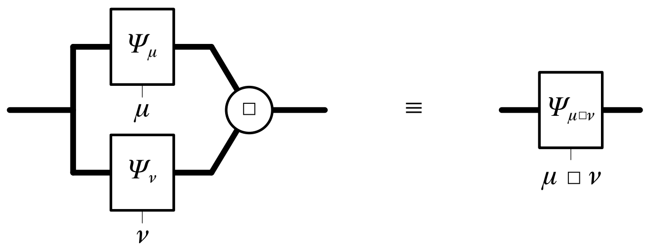


Fig. 4.8 – Morphism from measures to operators.

As a consequence of the previous propositions we have the following result.

Proposition 4.22 (closed subset of spatially invariant window operators of type 1) – Let K_1 be a non–empty set, and let E_1, E_2 and W be three non–empty subsets of $\mathbf{Z}^2$ such that $E_2 = E_1 \ominus W$.

1. If $(K_2, \Box)$ is a set equipped with an internal binary operation, then the subset $\mathrm{IW}_1({K_1}^{E_1}, {K_2}^{E_2})$ is $\Box_{\mathrm{Op}}$–closed.

2. If $(K_2, .)$ is a set equipped with an external binary operation with operand in a set C, then the subset $\mathrm{IW}_1({K_1}^{E_1}, {K_2}^{E_2})$ is $\cdot_{\mathrm{Op}}$–closed.

3. If (K_2, e_2) is a set equipped with a nullary operation with operand in a set C, then the subset $\mathrm{IW}_1({K_1}^{E_1}, {K_2}^{E_2})$ is E_{Op}–closed. □

Proof – By Proposition 4.20, the image of $\mathrm{Me}({K_1}^W, K_2)$ through $\mu \mapsto \Psi_\mu$ is $\mathrm{IW}_1({K_1}^{E_1}, {K_2}^{E_2})$.

1. By Part 1 of Proposition 4.21, $\mu \mapsto \Psi_\mu$ is a morphism from $(\mathrm{Me}({K_1}^W, K_2), \square)$ to $(\mathrm{Op}({K_1}^{E_1}, {K_2}^{E_2}), \square)$. Therefore, from Part 1 of Exercise 3.57, $\mathrm{IW}_1({K_1}^{E_1}, {K_2}^{E_2})$ is $\square_{\mathrm{Op}}$–closed.

2. By Part 2 of Proposition 4.21, $\mu \mapsto \Psi_\mu$ is a morphism from $(\mathrm{Me}({K_1}^W, K_2), \cdot)$ to $(\mathrm{Op}({K_1}^{E_1}, {K_2}^{E_2}), \cdot)$. Therefore, from Part 2 of Exercise 3.57, $\mathrm{IW}_1({K_1}^{E_1}, {K_2}^{E_2})$ is $\cdot_{\mathrm{Op}}$–closed.

3. By Proposition 4.21, $\Psi_\varepsilon = E$. This prove that E is in $\mathrm{IW}_1({K_1}^{E_1}, {K_2}^{E_2})$. □

4.3 Spatially invariant window morphism

In this section we will characterize the spatially invariant window operators which are morphisms. In the case of the window operators of type 1 we will present two characterizations, one in terms of measures and another one in terms of families of luts. In the case of type 2 we will present a characterization in terms of families of point spread functions.

Let us present the first the characterization in terms of measures for the spatially invariant window operators of type 1.

Let $(K_1, \square)$ and $(K_2, \square)$ be two sets equipped with internal binary operations. We denote by $\mathrm{IW}_1\mathrm{M}_\square({K_1}^{E_1}, {K_2}^{E_2})$, the set of spatially invariant window operators of type 1 which are morphisms for the extensions of the $\square$'s to ${K_1}^{E_1}$ and ${K_2}^{E_2}$.

In the same way, let $(K_1, \cdot)$ and $(K_2, \cdot)$ be two sets equipped with external binary operations with operand in C. We denote by $\mathrm{IW}_1\mathrm{M}_\cdot({K_1}^{E_1}, {K_2}^{E_2})$, the set of spatially invariant window operators of type 1 which are morphisms for the extensions of the $\cdot$'s to ${K_1}^{E_1}$ and ${K_2}^{E_2}$.

Finally, let (K_1, e_1) and (K_2, e_2) be two sets equipped with nullary operations. We denote by $\mathrm{IW}_1\mathrm{M}_e({K_1}^{E_1}, {K_2}^{E_2})$, the set of spatially invariant window operators of type 1 which are morphisms for the extensions of e_1 and e_2 to, respectively, ${K_1}^{E_1}$ and ${K_2}^{E_2}$.

Proposition 4.23 (characterization of the spatially invariant window operators of type 1 which are morphisms of one pair of operations – characterization in terms of measures) – Let E_1, E_2 and W be three non–empty subsets of $\mathbf{Z}^2$ such that $E_2 = E_1 \ominus W$.

1.1. Let $(K_1, \square)$ be a set equipped with an internal binary operation. If $(K_2, \square)$ is a commutative semigroup and if $(K_2, \cdot)$ is a set equipped with an external binary operation with operand in C which is distributive over $\square$, then the mapping $\Psi \mapsto \mu_\Psi$ is an isomorphism from $(\mathrm{IW}_1\mathrm{M}_\square({K_1}^{E_1}, {K_2}^{E_2}), \square, \cdot)$ to $(\mathrm{M}_\square({K_1}^W, K_2), \square, \cdot)$.

1.2. Let $(K_1, \square)$ be a set equipped with an internal binary operation. If $(K_2, \square, e_2)$ is a commutative monoid, then the mapping $\Psi \mapsto \mu_\Psi$ is an isomorphism from $(\mathrm{IW}_1\mathrm{M}_\square(K_1{}^{E_1}, K_2{}^{E_2}), \square, E)$ to $(\mathrm{M}_\square(K_1{}^W, K_2), \square, \varepsilon)$.

2. Let $(C, \cdot)$ be a set equipped with a commutative internal binary operation and let $(K_1, \cdot)$ and $(K_2, \cdot)$ be two sets equipped with external binary operations with operand in C. If $(K_2, \cdot)$ satisfies, for any t in K_2, and any α and β in C,

$$\alpha\cdot(\beta\cdot t) = (\alpha\cdot\beta)\cdot t, \qquad \text{(mixt associativity)}$$

and $(K_2, \square)$ is a set equipped with an internal binary operation such that $\cdot$ on K_2 is distributive over $\square$, then the mapping $\Psi \mapsto \mu_\Psi$ is an isomorphism from $(\mathrm{IW}_1\mathrm{M}_\cdot(K_1{}^{E_1}, K_2{}^{E_2}), \square, \cdot)$ to $(\mathrm{M}_\cdot(K_1{}^W, K_2), \square, \cdot)$.

3. Let (K_1, e_1) and (K_2, e_2) be two sets equipped with nullary operations. If $(K_2, \square, e_2)$ is a monoid, then the mapping $\Psi \mapsto \mu_\Psi$ is an isomorphism from $(\mathrm{IW}_1\mathrm{M}_e(K_1{}^{E_1}, K_2{}^{E_2}), \square, E)$ to $(\mathrm{M}_e(K_1{}^W, K_2), \square, \varepsilon)$. □

Proof – Let us divide the proof into four main steps.

1.1. Let Ψ be in $\mathrm{IW}_1\mathrm{M}_\square(K_1{}^{E_1}, K_2{}^{E_2})$. By definition, μ_Ψ is a mapping from $K_1{}^W$ to K_2. Let us prove that it is in $\mathrm{M}_\square(K_1{}^W, K_2)$. For any g_1 and g_2 in $K_1{}^W$, let y be any element of E_2, and let f_1 and f_2 be any images of $K_1{}^{E_1}$ such that $f_1 \circ b_y = g_1$ and $f_2 \circ b_y = g_2$, then, first, for any u in W,

$$\begin{aligned}
((f_1 \square f_2) \circ b_y)(u) &= (f_1 \square f_2)(b_y(u)) && \text{(composition definition)}\\
&= (f_1 \square f_2)(u + y) && (b_y \text{ definition})\\
&= f_1(u + y) \square f_2(u + y) && \text{(extension definition)}\\
&= f_1(b_y(u)) \square f_2(b_y(u)) && (b_y \text{ definition})\\
&= (f_1 \circ b_y)(u) \square (f_2 \circ b_y)(u) && \text{(composition definition)}\\
&= g_1(u) \square g_2(u) && \text{(hypothesis)}\\
&= (g_1 \square g_2)(u), && \text{(extension definition)}
\end{aligned}$$

that is, $(f_1 \square f_2) \circ b_y = g_1 \square g_2$. Second,

$$\begin{aligned}
\mu_\Psi(g_1 \square g_2) &= \Psi(f_1 \square f_2)(y) && \text{(first part of 1.1 and } \mu_\Psi \text{ def.)}\\
&= (\Psi(f_1) \square \Psi(f_2))(y) && (\Psi \text{ is a morphism})\\
&= \Psi(f_1)(y) \square \Psi(f_2)(y) && \text{(extension definition)}\\
&= \mu_\Psi(g_1) \square \mu_\Psi(g_2). && (\mu_\Psi \text{ definition})
\end{aligned}$$

1.2. Let Ψ be in $\mathrm{IW}_1\mathrm{M}_\cdot(K_1{}^{E_1}, K_2{}^{E_2})$. By definition, μ_Ψ is a mapping from $K_1{}^W$ to K_2. Let us prove that it is in $\mathrm{M}_\cdot(K_1{}^W, K_2)$. For any g in $K_1{}^W$, and any α in C, let y be any element of E_2, and let f be any image of $K_1{}^{E_1}$ such that $f \circ b_y = g$, then, first, for any u in W,

$$
\begin{aligned}
((\alpha.f) \circ b_y)(u) &= (\alpha.f)(b_y(u)) && \text{(composition definition)}\\
&= (\alpha.f)(u + y) && (b_y \text{ definition})\\
&= \alpha.f(u + y) && \text{(extension definition)}\\
&= \alpha.f(b_y(u)) && (b_y \text{ definition})\\
&= \alpha\cdot(f \circ b_y)(u) && \text{(composition. definition)}\\
&= \alpha\cdot g(u) && \text{(hypothesis)}\\
&= (\alpha\cdot g)(u), && \text{(extension definition)}
\end{aligned}
$$

that is, $(\alpha.f) \circ b_y = \alpha\cdot g$.

Second,

$$
\begin{aligned}
\mu_\Psi(\alpha\cdot g) &= \Psi(\alpha.f)(y) && \text{(first part of 2.1 and } \mu_\Psi \text{ def.)}\\
&= (\alpha\cdot\Psi(f))(y) && (\Psi \text{ is a morphism})\\
&= \alpha\cdot\Psi(f)(y) && \text{(extension definition)}\\
&= \alpha.\mu_\Psi(g). && (\mu_\Psi \text{ definition})
\end{aligned}
$$

1.3. First, let g_e be the unit element of $\square$ on $K_1{}^W$, and let f_e be the unit element of $\square$ on $K_1{}^{E_1}$, then, for any y in E_2 and u in W,

$$
\begin{aligned}
(f_e \circ b_y)(u) &= f_e(b_y(u)) && \text{(composition definition)}\\
&= f_e(u + y) && (b_y \text{ definition})\\
&= e_1 && \text{(extension definition)}\\
&= g_e(u), && \text{(extension definition)}
\end{aligned}
$$

that is, for any y in E_2, $f_e \circ b_y = g_e$. Second,

$$
\begin{aligned}
\mu_\Psi(g_e) &= \Psi(f_e)(y) && \text{(first part of 1.3 and } \mu_\Psi \text{ definition)}\\
&= e_2(y) && (f_1 \text{ is a unit and } \Psi \text{ is a morphism})\\
&= e_2. && \text{(extension definition)}
\end{aligned}
$$

2.1. Let μ be in $M_\square(K_1{}^W, K_2)$. By definition, Ψ_μ is an operator from $K_1{}^E$ to $K_2{}^E$. Let us prove that it is in $IW_1M_\square(K_1{}^{E_1}, K_2{}^{E_2})$. For any images f_1 and f_2 of $K_1{}^E$, and any y in E_2,

$$
\begin{aligned}
\Psi_\mu(f_1 \square f_2)(y) &= \mu((f_1 \square f_2) \circ b_y) && (\Psi_\mu \text{ definition})\\
&= \mu(f_1 \circ b_y \square f_2 \circ b_y)\text{(first part of 1.1)}\\
&= \mu(f_1 \circ b_y) \square \mu(f_2 \circ b_y) && (\mu \text{ is a morphism})\\
&= \Psi_\mu(f_1)(y) \square \Psi_\mu(f_2)(y) && (\Psi_\mu \text{ definition})\\
&= (\Psi_\mu(f_1) \square \Psi_\mu(f_2))(y). && \text{(extension definition)}
\end{aligned}
$$

2.2. Let μ be in $\mathrm{M}_{\cdot}({K_1}^W, K_2)$. By definition, Ψ_μ is an operator from ${K_1}^E$ to ${K_2}^E$. Let us prove that it is in $\mathrm{IW}_1\mathrm{M}_{\cdot}({K_1}^{E_1}, {K_2}^{E_2})$. For any images f of ${K_1}^E$, any α in C, and any y in E_2,

$$
\begin{aligned}
\Psi_\mu(\alpha{\cdot}f)(y) &= \mu((\alpha{\cdot}f) \circ b_y) \quad (\Psi_\mu \text{ definition}) \\
&= \mu(\alpha{\cdot}f \circ b_y) && \text{(first part of 1.2)} \\
&= \alpha{\cdot}\mu(f \circ b_y) && (\mu \text{ is a morphism}) \\
&= \alpha{\cdot}\Psi_\mu(f)(y) && (\Psi_\mu \text{ definition}) \\
&= (\alpha{\cdot}\Psi_\mu(f))(y). && \text{(extension definition)}
\end{aligned}
$$

2.3. Let g_1 be the unit element of $\square$ on ${K_1}^W$, and let f_1 be the unit element of $\square$ on ${K_1}^{E_1}$, then, for any y in E_2,

$$
\begin{aligned}
\Psi_\mu(f_1)(y) &= \mu(f_1 \circ b_y) && (\Psi_\mu \text{ definition}) \\
&= \mu(g_1) && \text{(first part of 1.2)} \\
&= e_2 && (\mu \text{ is a morphism}) \\
&= e_2(y). && \text{(extension definition)}
\end{aligned}
$$

3.1. From Parts 1.1 and 2.1 above, and by Exercise 1.26 and Proposition 4.20, $\Psi \mapsto \mu_\Psi$ from $\mathrm{IW}_1\mathrm{M}_\square({K_1}^{E_1}, {K_2}^{E_2})$ to $\mathrm{M}_\square({K_1}^W, K_2)$ is a bijection.

3.2. From Parts 1.2 and 2.2 above, and by Exercise 1.26 and Proposition 4.20, $\Psi \mapsto \mu_\Psi$ from $\mathrm{IW}_1\mathrm{M}_{\cdot}({K_1}^{E_1}, {K_2}^{E_2})$ to $\mathrm{M}_{\cdot}({K_1}^W, K_2)$ is a bijection.

3.3. From Parts 1.3 and 2.3 above, and by Exercise 1.26 and Proposition 4.20, $\Psi \mapsto \mu_\Psi$ from $\mathrm{IW}_1\mathrm{M}_e({K_1}^{E_1}, {K_2}^{E_2})$ to $\mathrm{M}_e({K_1}^W, K_2)$ is a bijection.

4.1.1. By Propositions 4.22 and 3.67, the subsets $\mathrm{IW}_1({K_1}^{E_1}, {K_2}^{E_2})$ and $\mathrm{M}_\square({K_1}^{E_1}, {K_2}^{E_2})$ are, respectively, both $\square_{\mathrm{Op}}$–closed and $\cdot_{\mathrm{Op}}$–closed. Consequently, by Proposition 3.55 their intersection $\mathrm{IW}_1\mathrm{M}_\square({K_1}^{E_1}, {K_2}^{E_2})$ is $\square_{\mathrm{Op}}$–closed and $\cdot_{\mathrm{Op}}$–closed. The set $\mathrm{M}_\square({K_1}^W, K_2)$ is $\square_{\mathrm{Me}}$–closed and $\cdot_{\mathrm{Me}}$–closed by Proposition 3.67. Finally, by Proposition 4.21, $\mu \mapsto \Psi_\mu$ is a morphism from $(\mathrm{Me}({K_1}^W, K_2), \square, \cdot)$ to $(\mathrm{Op}({K_1}^{E_1}, {K_2}^{E_2}), \square, \cdot)$, therefore, by Part 3.1, $\mu \mapsto \Psi_\mu$ from $\mathrm{M}_\square({K_1}^W, K_2)$ to $\mathrm{IW}_1\mathrm{M}_\square({K_1}^{E_1}, {K_2}^{E_2})$ is an isomorphism from $(\mathrm{M}_\square({K_1}^W, K_2), \square, \cdot)$ to $(\mathrm{IW}_1\mathrm{M}_\square({K_1}^{E_1}, {K_2}^{E_2}), \square, \cdot)$, that is, its inverse $\Psi \mapsto \mu_\Psi$ is an isomorphism from $(\mathrm{IW}_1\mathrm{M}_\square({K_1}^{E_1}, {K_2}^{E_2}), \square, \cdot)$ to $(\mathrm{M}_\square({K_1}^W, K_2), \square, \cdot)$.

4.1.2. By Propositions 4.22 and 3.67, the subsets $\mathrm{IW}_1(K_1{}^{E_1}, K_2{}^{E_2})$ and $\mathrm{M}_\square(K_1{}^{E_1}, K_2{}^{E_2})$ are, respectively, both $\square_{\mathrm{Op}}$–closed and E_{Op}–closed. Consequently, by Proposition 3.55 their intersection $\mathrm{IW}_1\mathrm{M}_\square(K_1{}^{E_1}, K_2{}^{E_2})$ is $\square_{\mathrm{Op}}$–closed and E_{Op}–closed. The set $\mathrm{M}_\square(K_1{}^{W}, K_2)$ is $\square_{\mathrm{Me}}$–closed and $\varepsilon_{\mathrm{Me}}$–closed by Proposition 3.67. Finally, by Proposition 4.21, $\mu \mapsto \Psi_\mu$ is a morphism from $(K_2{}^{K_1}, \square, e)$ to $(\mathrm{Op}(K_1{}^{E}, K_2{}^{E}), \square, \varepsilon)$, therefore, by Part 3.1, $l \mapsto \psi_l$ from $\mathrm{M}_\square(K_1{}^{W}, K_2)$ to $\mathrm{IW}_1\mathrm{M}_\square(K_1{}^{E_1}, K_2{}^{E_2})$ is an isomorphism from $(\mathrm{M}_\square(K_1{}^{W}, K_2), \square, \varepsilon)$ to $(\mathrm{IW}_1\mathrm{M}_\square(K_1{}^{E_1}, K_2{}^{E_2}), \square, E)$, that is, its inverse $\Psi \mapsto \mu_\Psi$ is an isomorphism from $(\mathrm{IW}_1\mathrm{M}_\square(K_1{}^{E_1}, K_2{}^{E_2}), \square, E)$ to $(\mathrm{M}_\square(K_1{}^{W}, K_2), \square, \varepsilon)$.

4.2. By Propositions 4.22 and 3.68, the subsets $\mathrm{IW}_1(K_1{}^{E_1}, K_2{}^{E_2})$ and $\mathrm{M}_\cdot(K_1{}^{E_1}, K_2{}^{E_2})$ are, respectively, both $\square_{\mathrm{Op}}$–closed and $\cdot_{\mathrm{Op}}$–closed. Consequently, by Proposition 3.55 their intersection $\mathrm{IW}_1\mathrm{M}_\cdot(K_1{}^{E_1}, K_2{}^{E_2})$ is $\square_{\mathrm{Op}}$–closed and $\cdot_{\mathrm{Op}}$–closed. The set $\mathrm{M}_\cdot(K_1{}^{W}, K_2)$ is $\square_{\mathrm{Me}}$–closed and $\cdot_{\mathrm{Me}}$–closed by Proposition 3.68. Finally, by Proposition 4.21, $\mu \mapsto \Psi_\mu$ is a morphism from $(\mathrm{Me}(K_1{}^{W}, K_2), \square, \cdot)$ to $(\mathrm{Op}(K_1{}^{E_1}, K_2{}^{E_2}), \square, \cdot)$, therefore, by Part 3.2, $\mu \mapsto \Psi_\mu$ is an isomorphism from $(\mathrm{M}_\cdot(K_1{}^{W}, K_2), \square, \cdot)$ to $(\mathrm{IW}_1\mathrm{M}_\cdot(K_1{}^{E_1}, K_2{}^{E_2}), \square, \cdot)$, that is, its inverse $\Psi \mapsto \mu_\Psi$ is an isomorphism from $(\mathrm{IW}_1\mathrm{M}_\cdot(K_1{}^{E_1}, K_2{}^{E_2}), \square, \cdot)$ to $(\mathrm{M}_\cdot(K_1{}^{W}, K_2), \square, \cdot)$.

4.3. By Propositions 4.22 and 3.69, the subsets $\mathrm{IW}_1(K_1{}^{E_1}, K_2{}^{E_2})$ and $\mathrm{M}_\cdot(K_1{}^{E_1}, K_2{}^{E_2})$ are, respectively, both $\square_{\mathrm{Op}}$–closed and E_{Op}–closed. Consequently, by Proposition 3.55 their intersection $\mathrm{IW}_1\mathrm{M}_e(K_1{}^{E_1}, K_2{}^{E_2})$ is $\square_{\mathrm{Op}}$–closed and $\cdot_{\mathrm{Op}}$–closed. The set $\mathrm{M}_e(K_1{}^{W}, K_2)$ is $\square_{\mathrm{Me}}$–closed and $\varepsilon_{\mathrm{Me}}$–closed by Proposition 3.69. Finally, by Proposition 4.21, $\mu \mapsto \Psi_\mu$ is a morphism from $(\mathrm{Me}(K_1{}^{W}, K_2), \square, \varepsilon)$ to $(\mathrm{Op}(K_1{}^{E_1}, K_2{}^{E_2}), \square, E)$, therefore, by Part 3.2, $\mu \mapsto \Psi_\mu$ is an isomorphism from $(\mathrm{M}_e(K_1{}^{W}, K_2), \square, \varepsilon)$ to $(\mathrm{IW}_1\mathrm{M}_e(K_1{}^{E_1}, K_2{}^{E_2}), \square, E)$, that is, its inverse $\Psi \mapsto \mu_\Psi$ is an isomorphism from $(\mathrm{IW}_1\mathrm{M}_e(K_1{}^{E_1}, K_2{}^{E_2}), \square, E)$ to $(\mathrm{M}_{e\cdot}(K_1{}^{W}, K_2), \square, \varepsilon)$. □

Figure 4.9 shows the isomorphism $\Psi \mapsto \mu_\Psi$ of Proposition 4.23 for the internal binary operations on the set of operators and on the set of measures.

Figure 4.10 illustrates graphically an aspect of Part 1.2 of Proposition 4.23. From Part 1.2 we know that if μ is a measure which is a morphism of internal binary operations on $K_1{}^{W}$ and K_2, then the spatially invariant window operator of type 1 constructed with this measure is also a morphism of internal binary operations on $K_1{}^{E}$ and $K_2{}^{E}$. This means that we have the two equivalent implementations shown in Figure 4.10. We observe that the right handside implementation is simpler than the left handside one.

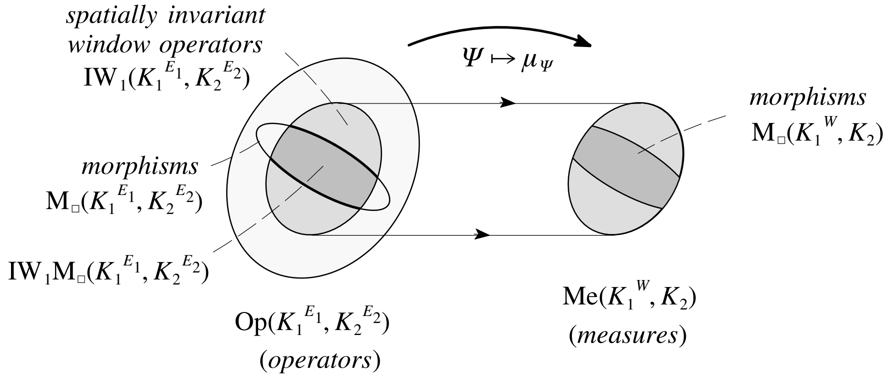


Fig. 4.9 – Isomorphism.

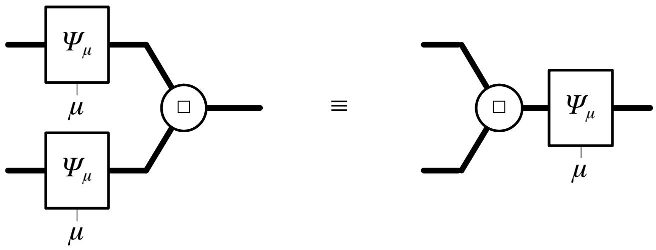


Fig. 4.10 – Two equivalent implementations when the measure is a morphism.

Recalling (see Part 1 of Proposition 3.67) that if the measures μ and ν are two morphisms for the internal binary operations $\square$'s on $K_1{}^W$ and K_2 then $\mu \,\square\, \nu$ is also a morphism, we can combine the equivalence of Figure 4.10 with the one shown in Figure 4.8, to get the equivalence between the two implementations shown in Figure 4.11. We observe that the right handside implementations is much simpler than the left handside one.

What will be useful is to combine, in one hand, the results of Parts 1.2 and 3 of Proposition 4.23 and, on the other hand the results of Parts 1.1 and 2 of that proposition. This leads to the following proposition.

Proposition 4.24 (characterization of the spatially invariant window operators of type 1 which are morphisms of more than one pair of operations – characterization in terms of measures) – Let E_1, E_2 and W be three non–empty subsets of $\mathbf{Z}^2$ such that $E_2 = E_1 \ominus W$.

1. Let $(K_1, \square, e_1)$ be a set equipped with an internal binary operation $\square$ and a nullary operation e_1. If $(K_2, \square, e_2)$ is a commutative monoid, then the mapping $\Psi \mapsto \mu_\Psi$ is an isomorphism from $(\mathrm{IW}_1\mathrm{M}_{\square,e}(K_1{}^{E_1}, K_2{}^{E_2}), \square, E)$ to $(\mathrm{M}_{\square,e}(K_1{}^W, K_2), \square, \varepsilon)$. Its inverse is $\mu \mapsto \Psi_\mu$.

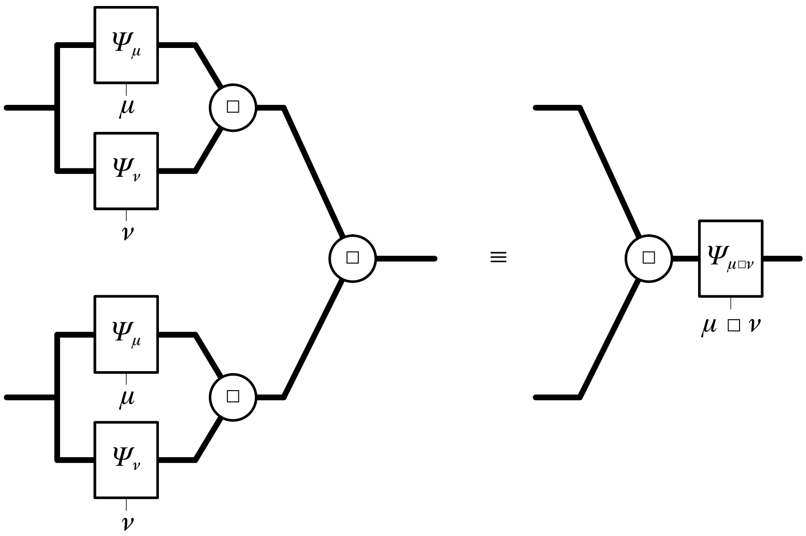


Fig. 4.11 – Two equivalent implementations when the measures are morphisms.

In particular, we have, for any Ψ and Φ in $\mathrm{IW}_1\mathrm{M}_{\Box,e}({K_1}^{E_1}, {K_2}^{E_2})$,

$$\mu_{\Psi \Box \Phi} = \mu_\Psi \Box \mu_\Phi \quad \text{and} \quad \mu_E = \varepsilon,$$

and, for any μ and ν in $\mathrm{M}_{\Box,e}({K_1}^W, K_2)$,

$$\Psi_{\mu \Box \nu} = \Psi_\mu \Box \Psi_\nu \quad \text{and} \quad \Psi_\varepsilon = E.$$

2. Let $(C, \cdot)$ be a set equipped with a commutative internal binary operation. Let $(K_1, \Box, \cdot, e_1)$ be a set equipped with an internal binary operation $\Box$, an external binary operation $\cdot$ and a nullary operation e_1. If $(K_2, \Box, \cdot, e_2)$ is such that $(K_2, \Box, e_2)$ is a commutative monoid and $\cdot$is an external binary operation (with operand in C) satisfying for any s and t in K_2, and any α and β in C,

$$\alpha\cdot e_2 = e_2, \quad \alpha\cdot(\beta\cdot t) = (\alpha\cdot\beta)\cdot t \quad \text{and} \quad \alpha\cdot(s \Box t) = \alpha\cdot s \Box \alpha\cdot t,$$

then the mapping $\Psi \mapsto \mu_\Psi$ is an isomorphism from $(\mathrm{IW}_1\mathrm{M}_{\Box,\cdot,e}({K_1}^{E_1}, {K_2}^{E_2}), \Box, \cdot, E)$ to $(\mathrm{M}_{\Box,\cdot,e}({K_1}^W, K_2), \Box, \cdot, \varepsilon)$. Its inverse is $\mu \mapsto \Psi_\mu$.

In particular, we have, for any Ψ in $\mathrm{IW}_1\mathrm{M}_{\Box,\cdot,e}({K_1}^{E_1}, {K_2}^{E_2})$, and α in C,

$$\mu_{\alpha.\Psi} = \alpha\cdot\mu_\Psi,$$

and, for any μ in $\mathrm{M}_{\Box,\cdot,e}({K_1}^W, K_2)$, and α in C,

$$\Psi_{\alpha\cdot\mu} = \alpha\cdot\Psi_\mu. \qquad \Box$$

Proof – We divide each part of the proof in four steps.

1.1. By Proposition 4.23, the subsets $\mathrm{IW}_1\mathrm{M}_\square(K_1{}^{E_1}, K_2{}^{E_2})$ and $\mathrm{IW}_1\mathrm{M}_e(K_1{}^{E_1}, K_2{}^{E_2})$ are $\square_{\mathrm{Op}}$–closed and E_{Op}–closed. Consequently, by Proposition 3.55 their intersection $\mathrm{IW}_1\mathrm{M}_{\square,e}(K_1{}^{E_1}, K_2{}^{E_2})$ is also $\square_{\mathrm{Op}}$–closed and E_{Op}–closed.

1.2. By Part 1 of Proposition 4.9, $\mathrm{M}_{\square,e}(K_1{}^W, K_2)$ is $\square_{\mathrm{Me}}$–closed and $\varepsilon_{\mathrm{Me}}$–closed.

1.3. From Proposition 4.20, $\Psi \mapsto \mu_\Psi$ is a bijection and from Proposition 4.23, $\mathrm{M}_\square(K_1{}^W, K_2)$ (resp. $\mathrm{M}_e(K_1{}^W, K_2)$) is the image of $\mathrm{IW}_1\mathrm{M}_\square(K_1{}^{E_1}, K_2{}^{E_2})$ (resp. $\mathrm{IW}_1\mathrm{M}_e(K_1{}^{E_1}, K_2{}^{E_2})$) through $\Psi \mapsto \mu_\Psi$, therefore, by Exercise 1.30, $\mathrm{M}_{\square,e}(K_1{}^W, K_2)$ is the image of $\mathrm{IW}_1\mathrm{M}_{\square,e}(K_1{}^{E_1}, K_2{}^{E_2})$ through $\Psi \mapsto \mu_\Psi$.

1.4. By Part 1.2 of Proposition 4.23, $\Psi \mapsto \mu_\Psi$ is an isomorphism from $(\mathrm{IW}_1\mathrm{M}_\square(K_1{}^{E_1}, K_2{}^{E_2}), \square, E)$ to $(\mathrm{M}_\square(K_1{}^W, K_2), \square, \varepsilon)$. As we see above in Part 1.1, $\mathrm{IW}_1\mathrm{M}_{\square,e}(K_1{}^{E_1}, K_2{}^{E_2})$ is $\square_{\mathrm{Op}}$–closed and E_{Op}–closed, therefore by Exercise 3.58 and Part 1.3 above, $\Psi \mapsto \mu_\Psi$ restricted to $\mathrm{IW}_1\mathrm{M}_{\square,e}(K_1{}^{E_1}, K_2{}^{E_2})$ and $\mathrm{M}_{\square,e}(K_1{}^W, K_2)$ is an isomorphism from $(\mathrm{IW}_1\mathrm{M}_{\square,e}(K_1{}^{E_1}, K_2{}^{E_2}), \square, E)$ to $(\mathrm{M}_{\square,e}(K_1{}^W, K_2), \square, \varepsilon)$.

2.1. By Proposition 4.23, the subsets $\mathrm{IW}_1\mathrm{M}_\square(K_1{}^{E_1}, K_2{}^{E_2})$ and $\mathrm{IW}_1\mathrm{M}_e(K_1{}^{E_1}, K_2{}^{E_2})$ are $\cdot_{\mathrm{Op}}$–closed. Consequently, by Proposition 3.55 their intersection $\mathrm{IW}_1\mathrm{M}_{\square,e}(K_1{}^{E_1}, K_2{}^{E_2})$ is also $\cdot_{\mathrm{Op}}$–closed. Proposition 3.68 applies to the set of operators as well, in other words, $\mathrm{IW}_1\mathrm{M}_{\cdot}(K_1{}^{E_1}, K_2{}^{E_2})$ is $\square_{\mathrm{Op}}$–closed, $\cdot_{\mathrm{Op}}$–closed and E_{Op}–closed. Consequently, considering Part 1.2 above, by Proposition 3.55 the intersection $\mathrm{IW}_1\mathrm{M}_{\square,\cdot,e}(K_1{}^{E_1}, K_2{}^{E_2})$ of $\mathrm{IW}_1\mathrm{M}_{\square,e}(K_1{}^{E_1}, K_2{}^{E_2})$ and $\mathrm{IW}_1\mathrm{M}_{\cdot}(K_1{}^{E_1}, K_2{}^{E_2})$ is $\square_{\mathrm{Op}}$–closed, $\cdot_{\mathrm{Op}}$–closed and E_{Op}–closed.

2.2. By Part 2 of Proposition 4.9, $\mathrm{M}_{\square,\cdot,e}(K_1{}^W, K_2)$ is $\square_{\mathrm{Me}}$–closed, $\cdot_{\mathrm{Me}}$–closed, and $\varepsilon_{\mathrm{Me}}$–closed set .

2.3. From Proposition 4.20, $\Psi \mapsto \mu_\Psi$ is a bijection and from Proposition 4.23, $\mathrm{M}_{\cdot}(K_1{}^W, K_2)$ is the image of $\mathrm{IW}_1\mathrm{M}_\square(K_1{}^{E_1}, K_2{}^{E_2})$ through $\Psi \mapsto \mu_\Psi$, therefore, considering Part 1.3 above, by Exercise 1.30, $\mathrm{M}_{\square,\cdot,e}(K_1{}^W, K_2)$ is the image of $\mathrm{IW}_1\mathrm{M}_{\square,\cdot,e}(K_1{}^{E_1}, K_2{}^{E_2})$ through $\Psi \mapsto \mu_\Psi$.

2.4. By Part 1.1 of Proposition 4.23, $\Psi \mapsto \mu_\Psi$ is an isomorphism from $(\mathrm{IW}_1\mathrm{M}_\square(K_1{}^{E_1}, K_2{}^{E_2}), \square, \cdot)$ to $(\mathrm{M}_\square(K_1{}^W, K_2), \square, \cdot)$. As we see above in Part 2.1, $\mathrm{IW}_1\mathrm{M}_{\square,\cdot,e}(K_1{}^{E_1}, K_2{}^{E_2})$ is $\square_{\mathrm{Op}}$–closed, $\cdot_{\mathrm{Op}}$–closed and E_{Op}–closed, therefore by Exercise 3.58 and Part 2.3 above, $\Psi \mapsto \mu_\Psi$ restricted to $\mathrm{IW}_1\mathrm{M}_{\square,\cdot,e}(K_1{}^{E_1}, K_2{}^{E_2})$ and $\mathrm{M}_{\square,\cdot,e}(K_1{}^W, K_2)$ is an isomorphism from $(\mathrm{IW}_1\mathrm{M}_{\square,\cdot,e}(K_1{}^{E_1}, K_2{}^{E_2}), \square, E)$ to $(\mathrm{M}_{\square,\cdot,e}(K_1{}^W, K_2), \square, \varepsilon)$. $\square$

Let us apply Proposition 4.24 to the continuous and discrete cases. We consider first the continuous case, that is the case of the linear operators.

Proposition 4.25 (characterization of the linear spatially invariant window operators of type 1 – characterization in term of measures) – Let E_1, E_2 and W be three non–empty subsets of $\mathbf{Z}^2$ such that $E_2 = E_1 \ominus W$, and let $(\mathbf{R}, +, \cdot, o)$ be the linear vector space of the real numbers. The mapping $\Psi \mapsto \mu_\Psi$ from the subspace $(\mathrm{IW_1L}(\mathbf{R}^E, \mathbf{R}^E), +, \cdot, o)$ of spatially invariant window operators of type 1 which are linear, to the subspace $(\mathrm{L}(\mathbf{R}^W, \mathbf{R}), +, \cdot, o)$ of measures which are linear, is an isomorphism of linear vector spaces. Its inverse is $\mu \mapsto \Psi_\mu$. In particular, we have, for any Ψ and Φ in $\mathrm{IW_1L}(\mathbf{R}^{E_1}, \mathbf{R}^{E_2})$, and α in $\mathbf{R}$,

$$\mu_{\Psi+\Phi} = \mu_\Psi + \mu_\Phi\,, \quad \mu_{\alpha.\Psi} = \alpha.\mu_\Psi \quad \text{and} \quad \mu_O = o,$$

and, for any μ and ν in $\mathrm{L}(\mathbf{R}^W, \mathbf{R})$, and α in $\mathbf{R}$,

$$\Psi_{\mu+\nu} = \Psi_\mu + \Psi_\nu\,, \quad \Psi_{\alpha.\mu} = \alpha.\Psi_\mu \quad \text{and} \quad \Psi_o = O. \qquad \square$$

Proof – The assumptions of Part 2 of Proposition 4.24 are satisfied for any linear vector space, consequently the result of Part 2 applies to the mapping $\Psi \mapsto \mu_\Psi$ from $(\mathrm{IW_1L}(\mathbf{R}^E, \mathbf{R}^E), +, \cdot, o)$ to $(\mathrm{L}(\mathbf{R}^W, \mathbf{R}), +, \cdot, o)$. $\square$

We now consider the discrete case.

Proposition 4.26 (characterization of the elementary morphological spatially invariant window operators of type 1 – characterization in term of measures) – Let E_1, E_2 and W be three non–empty subsets of $\mathbf{Z}^2$ such that $E_2 = E_1 \ominus W$, and let $(K_1, \leq)$ and $(K_2, \leq)$ be, respectively, the intervals $[0, k_1]$ and $[0, k_2]$ of natural numbers equipped with the usual order relation.

1. The mapping $\Psi \mapsto \mu_\Psi$ from the submonoid $(\mathrm{IWD}(K_1{}^E, K_2{}^E), \vee, o)$ (resp. $(\mathrm{IW_1E^a}(K_1{}^E, K_2{}^E), \vee, o)$) of spatially invariant window operators of type 1 which are dilations (resp. anti–erosions), to the submonoid $(\mathrm{D}(K_1{}^W, K_2), \vee, o)$ (resp. $(\mathrm{E^a}(K_1{}^W, K_2), \vee, o)$) of measures which are dilations (resp. anti–erosions), is an isomorphism of monoids. Its inverse is $\mu \mapsto \Psi_\mu$.

In particular, we have, for any Ψ and Φ in $\mathrm{IW_1D}(K_1{}^{E_1}, K_2{}^{E_2})$ (resp. $\mathrm{IW_1E^a}(K_1{}^{E_1}, K_2{}^{E_2})$),

$$\mu_{\Psi \vee \Phi} = \mu_\Psi \vee \mu_\Phi \quad \text{and} \quad \mu_O = o,$$

and, for any μ and ν in $\mathrm{D}(K_1{}^W, K_2)$ (resp. $\mathrm{E^a}(K_1{}^W, K_2)$),

$$\Psi_{\mu \vee \nu} = \Psi_\mu \vee \Psi_\nu \quad \text{and} \quad \Psi_o = O.$$

2. The mapping $\Psi \mapsto \mu_\Psi$ from the submonoid $(\mathrm{IW_1E}(K_1{}^E, K_2{}^E), \wedge, \iota)$ (resp. $(\mathrm{IW_1D^a}(K_1{}^E, K_2{}^E), \wedge, \iota)$) of spatially invariant window operators of type 1 which are erosions (resp. anti–dilations), to the submonoid $(\mathrm{E}(K_1{}^W, K_2), \wedge, i)$ (resp. $(\mathrm{D^a}(K_1{}^W, K_2), \wedge, i)$) of measures which are erosions (resp. anti–dilations), is an isomorphism of monoids. Its inverse is $\mu \mapsto \Psi_\mu$.

In particular, we have, for any Ψ and Φ in $\mathrm{IW}_1\mathrm{E}(K_1{}^{E_1}, K_2{}^{E_2})$ (resp. $\mathrm{IW}_1\mathrm{D^a}(K_1{}^{E_1}, K_2{}^{E_2})$),

$$\mu_{\Psi \wedge \Phi} = \mu_\Psi \wedge \mu_\Phi \quad \text{and} \quad \mu_I = \iota,$$

and, for any μ and ν in $\mathrm{E}(K_1{}^W, K_2)$ (resp. $\mathrm{D^a}(K_1{}^W, K_2)$),

$$\Psi_{\mu \wedge \nu} = \Psi_\mu \wedge \Psi_\nu \quad \text{and} \quad \Psi_\iota = I. \qquad \square$$

Proof – The assumptions of Part 1 of Proposition 4.24 are satisfied for any monoid, consequently the result of Part 1 applies to the mapping $\Psi \mapsto \mu_\Psi$ from $(\mathrm{IW}_1\mathrm{D}(K_1{}^E, K_2{}^E), \vee, o)$ to $(\mathrm{D}(K_1{}^W, K_2), \vee, o)$. The same is true for the erosions, the anti–dilations and the anti–erosions. $\square$

Let us present the second characterization for the spatially invariant window operators of type 1. By combining the previous measure and operator characterizations, we can state a characterization of the spatially invariant window operators of type 1 in terms of families of luts.

Let E_1, E_2 and W be three non–empty subsets of $\mathbf{Z}^2$ such that $E_2 = E_1 \ominus W$.

Let Ψ be in $\mathrm{IW}_1(K_1{}^{E_1}, K_2{}^{E_2})$ with window W. Let u be an element of W and s be an element of K_1, we denote by $l_\Psi(u)(s)$ the expression defined by

$$l_\Psi(u)(s) \triangleq \Psi(f_{y+u,s})(y),$$

where y is any element of E_2.

Let l be a mapping from W to $K_2{}^{K_1}$, we denote by Ψ_l the expression defined by

$$\Psi_l \triangleq \mathop{\square}_{u \in W} \psi_{l(u)} \circ T_{-u},$$

where, for any u in W^t, T_u is a spatially invariant window operator of type 1 from $K_1{}^{E_1}$ to $K_1{}^{E_2}$, called *translation by* u and defined by, for any image f in $K_1{}^{E_1}$,

$$T_u(f) \triangleq f \circ m_u,$$

where, for any u in W^t, m_u is a mapping from E_2 to E_1, defined by, for any y in E_2,

$$m_u(y) \triangleq y - u;$$

where ψ_l is spatially invariant pointwise operator from $K_1{}^{E_2}$ to $K_2{}^{E_2}$ characterized by the lut l from K_1 to K_2, i.e., $\psi_l(f) \triangleq l \circ f$ for any image f in $K_1{}^{E_2}$.

When E_1 is $\mathbf{Z}^2$, then E_2 is $\mathbf{Z}^2$ independently of W, therefore, in this case, we have simply $T_u(f)(y) = f(y - u)$ for any y in E_2 and u in W^t.

Proposition 4.27 (characterization of the spatially invariant window operators of type 1 which are morphisms of more than one pair of operations – characterization in terms of families of luts) – Let E_1, E_2 and W be three non–empty subsets of $\mathbf{Z}^2$ such that $E_2 = E_1 \ominus W$.

1. Let $(K_1, \square, e_1)$ be a set equipped with an internal binary operation $\square$ and a nullary operation e_1. If $(K_2, \square, e_2)$ is a commutative monoid, then the mapping $\Psi \mapsto l_\Psi$ is an isomorphism from $(\mathrm{IW}_1\mathrm{M}_{\square,e}({K_1}^{E_1}, {K_2}^{E_2}), \square, E)$ to $(\mathrm{Lu}(W, \mathrm{M}_{\square,e}(K_1, K_2)), \square, e)$. Its inverse is $l \mapsto \Psi_l$. In particular, we have, for any Ψ and Φ in $\mathrm{IW}_1\mathrm{M}_{\square,e}({K_1}^{E_1}, {K_2}^{E_2})$,

$$l_{\Psi \,\square\, \Phi} = l_\Psi \,\square\, l_\Phi \quad \text{and} \quad l_E = e,$$

and, for any l and m in $\mathrm{Lu}(W, \mathrm{M}_{\square,e}(K_1, K_2))$,

$$\Psi_{l \,\square\, m} = \Psi_l \,\square\, \Psi_m \quad \text{and} \quad \Psi_e = E.$$

2. Let $(C, \cdot)$ be a set equipped with a commutative internal binary operation. Let $(K_1, \square, \cdot, e_1)$ be a set equipped with an internal binary operation $\square$, an external binary operation $\cdot$ and a nullary operation e_1. If $(K_2, \square, \cdot, e_2)$ is such that $(K_2, \square, e_2)$ is a commutative monoid and $\cdot$ is an external binary operation (with operand in C) satisfying for any s and t in K_2, and any α and β in C,

$$\alpha{\cdot}e_2 = e_2, \quad \alpha{\cdot}(\beta{\cdot}t) = (\alpha{\cdot}\beta){\cdot}t \quad \text{and} \quad \alpha{\cdot}(s \,\square\, t) = \alpha{\cdot}s \,\square\, \alpha{\cdot}t,$$

then the mapping $\Psi \mapsto l_\Psi$ is an isomorphism from $(\mathrm{IW}_1\mathrm{M}_{\square,\cdot,e}({K_1}^{E_1}, {K_2}^{E_2}), \square, \cdot, E)$ to $(\mathrm{Lu}(W, \mathrm{M}_{\square,\cdot,e}(K_1, K_2)), \square, \cdot, e)$. Its inverse is $l \mapsto \Psi_l$. In particular, we have, for any Ψ in $\mathrm{IW}_1\mathrm{M}_{\square,\cdot,e}({K_1}^{E_1}, {K_2}^{E_2})$, and α in C,

$$l_{\alpha.\Psi} = \alpha{\cdot}l_\Psi,$$

and for any l in $\mathrm{Lu}(W, \mathrm{M}_{\square,\cdot,e}(K_1, K_2))$ and α in C,

$$\Psi_{\alpha \cdot l} = \alpha{\cdot}\Psi_l\,. \qquad \square$$

Proof – 1. Let us show that $\Psi \mapsto l_\Psi$ is the composition of $\Psi \mapsto \mu_\Psi$ and $\mu \mapsto l_\mu$.

1.1. Let y be an element of E_2, let u be an element of W, let s be an element of K_1, let $g_{u,s}$ be a pulse mapping of ${K_1}^W$, and let f be an image of ${K_1}^{E_1}$ such that $f \circ b_y = g_{u,s}$, then for any u' in $W + y$,

$$\begin{aligned}
f/(W + y)(u') &= f(u') && \text{(restriction definition)}\\
&= f(b_y(u' - y)) && (b_y \text{ definition})\\
&= (f \circ b_y)(u' - y) && \text{(composition definition)}\\
&= g_{u,s}(u' - y) && \text{(hypotheses on } f)\\
&= \begin{cases} s & \text{if} \quad u' - y = u \\ e_1 & \text{otherwise.} \end{cases} && \text{(pulse mapping definition)}
\end{aligned}$$

$$= \begin{cases} s & \text{if} \quad u' = u + y \\ e_1 & \text{otherwise.} \end{cases} \qquad \text{(+ is regular)}$$

$$= f_{u+y,s}(u') \qquad \text{(pulse mapping definition)}$$

$$= f_{u+y,s}/(W + y)(u'), \qquad \text{(restriction definition)}$$

that is, $f/(W + y) = f_{u+y,s}/(W + y)$.

1.2. Let y be an element of E_2, let u be an element of W, let s be an element of K_1, let $g_{u,s}$ be an pulse mapping of $K_1{}^W$, and let f be an image of $K_1{}^{E_1}$ such that $f \circ b_y = g_{u,s}$, then for any Ψ in $\mathrm{IW}_1(K_1{}^{E_1}, K_2{}^{E_2})$ with window W,

$$l_{\mu_\Psi}(u)(s) = \mu_\Psi(g_{u,s}) \qquad (l_\mu \text{ definition})$$

$$= \Psi(f)(y) \qquad (\mu_\Psi \text{ definition})$$

$$= \Psi(f_{u+y,s})(y) \qquad (\Psi \text{ is a window operator and Part 1.1})$$

$$= l_\Psi(u)(s), \qquad (l_\Psi \text{ definition})$$

that is, for any Ψ in $\mathrm{IW}_1(K_1{}^{E_1}, K_2{}^{E_2})$, $l_{\mu_\Psi} = l_\Psi$.

2. Let us show that $l \mapsto \Psi_l$ is the composition of $l \mapsto \mu_l$ and $\mu \mapsto \Psi_\mu$. Let f be an image of $K_1{}^{E_1}$ and let y be an element of E_2, then for any l in $\mathrm{Lu}(W, \mathrm{M}_{\square,e}(K_1, K_2))$ or in $\mathrm{Lu}(W, \mathrm{M}_{\square,\cdot,e}(K_1, K_2))$,

$$\Psi_{\mu_l}(f)(y) = \mu_l(f \circ b_y) \qquad (\Psi_\mu \text{ def.})$$

$$= \Big(\mathop{\square}_{u \in W} l(u) \circ \mu_u\Big)(f \circ b_y) \qquad (\mu_l \text{ def.})$$

$$= \mathop{\square}_{u \in W} (l(u) \circ \mu_u)(f \circ b_y) \qquad \text{(extension def.)}$$

$$= \mathop{\square}_{u \in W} l(u)(\mu_u(f \circ b_y)) \qquad \text{(composition def.)}$$

$$= \mathop{\square}_{u \in W} l(u)(f \circ b_y)(u)) \qquad (\mu_u \text{ def.})$$

$$= \mathop{\square}_{u \in W} l(u)(f(b_y(u))) \qquad \text{(composition def.)}$$

$$= \mathop{\square}_{u \in W} l(u)(f(u + y)) \qquad (b_y \text{ def.})$$

$$= \mathop{\square}_{u \in W} l(u)(f(m_{-u}(y))) \qquad (m_u \text{ def.})$$

$$= \bigsqcup_{u \in W} l(u)((f \circ m_{-u})(y)) \qquad \text{(composition def.)}$$

$$= \bigsqcup_{u \in W} l(u)(T_{-u}(f)(y)) \qquad (T_u \text{ def.})$$

$$= \bigsqcup_{u \in W} (l(u) \circ T_{-u}(f))(y) \qquad \text{(composition def.)}$$

$$= (\bigsqcup_{u \in W} l(u) \circ T_{-u}(f))(y) \qquad \text{(extension def.)}$$

$$= (\bigsqcup_{u \in W} \psi_{l(u)}(T_{-u}(f)))(y) \qquad (\psi_l \text{ def.})$$

$$= (\bigsqcup_{u \in W} (\psi_{l(u)} \circ T_{-u})(f))(y) \qquad \text{(composition def.)}$$

$$= (\bigsqcup_{u \in W} \psi_{l(u)} \circ T_{-u})(f)(y) \qquad \text{(extension def.)}$$

$$= \Psi_l(f)(y), \qquad (\Psi_l \text{ def.})$$

that is, for any l in $\mathrm{Lu}(W, \mathrm{M}_{\square,e}(K_1, K_2))$ or in $\mathrm{Lu}(W, \mathrm{M}_{\square,\cdot,e}(K_1, K_2))$, $\Psi_{\mu_l} = \Psi_l$.

3. From Part 1, $\Psi \mapsto l_\Psi$ from $\mathrm{IW}_1\mathrm{M}_{\square,e}(K_1{}^{E_1}, K_2{}^{E_2})$ to $\mathrm{Lu}(W, \mathrm{M}_{\square,e}(K_1, K_2))$ is an isomorphism as the composition of two mappings which are isomorphisms by Propositions 4.9 and 4.24. From Part 2, $l \mapsto \Psi_l$ from $\mathrm{Lu}(W, \mathrm{M}_{\square,e}(K_1, K_2))$ to $\mathrm{IW}_1\mathrm{M}_{\square,e}(K_1{}^{E_1}, K_2{}^{E_2})$ is its inverse as composition of inverses.

4. From Part 1, $\Psi \mapsto l_\Psi$ from $\mathrm{IW}_1\mathrm{M}_{\square,\cdot,e}(K_1{}^{E_1}, K_2{}^{E_2})$ to $\mathrm{Lu}(W, \mathrm{M}_{\square,\cdot,e}(K_1, K_2))$ is an isomorphism as the composition of two mappings which are isomorphisms by Propositions 4.9 and 4.24. From Part 2, $l \mapsto \Psi_l$ from $\mathrm{Lu}(W, \mathrm{M}_{\square,\cdot,e}(K_1, K_2))$ to $\mathrm{IW}_1\mathrm{M}_{\square,\cdot,e}(K_1{}^{E_1}, K_2{}^{E_2})$ is its inverse as composition of inverses. □

Figure 4.12 shows the isomorphism $\Psi \mapsto \mu_\Psi$ of Part 1 of Proposition 4.27.

A spatially invariant window operator of type 1 Ψ from $K_1{}^{E_1}$ to $K_2{}^{E_2}$ is called *separable with respect to an internal binary operation* $\square$ *on* $\mathrm{Op}(K_1{}^{E_1}, K_2{}^{E_2})$, iff there exists a family l of luts from W to $K_2{}^{K_1}$ such that

$$\Psi = \bigsqcup_{u \in W} \psi_{l(u)} \circ T_{-u}. \tag{4.2}$$

The spatially invariant window operators of type 1 which are morphisms, are separable. Figure 4.13 shows a separable operator Ψ.

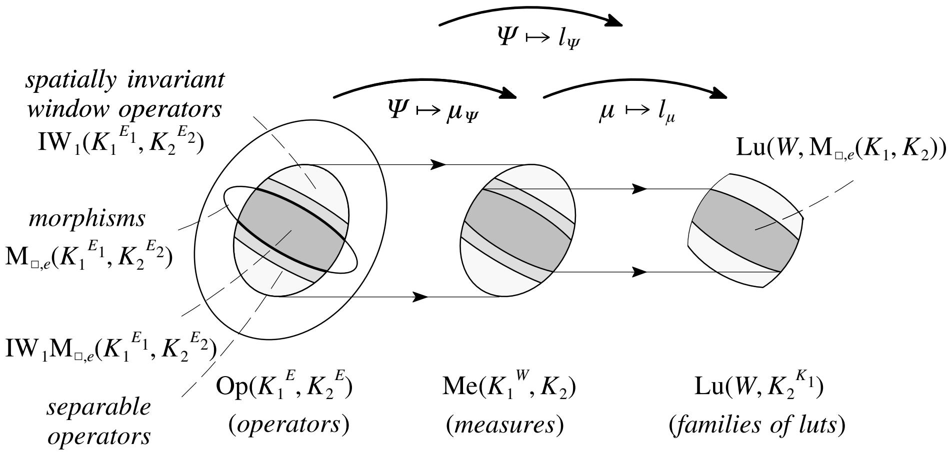


Fig. 4.12 – Composition of isomorphisms.

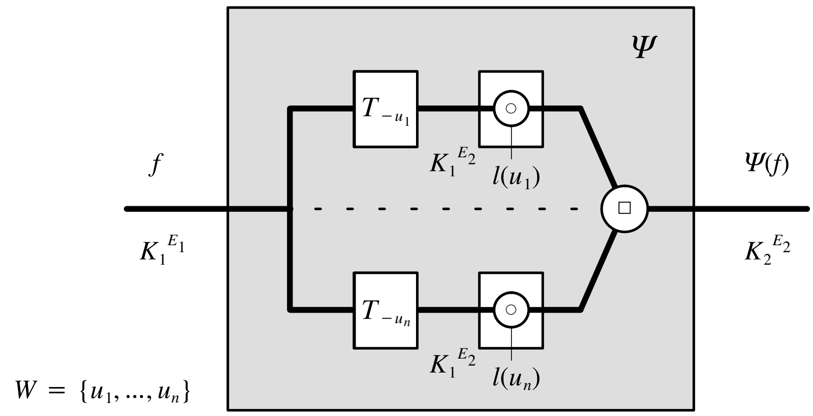


Fig. 4.13 – A separable operator.

By applying Proposition 4.27 to the continuous case, we can state the following result.

Proposition 4.28 (characterization of the linear spatially invariant window operators of type 1 – characterization in term of families of luts) – Let E_1, E_2 and W be three non–empty subsets of $\mathbf{Z}^2$ such that $E_2 = E_1 \ominus W$, and let $(\mathbf{R}, +, \cdot, o)$ be the linear vector space of the real numbers. The mapping $\Psi \mapsto l_\Psi$ from the set $\mathrm{IW}_1\mathrm{L}(\mathbf{R}^{E_1}, \mathbf{R}^{E_2})$ of linear spatially invariant window operators of type 1 with window W to the set $\mathrm{Lu}(W, \mathrm{L}(\mathbf{R}, \mathbf{R}))$ of families of luts which are linear, is an isomorphism of linear vector spaces. Its inverse is $l \mapsto \Psi_l$.

In particular, we have, for any operators Ψ and Φ in $\mathrm{IW}_1\mathrm{L}(\mathbf{R}^{E_1}, \mathbf{R}^{E_2})$, and α in $\mathbf{R}$,

$$l_{\Psi+\Phi} = l_\Psi + l_\Phi\,, \quad l_{\alpha.\Psi} = \alpha{\cdot}l_\Psi \quad \text{and} \quad l_O = o,$$

and, for any l and m in $\mathrm{Lu}(W, \mathrm{L}(\mathbf{R}, \mathbf{R}))$, and α in $\mathbf{R}$,

$$\Psi_{l+m} = \Psi_l + \Psi_m\,, \quad \Psi_{\alpha.l} = \alpha{\cdot}\Psi_l \quad \text{and} \quad \Psi_o = O. \qquad \square$$

Proposition 4.28 shows that any linear spatially invariant window operator of type 1 with window W, is a linear combination of translation operators T_{-u} when u runs over W. Furthermore, since the set of these operators is linearly independent, these operators form a base for the linear vector space of the linear spatially invariant window operators of type 1. Furthermore, for any u in W, the coordinate in this base of an operator Ψ relative to the translation T_{-u} is $\Psi(f_{y+u,\,1})(y)$, where y is any element of E_2.

By applying Proposition 4.27 to the discrete case, we can state the following result.

Proposition 4.29 (characterization of the elementary morphological spatially invariant window operators of type 1 – characterization in term of families of luts) – Let E_1, E_2 and W be three non–empty subsets of $\mathbf{Z}^2$ such that $E_2 = E_1 \ominus W$, and let $(K_1, \leq)$ and $(K_2, \leq)$ be, respectively, the intervals $[0, k_1]$ and $[0, k_2]$ of natural numbers equipped with the usual order relation.

1. The mapping $\Psi \mapsto l_\Psi$ from the set $\mathrm{IW}_1\mathrm{D}(K_1^{E_1}, K_2^{E_2})$ (resp. $\mathrm{IW}_1\mathrm{E}^{\mathrm{a}}(K_1^{E_1}, K_2^{E_2})$) of spatially invariant window operators of type 1 which are dilations (resp. anti–erosions) to the set $\mathrm{Lu}(W, \mathrm{D}(K_1, K_2))$ of families of luts which are dilations is an isomorphism of monoids. Its inverse is $l \mapsto \Psi_l$.

In particular, we have, for any operators Ψ and Φ in $\mathrm{IW}_1\mathrm{D}(K_1^{E_1}, K_2^{E_2})$ (resp. $\mathrm{IW}_1\mathrm{E}^{\mathrm{a}}(K_1^{E_1}, K_2^{E_2})$),

$$l_{\Psi\vee\Phi} = l_\Psi \vee l_\Phi \quad \text{and} \quad l_O = o,$$

and, for any l and m in $\mathrm{Lu}(W, \mathrm{D}(K_1, K_2))$ (resp. $\mathrm{Lu}(W, \mathrm{E}^{\mathrm{a}}(K_1, K_2))$),

$$\Psi_{l\vee m} = \Psi_l \vee \Psi_m \quad \text{and} \quad \Psi_o = O.$$

2. The mapping $\Psi \mapsto l_\Psi$ from the set $\mathrm{IW}_1\mathrm{E}(K_1^{E_1}, K_2^{E_2})$ (resp. $\mathrm{IW}_1\mathrm{D}^{\mathrm{a}}(K_1^{E_1}, K_2^{E_2})$) of spatially invariant window operators of type 1 which are erosions (resp. anti–dilations) to the set $\mathrm{Lu}(W, \mathrm{E}(K_1, K_2))$ of families of luts which are erosions is an isomorphism of monoids. Its inverse is $l \mapsto \Psi_l$.

In particular, we have, for any operators Ψ and Φ in $\mathrm{IW}_1\mathrm{E}(K_1^{E_1}, K_2^{E_2})$ (resp. $\mathrm{IW}_1\mathrm{D}^{\mathrm{a}}(K_1^{E_1}, K_2^{E_2})$),

$$l_{\Psi\wedge\Phi} = l_\Psi \wedge l_\Phi \quad \text{and} \quad l_I = i,$$

and, for any l and m in $\mathrm{Lu}(W, \mathrm{E}(K_1, K_2))$ (resp. $\mathrm{Lu}(W, \mathrm{D}^{\mathrm{a}}(K_1, K_2))$),

$$\Psi_{l\wedge m} = \Psi_l \wedge \Psi_m \quad \text{and} \quad \Psi_i = I. \qquad \square$$

Finally, let us present the characterization in terms of families of point spread functions for the spatially invariant window operators of type 2.

Let Φ be a spatially invariant window operator of type 2 (with window W) from ${K_1}^{E_1}$ to ${K_2}^{E_2}$. For any s in K_1, we denote by $h_\Phi(s)$ the expression defined by

$$h_\Phi(s) \triangleq \Phi(f_{x,s}) \circ a_x,$$

where x is any element of E_1, and where a_x is the mapping of Definition 4.19.

The expression $h_\Phi(s)$ is called the *point spread function* of the spatially invariant window operator of type 2, Φ for the intensity s.

We denote by $\mathrm{PSF}(K_1, {K_2}^W)$ the set of mappings from K_1 to ${K_2}^W$. Let h be one of these mappings. For any f in ${K_1}^{E_1}$, we denote by $\Phi_h(f)$ the expression defined by

$$\Phi_h(f) \triangleq \mathop{\Box}_{x \in E_1} (h(f(x)) + x),$$

where the operation $+$ is a mapping from ${K_2}^W \times E_1$ to ${K_2}^{E_2}$ defined by, for any g in ${K_2}^W$, any x in E_1 and any y in E_2,

$$(g + x)(y) \triangleq \begin{cases} g(y - x) & \text{if } y \in W + x \\ e_2 & \text{otherwise,} \end{cases}$$

where e_2 is an element of K_2.

Exercise 4.30 (left inverse) – Prove that the mapping $f \mapsto f \circ a_x$ from ${K_2}^{E_2}$ to ${K_2}^W$ is the left inverse of the mapping $g \mapsto g + x$ from ${K_2}^W$ to ${K_2}^{E_2}$ whenever x is in E_1. □

Exercise 4.31 (distributivity of $+$ over an internal binary operation) – Let $(K_2, \Box, e_2)$ be a set equipped with an internal binary operation and a nullary operation such that $e_2 = e_2 \Box e_2$. Prove that for any g_1 and g_2 in ${K_2}^W$, and any x in E_1,

$$(g_1 \Box_{{K_2}^W} g_2) + x = (g_1 + x) \Box_{{K_2}^{E_2}} (g_2 + x). \qquad \Box$$

Exercise 4.32 (mixt associativity of $+$ and of an external binary operation) – Let $(K_2, \cdot, e_2)$ be a set equipped with an external binary operation (with operand in C) and a nullary operation such that $e_2 = \alpha \cdot e_2$ for any α in C. Prove that for any g in ${K_2}^W$, any α in C, and any x in E_1,

$$(\alpha \cdot_{{K_2}^W} g) + x = \alpha \cdot_{{K_2}^{E_2}} (g_2 + x). \qquad \Box$$

Let $(K_1, \Box, e_1)$ and $(K_2, \Box, e_2)$ be two sets equipped with internal binary operations and with nullary operations. We denote by $\mathrm{IW_2M}_e({K_1}^{E_1}, {K_2}^{E_2})$ the set of spatially invariant window operators of type 2 which are morphisms for the extensions of e_1 and e_2 to ${K_1}^{E_1}$ and ${K_2}^{E_2}$, and we denote by $\mathrm{IW_2M}_{\Box,e}({K_1}^{E_1}, {K_2}^{E_2})$ the set of the previous operators which are also morphisms for the extensions of the $\Box$'s.

Finally, we denote by $\mathrm{M}_{\Box,e}(K_1, {K_2}^W)$ the set of families of point spread functions which are morphisms for the binary and nullary operations on K_1 and ${K_2}^W$.

Exercise 4.33 (property of the spatially invariant window operators of type 2 which are morphisms of nullary operations) – Let Φ be in $\mathrm{IW}_2\mathrm{M}_e({K_1}^{E_1}, {K_2}^{E_2})$. Prove that for any x in E_1 and s in K_1,

$$\Phi(f_{x,s}) \circ a_x + x = \Phi(f_{x,s}) \qquad \square$$

Solution – For any x in E_1, let denote by C_x the expression $W + x$ and let denote by D_x the expression $E_2 - C_x$. For any x in E_1, C_x and D_x form a partition of E_2. Let us divide the proof in two parts.

1. For any x in E_1, s in K_1 and y in C_x,

$$\begin{aligned}
(\Phi(f_{x,s}) \circ a_x + x)(y) &= (\Phi(f_{x,s}) \circ a_x)(y - x) && (+ \text{ definition})\\
&= \Phi(f_{x,s})(a_x(y - x)) && (\text{composition definition})\\
&= \Phi(f_{x,s})(y) && (a_x \text{ definition})
\end{aligned}$$

2. For any x in E_1, s in K_1 and y in D_x,

$$\begin{aligned}
(\Phi(f_{x,s}) \circ a_x + x)(y) &= e_2 && (+ \text{ definition})\\
&= e_2(y) && (e_2 \text{ is the mapping } y \mapsto e_2)\\
&= e_2/D_x(y) && (y \text{ belongs to } D_x)\\
&= \Phi(e_1)/D_x(y) && (\Phi \text{ is a morphism})\\
&= \Phi(f_{x,e_1})/D_x(y) && (f_{x,s} \text{ definition})\\
&= \Phi(f_{x,s})/D_x(y) && (\Phi \text{ is a window op. of type 2})\\
&= \Phi(f_{x,s})(y) && (y \text{ belongs to } D_x)
\end{aligned}$$

$\square$

Proposition 4.34 (characterization of the spatially invariant window operators of type 2 which are morphisms – characterization in terms of families of point spread functions) – Let E_1, E_2 and W be three non–empty subsets of $\mathbf{Z}^2$ such that $E_2 = E_1 \oplus W$.

1. Let $(K_1, \square, e_1)$ be a set equipped with an internal binary operation $\square$ and a nullary operation e_1. If $(K_2, \square, e_2)$ is a commutative monoid, then the mapping $\Phi \mapsto h_\Phi$ is an isomorphism from $(\mathrm{IW}_2\mathrm{M}_{\square,e}({K_1}^{E_1}, {K_2}^{E_2}), \square, E)$ to $(\mathrm{M}_{\square,e}(K_1, {K_2}^{W}), \square, e)$. Its inverse is $h \mapsto \Phi_h$.

In particular, we have, for any Ψ and Φ in $\mathrm{IW}_2\mathrm{M}_{\square,e}({K_1}^{E_1}, {K_2}^{E_2})$,

$$h_{\Psi \square \Phi} = h_\Psi \square h_\Phi \quad \text{and} \quad h_E = e,$$

and, for any h and i in $\mathrm{M}_{\square,e}(K_1, {K_2}^{W})$,

$$\Phi_{h \square i} = \Phi_h \square \Phi_i \quad \text{and} \quad \Phi_e = E.$$

2. Let $(C, \cdot)$ be a set equipped with a commutative internal binary operation. Let $(K_1, \square, \cdot, e_1)$ be a set equipped with an internal binary operation $\square$, an external binary operation $\cdot$ and a nullary operation e_1. If $(K_2, \square, \cdot, e_2)$ is such that $(K_2, \square, e_2)$ is a commutative monoid and $\cdot$ is an external binary operation (with operand in C) satisfying for any s and t in K_2, and any α and β in C,

$$\alpha \cdot e_2 \ = \ e_2, \quad \alpha \cdot (\beta \cdot t) \ = \ (\alpha \cdot \beta) \cdot t \quad \text{and} \quad \alpha \cdot (s \,\square\, t) \ = \ \alpha \cdot s \,\square\, \alpha \cdot t,$$

then the mapping $\Phi \mapsto h_\Phi$ is an isomorphism from $(\mathrm{IW}_2\mathrm{M}_{\square,\cdot,e}({K_1}^{E_1}, {K_2}^{E_2}), \square, \cdot, E)$ to $(\mathrm{M}_{\square,\cdot,e}(K_1, {K_2}^{W}), \square, \cdot, \varepsilon)$. Its inverse is $h \mapsto \Phi_h$.

In particular, we have, for any Φ in $\mathrm{IW}_2\mathrm{M}_{\square,\cdot,e}({K_1}^{E_1}, {K_2}^{E_2})$, and α in C,

$$h_{\alpha.\Phi} \ = \ \alpha \cdot h_\Phi,$$

and, for any h in $\mathrm{M}_{\square,\cdot,e}(K_1, {K_2}^{W})$, and α in C,

$$\Phi_{\alpha.h} \ = \ \alpha \cdot \Phi_h. \qquad \square$$

Proof – We divide the proof in nine parts.

1. In this part, we show that the h_Φ's are morphisms.

1.1. Let Φ be in $\mathrm{M}_\square({K_1}^{E_1}, {K_2}^{E_2})$. Let us show that h_Φ is in $\mathrm{M}_\square(K_1, {K_2}^{W})$. For any x in E_1, and s_1 and s_2 in K_1,

$$\begin{aligned}
h_\Phi(s_1 \,\square\, s_2) \ &= \ \Phi(f_{x, s_1 \square s_2}) \circ a_x && (h_\Phi \text{ definition}) \\
&= \ \Phi(f_{x, s_1} \,\square\, f_{x, s_2}) \circ a_x && (\text{Proposition 4.5}) \\
&= \ (\Phi(f_{x, s_1}) \,\square\, \Phi(f_{x, s_2})) \circ a_x && (\Phi \text{ is a morphism}) \\
&= \ (\Phi(f_{x, s_1}) \circ a_x) \,\square\, (\Phi(f_{x, s_2}) \circ a_x) && (\text{Exercise 2.49}) \\
&= \ h_\Phi(s_1) \,\square\, h_\Phi(s_2). && (h_\Phi \text{ definition})
\end{aligned}$$

1.2. Let e_1 be the element of K_1 selected by the nullary operation e_1 on K_1, and let e_2 be the element of K_2 selected by the nullary operation e_2 on K_2. Let Φ be in $\mathrm{M}_e({K_1}^{E_1}, {K_2}^{E_2})$ (the set of morphism of the nullary operations). Let us show that h_Φ is in $\mathrm{M}_e(K_1, {K_2}^{W})$. For any x in E_1,

$$\begin{aligned}
h_\Phi(e_1) \ &= \ \Phi(f_{x, e_1}) \circ a_x && (h_\Phi \text{ definition}) \\
&= \ \Phi(e_1) \circ a_x && (f_{x, e_1} \text{ reduces to the constant image } e_1) \\
&= \ e_2 \circ a_x && (\Phi \text{ is a morphism}) \\
&= \ e_2. && (\text{constant image in } {K_2}^{W})
\end{aligned}$$

1.3. Let Φ be in $\mathrm{M}_\cdot({K_1}^{E_1}, {K_2}^{E_2})$. Let us show that h_Φ is in $\mathrm{M}_\cdot(K_1, {K_2}^{W})$. For any x in E_1, s in K_1 and a in C,

$$\begin{aligned}
h_\Phi(\alpha{\cdot}s) &= \Phi(f_{x,\alpha.s}) \circ a_x && (h_\Phi \text{ definition})\\
&= \Phi(\alpha.f_{x,s}) \circ a_x && (\text{Proposition 4.5})\\
&= (\alpha{\cdot}\Phi(f_{x,s})) \circ a_x && (\Phi \text{ is a morphism})\\
&= \alpha{\cdot}(\Phi(f_{x,s}) \circ a_x) && (\text{Exercise 2.49})\\
&= \alpha{\cdot}\, h_\Phi(s). && (h_\Phi \text{ definition})
\end{aligned}$$

2. In this part, we show that the Φ_h's are morphisms.

2.1. Let $(K_2, \square, e_2)$ be a set equipped with an associative internal binary operation and a nullary operation such that $e_2 = e_2 \square e_2$. Let h be in $\mathrm{M}_\square(K_1, K_2{}^W)$. Let us show that Φ_h is in $\mathrm{M}_\square(K_1{}^{E_1}, K_2{}^{E_2})$. For any f_1 and f_2 in $K_1{}^{E_1}$,

$$\begin{aligned}
\Phi_h(f_1 \square f_2) &= \mathop{\square}_{u \in E_1} (h((f_1 \square f_2)(u)) + u) && (\Phi_h \text{ definition})\\
&= \mathop{\square}_{u \in E_1} (h(f_1(u) \square f_2(u)) + u) && (\text{extension definition})\\
&= \mathop{\square}_{u \in E_1} ((h(f_1(u)) \square h(f_2(u))) + u) && (h \text{ is a morphism})\\
&= \mathop{\square}_{u \in E_1} ((h(f_1(u)) + u) \square (h(f_2(u)) + u)) && (e_2 = e_2 \square e_2, \text{ Exercise 4.31})\\
&= \mathop{\square}_{u \in E_1} (h(f_1(u)) + u) \square \mathop{\square}_{u \in E_1} (h(f_2(u)) + u) && (\text{associativity of } \square)\\
&= \Phi_h(f_1) \square \Phi_h(f_2). && (\Phi_h \text{ definition})
\end{aligned}$$

2.2. Let $(K_2, \square, e_2)$ be a set equipped with an internal binary operation and a nullary operation such that $e_2 = e_2 \square e_2$. Let h be in $\mathrm{M}_e(K_1, K_2{}^W)$. Let us show that Φ_h is in $\mathrm{M}_e(K_1{}^{E_1}, K_2{}^{E_2})$. We denote by e_1 (resp. e and e_2) the constant image of $K_1{}^{E_1}$ (resp. $K_2{}^W$ and $K_2{}^{E_2}$) assuming value e_2.

$$\begin{aligned}
\Phi_h(e_1) &= \mathop{\square}_{u \in E_1} (h(e_1(u)) + u) && (\Phi_h \text{ definition})\\
&= \mathop{\square}_{u \in E_1} (e + u) && (h \text{ is a morphism; } e \in K_2{}^W)\\
&= \mathop{\square}_{u \in E_1} e_2 && (+ \text{ definition; } e_2 \in K_2{}^{E_2})\\
&= e_2. && (e_2 = e_2 \square e_2)
\end{aligned}$$

2.3. Let $(K_2, \square, \cdot, e_2)$ be a set equipped with an internal binary operation, an external binary operation (with operand in C) and a nullary operation such that $\alpha \cdot (s \square t) = \alpha \cdot s \square \alpha \cdot t$ for any s and t in K_2, and $e_2 = \alpha \cdot e_2$ for any α in C. Let h be in $\mathrm{M}_{\cdot}(K_1, K_2^W)$. Let us show that Φ_h is in $\mathrm{M}_{\cdot}(K_1^{E_1}, K_2^{E_2})$. For any f in $K_1^{E_1}$ and any α in C,

$$\begin{aligned}
\Phi_h(\alpha \cdot f) &= \mathop{\square}_{u \in E_1} (h((\alpha \cdot f)(u)) + u) && (\Phi_h \text{ definition}) \\
&= \mathop{\square}_{u \in E_1} (h(\alpha \cdot f(u)) + u) && (\text{extension definition}) \\
&= \mathop{\square}_{u \in E_1} (\alpha \cdot h(f(u)) + u) && (h \text{ is a morphism}) \\
&= \mathop{\square}_{u \in E_1} (\alpha \cdot (h(f(u)) + u)) && (e_2 = \alpha \cdot e_2, \text{ Exercise 4.32}) \\
&= \alpha \cdot (\mathop{\square}_{u \in E_1} (h(f(u)) + u)) && (\text{distributivity of } \cdot \text{ over } \square) \\
&= \alpha \cdot \Phi_h(f). && (\Phi_h \text{ definition})
\end{aligned}$$

3. In this part, we show that the Φ_h's are spatially invariant window operators of type 2. The proof is divided in two steps.

3.1. Let $(K_2, \square, e_2)$ be a commutative monoid. For any x in E_1, let denote by D_x the expression $E_2 - (W + x)$. For any f and g in $K_1^{E_1}$ and any x in E_1, such that $f/(E_1 - \{x\}) = g/(E_1 - \{x\})$,

$$\begin{aligned}
\Phi_h(f)/D_x &= (\mathop{\square}_{u \in E_1} (h(f(u)) + u))/D_x && (\Phi_h \text{ definition}) \\
&= ((\mathop{\square}_{u \in E_1 - \{x\}} (h(f(u)) + u)) \square (h(f(x)) + x))/D_x && (\square \text{ properties}) \\
&= ((\mathop{\square}_{u \in E_1 - \{x\}} (h(f(u)) + u))/D_x) \square ((h(f(x)) + x)/D_x) && (\text{Exercise 2.50}) \\
&= (\mathop{\square}_{u \in E_1 - \{x\}} (h(f(u)) + u))/D_x && \begin{array}{c}(h(f(x)) + x \text{ assumes value } e_2 \text{ in } D_x \\ \text{and } e_2 \text{ is unit element for } \square)\end{array} \\
&= (\mathop{\square}_{u \in E_1 - \{x\}} (h(g(u)) + u))/D_x && (\text{hypothesis}) \\
&= \Phi_h(g)/D_x. && (\text{same arguments as above})
\end{aligned}$$

That is, Φ_h is a window operator of type 2 w.r.t. $x \mapsto (W + x)$.

3.2. Let $(K_2, \square, e_2)$ be a commutative monoid. For any s in K_1, and any x_1 and x_2 in E_1,

$$
\begin{aligned}
\Phi_h(f_{x_1,s}) \circ a_{x_1} &= (\mathop{\square}_{u \in E_1} (h(f_{x_1,s}(u)) + u)) \circ a_{x_1} && (\Phi_h \text{ definition})\\
&= ((\mathop{\square}_{u \in E_1 - \{x_1\}} (h(f_{x_1,s}(u)) + u)) \square (h(f_{x_1,s}(x_1)) + x_1)) \circ a_{x_1} && (\square \text{ prop.})\\
&= ((\mathop{\square}_{u \in E_1 - \{x_1\}} (h(e_1) + u)) \square (h(s) + x_1)) \circ a_{x_1} && (\text{pulse definition})\\
&= ((\mathop{\square}_{u \in E_1 - \{x_1\}} (e_2 + u)) \square (h(s) + x_1)) \circ a_{x_1} && (h \text{ is a morphism})\\
&= ((\mathop{\square}_{u \in E_1 - \{x_1\}} e_2) \square (h(s) + x_1)) \circ a_{x_1} && (+\text{definition})\\
&= (h(s) + x_1) \circ a_{x_1} && (e_2 \text{ is unit element for } \square)\\
&= h(s) && (\text{Exercise 4.30})\\
&= \Phi_h(f_{x_2,s}) \circ a_{x_2}. && (\text{same arguments as above})
\end{aligned}
$$

That is, Φ_h is a spatially invariant window operator of type 2 with window W.

4. In this part, we show that the spatially invariant window operators of type 2. form closed subsets.

4.1. Let us prove that $\mathrm{IW}_2({K_1}^{E_1}, {K_2}^{E_2})$ is $\square_{\mathrm{Op}}$–closed. Let Ψ and Φ be in $\mathrm{IW}_2({K_1}^{E_1}, {K_2}^{E_2})$. For any x_1 and x_2 in E_1, and s in K_1,

$$
\begin{aligned}
(\Psi \square \Phi)(f_{x_1,s}) \circ a_{x_1} &= (\Psi(f_{x_1,s}) \square \Phi(f_{x_1,s})) \circ a_{x_1} && (\text{extension definition})\\
&= (\Psi(f_{x_1,s}) \circ a_{x_1}) \square (\Phi(f_{x_1,s})) \circ a_{x_1}) && (\text{Exercise 2.49})\\
&= (\Psi(f_{x_2,s}) \circ a_{x_2}) \square (\Phi(f_{x_2,s})) \circ a_{x_2}) && (\text{spatial invariance})\\
&= (\Psi \square \Phi)(f_{x_2,s}) \circ a_{x_2}. && (\text{same arguments as above})
\end{aligned}
$$

4.2. Let us prove that $\mathrm{IW}_2({K_1}^{E_1}, {K_2}^{E_2})$ is E_{Op}–closed. For any x_1 and x_2 in E_1, and s in K_1,

$$
\begin{aligned}
E\,(f_{x_1,s}) \circ a_{x_1} &= e_2 \circ a_{x_1} && (\text{extension definition})\\
&= e_2 \circ a_{x_2} && (e_2 \text{ is a constant image})\\
&= E\,(f_{x_2,s}) \circ a_{x_2}. && (\text{extension definition})
\end{aligned}
$$

4.3. Let us prove that $\mathrm{IW}_2(K_1{}^{E_1}, K_2{}^{E_2})$ is $\cdot_{\mathrm{Op}}$–closed. Let Φ be in $\mathrm{IW}_2(K_1{}^{E_1}, K_2{}^{E_2})$ and let a in C. For any x_1 and x_2 in E_1, and s in K_1,

$$\begin{aligned}
(\alpha{\cdot}\Phi)(f_{x_1,s}) \circ a_{x_1} &= (\alpha{\cdot}\Phi(f_{x_1,s})) \circ a_{x_1} && \text{(extension definition)}\\
&= \alpha{\cdot}(\Phi(f_{x_1,s}) \circ a_{x_1}) && \text{(Exercise 2.49)}\\
&= \alpha{\cdot}(\Phi(f_{x_2,s}) \circ a_{x_2}) && \text{(spatial invariance)}\\
&= (\alpha{\cdot}\Phi)(f_{x_2,s}) \circ a_{x_2}. && \text{(same arguments as above)}
\end{aligned}$$

5. In this part, we show that the mappings $\Phi \mapsto h_\Phi$ are morphisms

5.1. Let us show that $\Phi \mapsto h_\Phi$ is a morphism from $(\mathrm{IW}_2(K_1{}^{E_1}, K_2{}^{E_2}), \square)$ to $(\mathrm{PSF}(K_1, K_2{}^W)), \square)$. For any Ψ and Φ in $\mathrm{IW}_2(K_1{}^{E_1}, K_2{}^{E_2})$, any x in E_1, and s in K_1,

$$\begin{aligned}
(h_\Psi \,\square\, h_\Phi)(s) &= h_\Psi(s) \,\square\, h_\Phi(s) && \text{(extension definition)}\\
&= \Phi(f_{x,s}) \circ a_x \,\square\, \Psi(f_{x,s}) \circ a_x && (h_\Phi \text{ definition})\\
&= (\Phi(f_{x,s}) \,\square\, \Psi(f_{x,s})) \circ a_x && \text{(Exercise 2.49)}\\
&= (\Psi \,\square\, \Phi)(f_{x,s}) \circ a_x && \text{(extension definition)}\\
&= h_{\Psi \square \Phi}(s). && (h_\Phi \text{ definition})
\end{aligned}$$

5.2. Let us show that $\Phi \mapsto h_\Phi$ is a morphism from $(\mathrm{IW}_2(K_1{}^{E_1}, K_2{}^{E_2}), E)$ to $(\mathrm{PSF}(K_1, K_2{}^W)), e)$. We denote by e (resp. e_2) the constant image of $K_2{}^W$ (resp. $K_2{}^{E_2}$) assuming value e_2. Let For any x in E_1 and s in K_1,

$$\begin{aligned}
h_E(s) &= E(f_{x,s}) \circ a_x && (h_\Phi \text{ definition})\\
&= e_2 \circ a_x && (E \text{ definition})\\
&= e && (a_x \text{ and } e \text{ definitions})
\end{aligned}$$

5.3. Let us show that $\Phi \mapsto h_\Phi$ is a morphism from $(\mathrm{IW}_2(K_1{}^{E_1}, K_2{}^{E_2}), \cdot)$ to $(\mathrm{PSF}(K_1, K_2{}^W)), \cdot)$. For any Φ in $\mathrm{IW}_2(K_1{}^{E_1}, K_2{}^{E_2})$, any α in C, any x in E_1, and s in K_1,

$$\begin{aligned}
(\alpha{\cdot}h_\Phi)(s) &= \alpha{\cdot}h_\Phi(s) && \text{(extension definition)}\\
&= \alpha{\cdot}(\Psi(f_{x,s}) \circ a_x) && (h_\Phi \text{ definition})\\
&= (\alpha{\cdot}\Psi(f_{x,s})) \circ a_x && \text{(Exercise 2.49)}\\
&= (\alpha{\cdot}\Psi)(f_{x,s}) \circ a_x && \text{(extension definition)}\\
&= h_{\alpha{\cdot}\Phi}(s). && (h_\Phi \text{ definition})
\end{aligned}$$

6. In this part, we show that the mappings $h \mapsto \Phi_h$ are morphisms

6.1. Let $(K_2, \square, e_2)$ be a set equipped with an associative and commutative internal binary operation and a nullary operation such that $e_2 = e_2 \square e_2$. Let us show that $h \mapsto \Phi_h$ is a morphism from $(\mathrm{PSF}(K_1, {K_2}^W)), \square)$ to $(\mathrm{IW}_2({K_1}^{E_1}, {K_2}^{E_2}), \square)$. For any h and i in $\mathrm{PSF}(K_1, {K_2}^W)$, any x in E_1, and f in ${K_1}^{E_1}$,

$$
\begin{aligned}
(\Phi_h \square \Phi_i)(f) &= \Phi(f) \square \Phi_i(f) && \text{(extension definition)}\\
&= \mathop{\square}_{x \in E_1} (h(f(x)) + x) \square \mathop{\square}_{x \in E_1} (i(f(x)) + x) && (\Phi_h \text{ def.})\\
&= \mathop{\square}_{x \in E_1} ((h(f(x)) + x) \square (i(f(x)) + x)) && (\square \text{ properties})\\
&= \mathop{\square}_{x \in E_1} ((h(f(x)) \square i(f(x))) + x) && (e_2 = e_2 \square e_2, \text{Ex. 4.31})\\
&= \mathop{\square}_{x \in E_1} ((h \square i)(f(x)) + x) && \text{(extension definition)}\\
&= \Phi_{h \square i}. && (\Phi_h \text{ def.})
\end{aligned}
$$

6.2. Let $(K_2, \square, e_2)$ be a set equipped with an internal binary operation and a nullary operation such that $e_2 = e_2 \square e_2$. Let us show that $h \mapsto \Phi_h$ is a morphism from $(\mathrm{PSF}(K_1, {K_2}^W)), e)$ to $(\mathrm{IW}_2({K_1}^{E_1}, {K_2}^{E_2}), E)$. We denote by e (resp. e_2) the constant image of ${K_2}^W$ (resp. ${K_2}^{E_2}$) assuming value e_2. For any x in E_1, and f in ${K_1}^{E_1}$,

$$
\begin{aligned}
\Phi_e(f) &= \mathop{\square}_{x \in E_1} (e(f(x)) + x) && (\Phi_h \text{ def.})\\
&= \mathop{\square}_{x \in E_1} (e + x) && (e \text{ selects the constant image } e)\\
&= e_2 && (+\text{definition and } \square \text{ properties})\\
&= E(f). && (E \text{ definition})
\end{aligned}
$$

6.3. Let $(K_2, \square, \cdot)$ be a set equipped with an internal binary operation, a distributive external binary operation (with operand in C) and a nullary operation such that $e_2 = \alpha \cdot e_2$, for any α in C. Let us show that $h \mapsto \Phi_h$ is a morphism from $(\mathrm{PSF}(K_1, {K_2}^W)), \cdot)$ to $(\mathrm{IW}_2({K_1}^{E_1}, {K_2}^{E_2}), \cdot)$. For any h in $\mathrm{PSF}(K_1, {K_2}^W)$, α in C, any x in E_1, and f in ${K_1}^{E_1}$,

$$\begin{aligned}
(\alpha \cdot \Phi_h)(f) &= \alpha \cdot \Phi_h(f) && \text{(extension definition)}\\
&= \alpha \cdot (\mathop{\Box}_{x \in E_1} (h(f(x)) + x)) && (\Phi_h \text{ def.})\\
&= \mathop{\Box}_{x \in E_1} ((\alpha \cdot (h(f(x)) + x)) && (\cdot \text{ property})\\
&= \mathop{\Box}_{x \in E_1} ((\alpha \cdot h(f(x))) + x) && (e_2 = \alpha \cdot e_2, \text{ Exercise 4.32})\\
&= \mathop{\Box}_{x \in E_1} ((\alpha \cdot h)(f(x)) + x) && \text{(extension definition)}\\
&= \Phi_{\alpha . h}. && (\Phi_h \text{ def.})
\end{aligned}$$

7. In this part, we show that $\Phi \mapsto h_\Phi$ is a bijection from $\mathrm{IW}_2\mathrm{M}_{\Box,e}(K_1{}^{E_1}, K_2{}^{E_2})$ to $\mathrm{M}_{\Box,e}(K_1, K_2{}^W)$. The proof is divided in four steps.

7.1. Let Φ be in $\mathrm{IW}_2\mathrm{M}_{\Box,e}(K_1{}^{E_1}, K_2{}^{E_2})$, by definition, h_Φ is a mapping from K_1 to $K_2{}^W$. Parts 1.1 and 1.2 imply that h_Φ is in $\mathrm{M}_{\Box,e}(K_1, K_2{}^W)$.

7.2. Let $(K_2, \Box, e_2)$ be a commutative monoid. Let h be in $\mathrm{M}_{\Box,e}(K_1, K_2{}^W)$, by definition, Φ_h is a mapping from $K_1{}^{E_1}$ to $K_2{}^{E_2}$. Parts 2.1, 2.2 and 3 imply that Φ_h is in $\mathrm{IW}_2\mathrm{M}_{\Box,e}(K_1{}^{E_1}, K_2{}^{E_2})$.

7.3. Let Φ be in $\mathrm{IW}_2\mathrm{M}_{\Box,e}(K_1{}^{E_1}, K_2{}^{E_2})$. For any f in $K_1{}^{E_1}$,

$$\begin{aligned}
\Phi_{h_\Phi}(f) &= \mathop{\Box}_{u \in E_1} (h_\Phi(f(u)) + u) && (\Phi_h \text{ definition})\\
&= \mathop{\Box}_{u \in E_1} (\Phi(f_{x,f(u)}) \circ a_x + u) && (h_\Phi \text{ definition; } x \in E_1)\\
&= \mathop{\Box}_{u \in E_1} (\Phi(f_{u,f(u)}) \circ a_u + u) && (\Phi \text{ is spatially invariant})\\
&= \mathop{\Box}_{u \in E_1} \Phi(f_{u,f(u)}) && \text{(Exercise 4.33)}\\
&= \Phi(\mathop{\Box}_{u \in E_1} f_{u,f(u)}) && (\Phi \text{ is a morphism})\\
&= \Phi(f), && \text{(Proposition 4.6)}
\end{aligned}$$

that is, $\Phi \mapsto h_\Phi$ is an injection from $\mathrm{IW}_2\mathrm{M}_{\Box,e}(K_1{}^{E_1}, K_2{}^{E_2})$ to $\mathrm{M}_{\Box,e}(K_1, K_2{}^W)$.

7.4. Let $(K_2, \Box, e_2)$ be a commutative monoid. Let h be in $\mathrm{M}_{\Box,e}(K_1, K_2{}^W)$. For any s in K_1,

$$h_{\Phi_h}(s) \;=\; \Phi_h(f_{x,s}) \circ a_x \qquad (h_\Phi \text{ definition})$$

$$= h(s), \qquad \text{(same arguments as in Part 3.2)}$$

that is, $\Phi \mapsto h_\Phi$ is a surjection from $\mathrm{IW}_2\mathrm{M}_{\Box,e}(K_1{}^{E_1}, K_2{}^{E_2})$ to $\mathrm{M}_{\Box,e}(K_1, K_2{}^W)$.

8. In this part, we show that $\Phi \mapsto h_\Phi$ is a bijection from $\mathrm{IW}_2\mathrm{M}_{\Box,\cdot,e}(K_1{}^{E_1}, K_2{}^{E_2})$ to $\mathrm{M}_{\Box,\cdot,e}(K_1, K_2{}^W)$. The proof is divided in four steps.

8.1. Let Φ be in $\mathrm{IW}_2\mathrm{M}_{\Box,\cdot,e}(K_1{}^{E_1}, K_2{}^{E_2})$, by definition, h_Φ is a mapping from K_1 to $K_2{}^W$. Parts 1.1, 1.2 and 1.3 imply that h_Φ is in $\mathrm{M}_{\Box,\cdot,e}(K_1, K_2{}^W)$.

8.2. Let $(K_2, \Box, \cdot, e_2)$ be a set equipped with an internal binary operation, an external binary operation (with operand in C) and a nullary operation such that $(K_2, \Box, e_2)$ is a commutative monoid, $\alpha\cdot(s \Box t) = \alpha\cdot s \Box \alpha\cdot t$ for any s and t in K_2, and $e_2 = \alpha\cdot e_2$ for any α in C. Let h be in $\mathrm{M}_{\Box,\cdot,e}(K_1, K_2{}^W)$, by definition, Φ_h is a mapping from $K_1{}^{E_1}$ to $K_2{}^{E_2}$. Parts 2.1, 2.2, 2.3 and 3 imply that Φ_h is in $\mathrm{IW}_2\mathrm{M}_{\Box,\cdot,e}(K_1{}^{E_1}, K_2{}^{E_2})$.

8.3 By the same arguments of 7.3, $\Phi \mapsto h_\Phi$ is an injection from $\mathrm{IW}_2\mathrm{M}_{\Box,\cdot,e}(K_1{}^{E_1}, K_2{}^{E_2})$ to $\mathrm{M}_{\Box,\cdot,e}(K_1, K_2{}^W)$.

8.4 By the same arguments of 7.4, $\Phi \mapsto h_\Phi$ is an surjection from $\mathrm{IW}_2\mathrm{M}_{\Box,\cdot,e}(K_1{}^{E_1}, K_2{}^{E_2})$ to $\mathrm{M}_{\Box,\cdot,e}(K_1, K_2{}^W)$.

9. In this part, we study the closure properties of $\mathrm{IW}_2\mathrm{M}_{\Box,e}(K_1{}^{E_1}, K_2{}^{E_2})$, $\mathrm{M}_{\Box,e}(K_1, K_2{}^W)$, $\mathrm{IW}_2\mathrm{M}_{\Box,\cdot,e}(K_1{}^{E_1}, K_2{}^{E_2})$ and $\mathrm{M}_{\Box,\cdot,e}(K_1, K_2{}^W)$.

9.1. By Part 4.1, $\mathrm{IW}_2(K_1{}^{E_1}, K_2{}^{E_2})$ is $\Box_{\mathrm{Op}}$–closed and E_{Op}–closed. By Proposition 3.67, $\mathrm{M}_\Box(K_1{}^{E_1}, K_2{}^{E_2})$ is $\Box_{\mathrm{Op}}$–closed and E_{Op}–closed. By Proposition 3.69, $\mathrm{M}_e(K_1{}^{E_1}, K_2{}^{E_2})$ is $\Box_{\mathrm{Op}}$–closed and E_{Op}–closed. Consequently, by Proposition 3.55 their intersection $\mathrm{IW}_2\mathrm{M}_{\Box,e}(K_1{}^{E_1}, K_2{}^{E_2})$ is $\Box_{\mathrm{Op}}$–closed and E_{Op}–closed.

9.2. By Proposition 3.67, $\mathrm{M}_\Box(K_1, K_2{}^W)$ is $\Box_{\mathrm{PSF}}$–closed and e_{PSF}–closed. By Proposition 3.69, $\mathrm{M}_e(K_1, K_2{}^W)$ is $\Box_{\mathrm{PSF}}$–closed and e_{PSF}–closed. Consequently, by Proposition 3.55 their intersection $\mathrm{M}_{\Box,e}(K_1, K_2{}^W)$ is $\Box_{\mathrm{PSF}}$–closed and e_{PSF}–closed.

9.3. By Part 4.1, $\mathrm{IW}_2(K_1{}^{E_1}, K_2{}^{E_2})$ is $\Box_{\mathrm{Op}}$–closed, E_{Op}–closed and $\cdot_{\mathrm{Op}}$–closed. By Proposition 3.67, $\mathrm{M}_\Box(K_1{}^{E_1}, K_2{}^{E_2})$ is $\Box_{\mathrm{Op}}$–closed, E_{Op}–closed and $\cdot_{\mathrm{Op}}$–closed. By Proposition 3.68, $\mathrm{M}_\cdot(K_1{}^{E_1}, K_2{}^{E_2})$ is $\Box_{\mathrm{Op}}$–closed, E_{Op}–closed and $\cdot_{\mathrm{Op}}$–closed. By Proposition 3.69, $\mathrm{M}_e(K_1{}^{E_1}, K_2{}^{E_2})$ is $\Box_{\mathrm{Op}}$–closed, E_{Op}–closed and $\cdot_{\mathrm{Op}}$–closed. Consequently, by Proposition 3.55 their intersection $\mathrm{IW}_2\mathrm{M}_{\Box,\cdot,e}(K_1{}^{E_1}, K_2{}^{E_2})$ is $\Box_{\mathrm{Op}}$–closed, E_{Op}–closed and $\cdot_{\mathrm{Op}}$–closed.

9.4. By Proposition 3.67, $\mathrm{M}_{\square}(K_1, K_2{}^W)$ is $\square_{\mathrm{PSF}}$–closed, $\cdot_{\mathrm{PSF}}$–closed and e_{PSF}–closed. By Proposition 3.68, $\mathrm{M}_{\cdot}(K_1, K_2{}^W)$ is $\square_{\mathrm{PSF}}$–closed, $\cdot_{\mathrm{PSF}}$–closed and e_{PSF}–closed. By Proposition 3.69, $\mathrm{M}_e(K_1, K_2{}^W)$ is $\square_{\mathrm{PSF}}$–closed, $\cdot_{\mathrm{PSF}}$–closed and e_{PSF}–closed. Consequently, by Proposition 3.55 their intersection $\mathrm{M}_{\square,\cdot,e}(K_1, K_2{}^W)$ is $\square_{\mathrm{PSF}}$–closed, $\cdot_{\mathrm{PSF}}$–closed and e_{PSF}–closed. □

Before ending this section, we comment the relationship between the two types of window operators. First we observe that (in the finite case), the number of families of luts which characterize the spatially invariant window morphisms of type 1 is equal to the number of families of point spread functions which characterize the spatially invariant window morphisms of type 2. If the input and output gray–scale are, respectively K_1 and K_2, and the window is W, then this number is $\#K_2^{\#K_1 \#W}$.

As a consequence of the above observation, there exists a bijection between the set of families of luts $\mathrm{Lu}(W, K_2{}^{K_1})$ and the set of families of point spread functions $\mathrm{PSF}(K_1, K_2{}^W)$, in other words, $\mathrm{Lu}(W^t, K_2{}^{K_1}) \cong \mathrm{PSF}(K_1, K_2{}^W)$. One possible bijection from $\mathrm{PSF}(K_1, K_2{}^W)$ to $\mathrm{Lu}(W^t, K_2{}^{K_1})$ is $h \mapsto l_h$, where, l_h is the family of luts defined by, for any u in W^t and s in K_1,

$$l_h(u)(s) \triangleq h(s)(-u).$$

The existence of such bijection implies that there exists a bijection between the set $\mathrm{IW_1M}(K_1{}^{E_1}, K_2{}^{E_2})$ of spatially invariant window morphisms of type 1 (with window W^t) and the set $\mathrm{IW_2M}(K_1{}^{F_1}, K_2{}^{F_2})$ of spatially invariant window morphisms of type 2 (with window W), but this doesn't mean that we can find a spatially invariant window morphism of type 1 equals to a spatially invariant window morphism of type 2.

Nevertheless, this is true when the image domains are equal to $\mathbf{Z}^2$ as it is shown in the last proposition of this section.

Proposition 4.35 (relationship between window operators of type 1 and 2) – Let E_1 and E_2 be Z^2 and let W be a subset of $\mathbf{Z}^2$. Let $(K_2, \square, e_2)$ be a commutative monoid. For any h in $\mathrm{PSF}(K_1, K_2{}^W)$, Φ_h is a spatially invariant operator of type 2 with window W and Ψ_{l_h} is a spatially invariant operator of type 1 with window W^t, and we have,

$$\Psi_{l_h} = \Phi_h. \qquad \square$$

Proof – If $E_1 = E_2 = \mathbf{Z}^2$ then $E_2 = E_1 \oplus W$ and $E_2 = E_1 \ominus W$ for any subset W of $\mathbf{Z}^2$.

Furthermore, for any f in $K_1{}^{\mathbf{Z}^2}$ and y in $\mathbf{Z}^2$,

$$\begin{aligned}
\Psi_{l_h}(f)(y) &= (\mathop{\square}_{u \in W^t} \psi_{l_h(u)} \circ T_{-u})(f)(y) && (\Psi_l \text{ definition with window } W^t)\\
&= \mathop{\square}_{u \in W^t} (\psi_{l_h(u)} \circ T_{-u})(f)(y) && (\text{extension definition})\\
&= \mathop{\square}_{u \in W^t} \psi_{l_h(u)}(T_{-u}(f))(y) && (\text{composition definition})
\end{aligned}$$

$$= \square_{u \in W^t} (l_h(u) \circ T_{-u}(f))(y) \qquad (\psi_l \text{ definition})$$

$$= \square_{u \in W^t} l_h(u)(T_{-u}(f)(y)) \qquad (\text{composition definition})$$

$$= \square_{u \in W^t} l_h(u)(f(y + u)) \qquad (T_u \text{ definition})$$

$$= \square_{u \in W^t} h(f(y + u))(-u) \qquad (l_h \text{ definition})$$

$$= \square_{x - y \in W^t} h(f(x))(y - x) \qquad (x = y + u)$$

$$= \square_{x \in W^t + y} h(f(x))(y - x) \qquad (\text{translate definition})$$

$$= \square_{y \in W + x} h(f(x))(y - x) \qquad (\text{Exercise 4.42})$$

$$= \square_{y \in W + x} (h(f(x)) + x)(y) \qquad (+ \text{ definition})$$

$$= \square_{x \in W^t + y} (h(f(x)) + x)(y) \qquad (\text{Exercise 4.42})$$

$$= \square_{x \in \mathbf{Z}^2} (h(f(x)) + x)(y) \qquad (+ \text{ prop. and } e_2 \text{ is unit element})$$

$$= (\square_{x \in \mathbf{Z}^2} (h(f(x)) + x))(y) \qquad (\text{extension definition})$$

$$= \Phi_h(f)(y). \qquad (\Phi_h \text{ definition})$$

□

Proposition 4.35 shows that the distinction between types 1 and 2 becomes meaningless when the image domain becomes infinite ($\mathbf{Z}^2$).

4.4 Linear and morphological window operators

In Section 3.1 we have seen that under some specific conditions, each lut which is morphism for an external binary operation with operand in the gray–scale can be characterized in terms of a parameter, the value of which being a gray–level.

In this section, we will begin by studying a similar situation in which the luts are replaced by families of point spread functions. In other words, we will characterize the families of point spread functions which are morphisms of an external binary operation.

Let $(K, \cdot, e')$ be a commutative monoid. From the morphism definition, a family of point spread functions h from K to K^W is a morphism for the operations $\cdot$ on K and its extension to K^W, denoted by the same symbol $\cdot$, iff, for any s and β in K,

$$h(\beta \cdot s) = \beta \cdot h(s).$$

Let h be a mapping from K to K^W, we denote by α_h the expression defined by

$$\alpha_h \triangleq h(e').$$

Let s be an element in K and α be an element in K^W, we denote by $h_\alpha(s)$ the expression defined by

$$h_\alpha(s) \triangleq s \cdot \alpha.$$

Proposition 4.36 (characterization of the families of point spread functions which are morphisms of an external operation) – Let $(K, \cdot, e')$ be a monoid and let $\mathrm{M}_\cdot(K, K^W)$ be the set of families of point spread functions from K to K^W which are morphisms for the operations $\cdot$ on K and its extension to K^W, then the mapping $h \mapsto \alpha_h$ from $\mathrm{M}_\cdot(K, K^W)$ to K^W is a bijection and its inverse is $\alpha \mapsto h_\alpha$. □

Proof – 1. For any h in $\mathrm{M}_\cdot(K, K^W)$, α_h is an element of K^W.

2. For any α in K^W, the mapping $s \mapsto h_\alpha(s)$ is in $\mathrm{M}_\cdot(K, K^W)$ as we can see below. For any s and β in K, and u in W,

$$\begin{aligned}
h_\alpha(\beta \cdot s)(u) &= ((\beta \cdot s) \cdot \alpha)(u) && \text{(definition of } h_\alpha(s))\\
&= (\beta \cdot s) \cdot \alpha(u) && \text{(extension definition)}\\
&= \beta \cdot (s \cdot \alpha(u)) && \text{(associativity)}\\
&= \beta \cdot (s \cdot \alpha)(u) && \text{(extension definition)}\\
&= \beta \cdot h_\alpha(s)(u) && \text{(definition of } h_\alpha(s))\\
&= (\beta \cdot h_\alpha(s))(u), && \text{(extension definition)}
\end{aligned}$$

that is, $h_\alpha(\beta \cdot s) = \beta \cdot h_\alpha(s)$.

3. Let h be in $\mathrm{M}_\cdot(K, K^W)$, for any s in K

$$\begin{aligned}
h_{\alpha_h}(s) &= s \cdot \alpha_h && \text{(definition of } h_\alpha(s))\\
&= s \cdot h(e') && \text{(definition of } \alpha_h)\\
&= h(s \cdot e') && (h \text{ is in } \mathrm{M}_\cdot(K, K^W))\\
&= h(s), && (e' \text{ is a unit element of } \cdot)
\end{aligned}$$

that is, $l \mapsto \alpha_l$ is an injection.

4. Let α be in K^W, and u in W,

$$\begin{aligned} \alpha_{h_\alpha}(u) &= h_\alpha(e')(u) && \text{(definition of } \alpha_h\text{)} \\ &= (e' \cdot \alpha)(u) && \text{(definition of } h_\alpha(s)\text{)} \\ &= e' \cdot \alpha(u) && \text{(extension definition)} \\ &= \alpha(u) && (e' \text{ is a unit element of } \cdot) \end{aligned}$$

that is, $l \mapsto \alpha_l$ is a surjection. □

The characterization presented in Proposition 4.36 is important because it shows that each family of point spread functions which is a morphism as described in this proposition reduces to a unique point spread function.

We can apply this result to the case of linear operators on real images and to the case of the morphological operators (dilations and erosions) on binary images.

When the image domain is $\mathbf{Z}^2$, the characterization of the linear operators may be written in terms of a convolution product as shown in the next proposition.

Proposition 4.37 (linear operators and convolution product) – If Φ is a linear spatially invariant window operator from $\mathbf{R}^{\mathbf{Z}^2}$ to $\mathbf{R}^{\mathbf{Z}^2}$, then there exists a mapping η in $\mathbf{R}^{\mathbf{Z}^2}$ such that for any image f in $\mathbf{R}^{\mathbf{Z}^2}$,

$$\Phi(f) = f * \eta.$$ □

Proof – If Φ is a linear spatially invariant window operator from $\mathbf{R}^{\mathbf{Z}^2}$ to $\mathbf{R}^{\mathbf{Z}^2}$ with window W, then (by Part 2 of Proposition 4.34) there exists a linear family $h : \mathbf{R} \to \mathbf{R}^W$ of point spread functions such that $\Phi = \Phi_h$, namely $h = h_\Phi$, and (by Proposition 4.36) there exists a point spread function $\alpha : W \to \mathbf{R}$ such that $h(s) = s \cdot \alpha$, namely $\alpha = \alpha_h$. We denote by η the mapping in $\mathbf{R}^{\mathbf{Z}^2}$ defined by, for any y in $\mathbf{Z}^2$, .

$$\eta(y) \triangleq \begin{cases} \alpha(y) & \text{if } y \in W \\ 0 & \text{otherwise} \end{cases}.$$

For any f in $\mathbf{R}^{\mathbf{Z}^2}$ and any y in $\mathbf{Z}^2$,

$$\begin{aligned} \Phi(f)(y) &= \Phi_h(f)(y) && \text{(Proposition 4.34)} \\ &= \Big(\sum_{x \in \mathbf{Z}^2} (h(f(x)) + x)\Big)(y) && (\Phi_h \text{ definition}) \\ &= \sum_{x \in \mathbf{Z}^2} (h(f(x)) + x)(y) && \text{(extension definition)} \\ &= \sum_{x \in W^t + y} (h(f(x))(y - x) && (+ \text{ definition}) \end{aligned}$$

$$= \sum_{x \in W^t + y} (f(x) \cdot \alpha_h)(y - x) \qquad \text{(Proposition 4.36)}$$

$$= \sum_{x \in W^t + y} f(x) \cdot \alpha_h(y - x) \qquad \text{(extension definition)}$$

$$= \sum_{y \in W + x} f(x) \cdot \alpha_h(y - x) \qquad \text{(Exercise 4.42)}$$

$$= \sum_{y - x \in W} f(x) \cdot \alpha_h(y - x) \qquad \text{(translate definition)}$$

$$= \sum_{y - x \in \mathbf{Z}^2} f(x) \cdot \eta(y - x) \qquad \text{(η definition)}$$

$$= \sum_{x \in \mathbf{Z}^2} f(x) \cdot \eta(y - x) \qquad \text{(property of $\mathbf{Z}^2$)}$$

$$= (f * \eta)(y). \qquad \text{(convolution product definition)}$$

□

When the image domain is $\mathbf{Z}^2$ and the input and output images are binary, the characterization of the dilations may be written in terms of a Minkowski addition as shown in the next proposition.

Let f be a binary image in $\{0, 1\}^{\mathbf{Z}^2}$. We denote by supp(f) the inverse image of $\{1\}$ through f. The mapping $f \mapsto \text{supp}(f)$ is a bijection, its inverse is the mapping $X \mapsto 1_X$ where 1_X is the indicator function of the subset X of $\mathbf{Z}^2$ (see Banon & Barrera, 1998).

Proposition 4.38 (dilation and Minkowski addition) – If Φ is a spatially invariant window operator from $\{0, 1\}^{\mathbf{Z}^2}$ to $\{0, 1\}^{\mathbf{Z}^2}$ with window W, and is a dilation, then there exists a subset B of W such that, for any image f in $\{0, 1\}^{\mathbf{Z}^2}$,

$$\Phi(f) = 1_{\text{supp}(f) \oplus B}.$$

□

Proof – If Φ is a spatially invariant window operator from $\{0, 1\}^{\mathbf{Z}^2}$ to $\{0, 1\}^{\mathbf{Z}^2}$ with window W, and is a dilation, then (by Part 1 of Proposition 4.34) there exists a family $h : \{0, 1\} \to \{0, 1\}^W$ of point spread functions which is a dilation, such that $\Phi = \Phi_h$, and (by Proposition 4.36) there exists a point spread function $\alpha : W \to \{0, 1\}$ such that $h(s) = s \wedge \alpha$. We denote by B the subset of W defined by, .

$$B = \text{supp}(\alpha).$$

For any f in $\{0,1\}^{\mathbf{Z}^2}$ and any y in $\mathbf{Z}^2$,

$$
\begin{aligned}
\Phi(f)(y) &= \Phi_h(f)(y) && \text{(Proposition 4.34)}\\
&= \Big(\bigvee_{x \in \mathbf{Z}^2} (h(f(x)) + x)\Big)(y) && (\Phi_h \text{ definition})\\
&= \bigvee_{x \in \mathbf{Z}^2} (h(f(x)) + x)(y) && \text{(extension definition)}\\
&= \bigvee_{x \in W^t + y} (h(f(x))(y - x) && (+ \text{ definition})\\
&= \bigvee_{x \in W^t + y} (f(x) \wedge \alpha)(y - x) && \text{(Proposition 4.36)}\\
&= \bigvee_{x \in W^t + y} f(x) \wedge \alpha(y - x) && \text{(extension definition)}\\
&= \bigvee_{y \in W + x} f(x) \wedge \alpha(y - x) && \text{(Exercise 4.42)}\\
&= \bigvee_{y - x \in W} f(x) \wedge \alpha(y - x) && \text{(translate definition)}\\
&= \bigvee_{u \in W} f(y - u) \wedge \alpha(u) && (u = y - x)\\
&= \bigvee_{u \in B} f(y - u) && (B \text{ definition})\\
&= \bigvee_{u \in B} (f + u)(y) && \text{(translate definition)}\\
&= \Big(\bigvee_{u \in B} (f + u)\Big)(y), && \text{(extension definition)}
\end{aligned}
$$

that is, for any f in $\{0,1\}^{\mathbf{Z}^2}$,

$$
\begin{aligned}
\Phi(f) &= \bigvee_{u \in B} (f + u) && \text{(above result)}\\
&= \bigvee_{u \in B} 1_{\mathrm{supp}(f + u)} && (f \mapsto \mathrm{supp}(f) \text{ is injective})\\
&= \bigvee_{u \in B} 1_{\mathrm{supp}(f) + u} && \text{(supp commute with translation)}\\
&= 1_{\bigcup_{u \in B} \mathrm{supp}(f) + u} && (1_X \text{ commute with union})\\
&= 1_{\mathrm{supp}(f) \oplus B}. && \text{(Minkowski addition definition)}
\end{aligned}
$$

□

When the image domain is $\mathbf{Z}^2$ and the input and output images are binary, the characterization of the erosions may be written in terms of a Minkowski subtraction as shown in the next exercise.

Exercise 4.39 (erosion and Minkowski subtraction) – Prove that if Ψ is a spatially invariant window operator from $\{0, 1\}^{\mathbf{Z}^2}$ to $\{0, 1\}^{\mathbf{Z}^2}$ with window W, and is an erosion, then there exists a subset B of W such that, for any image f in $\{0, 1\}^{\mathbf{Z}^2}$,

$$\Psi(f) = 1_{\mathrm{supp}(f) \ominus B}. \qquad \square$$

4.5 Some mathematical definitions and properties

To help the understanding of the definitions given in the previous sections, we recall some important mathematical definitions such as reduction, transpose and translate, and we recall some properties of the morphism composition and closure with respect to an extended operation.

Reduction

Let A be a finite set with n elements, and $(B, \Box)$ a commutative semigroup. Let $i \mapsto a_i$ be bijection from $\mathbf{n}$ to A and f a mapping from A to B, we denote by $\underset{i \in \mathbf{n}}{\Box} f(a_i)$ the expression defined by

$$\underset{i \in \mathbf{n}}{\Box} f(a_i) \triangleq (...((f(a_1) \Box f(a_2)) \Box f(a_3))...) \Box f(a_n).$$

Since A is finite and $\Box$ is associative and commutative, the above expression does not depend on the bijection and we denote it, alternatively, by $\underset{a \in A}{\Box} f(a)$ or simply by $\Box f$.

The mapping from B^A to B given by $f \mapsto \Box f$ is called a *reduction* on B.

If A is a finite set, C is a non–empty set, and $(B, \Box)$ is a commutative semigroup, then by extension definition, we have, for any f from A to B^C, and any c in C,

$$(\underset{a \in A}{\Box} f(a))(c) = \underset{a \in A}{\Box} f(a)(c),$$

that is, the reduction $\Box$ on B^C is an extension of the reduction $\Box$ on B.

Morphism composition

The morphism property is preserved under composition.

Proposition 4.40 (morphism composition) – 1. Let $(A, \square)$, $(B, \square)$ and $(C, \square)$ be three sets equipped with internal binary operations. Let f be a morphism from $(A, \square)$ to $(B, \square)$, and let g be a morphism from $(B, \square)$ to $(C, \square)$, then $g \circ f$ is a morphism from $(A, \square)$ to $(C, \square)$. The same result is true for external binary operations with operand in a set K, and nullary operations. □

Proof – 1. For any f from A to B, any g from B to C, and any x and y in A,

$$\begin{aligned}(g \circ f)(x \,\square\, y) &= g(f(x \,\square\, y)) && \text{(composition def.)}\\ &= g(f(x) \,\square\, f(y)) && (f \text{ is a morphism})\\ &= g(f(x)) \,\square\, g(f(y)) && (g \text{ is a morphism})\\ &= (g \circ f)(x) \,\square\, (g \circ f)(y). && \text{(composition def.)}\end{aligned}$$

2. For any f from A to B, any g from B to C, any x in A and any α in K,

$$\begin{aligned}(g \circ f)(\alpha{\cdot}x) &= g(f(\alpha{\cdot}x)) && \text{(composition def.)}\\ &= g(\alpha{\cdot}f(x)) && (f \text{ is a morphism})\\ &= \alpha{\cdot}g(f(x)) && (g \text{ is a morphism})\\ &= \alpha{\cdot}(g \circ f)(x). && \text{(composition def.)}\end{aligned}$$

3. For any f from A to B, and any g from B to C,

$$\begin{aligned}(g \circ f)(e_A) &= g(f(e_A)) && \text{(composition def.)}\\ &= g(e_B) && (f \text{ is a morphism})\\ &= e_C. && (g \text{ is a morphism})\end{aligned}$$

□

Extension versus closure

The next proposition shows how extension and closure can be combined.

Proposition 4.41 (extension versus closure) – Let A be a non–empty set.

1. Let $(B, \square)$ be a set equipped with an internal binary operation, and let X be a subset of B $\square$–closed. Then the set of mappings f from A to B such that $f(A) \subset X$ is closed under the extension to B^A of $\square$ on B.

2. Let $(B, \cdot)$ be a set equipped with an external binary operation with operand in a set C, and let X be a subset of B $\cdot$–closed. Then the set of mappings f from A to B such that $f(A) \subset X$ is closed under the extension to B^A of $\cdot$ on B.

3. Let (B, e) be a set equipped with a nullary operation, and let X be a subset of B e–closed. Then the set of mappings f from A to B such that $f(A) \subset X$ is closed under the extension to B^A of e on B. □

Proof – 1. For any f and g from A to B such that $f(A) \subset X$ and $g(A) \subset X$, and any a in A,

$$\begin{aligned}\text{true} &\Leftrightarrow f(a) \text{ and } g(a) \in X && \text{(image definition)}\\ &\Leftrightarrow f(a) \,\square\, g(a) \in X && (X \text{ is } \square\text{–closed})\\ &\Leftrightarrow (f \,\square\, g)(a) \in X, && \text{(extension definition)}\end{aligned}$$

that is, by image definition, $f \,\square\, g$ is a mapping from A to B such that $(f \,\square\, g)(A) \subset X$.

2. For any f from A to B such that $f(A) \subset X$, any α in C, and any a in A,

$$\begin{aligned} \text{true} &\Leftrightarrow f(a) \in X && \text{(image definition)} \\ &\Leftrightarrow \alpha.f(a) \in X && (X \text{ is } \square\text{–closed}) \\ &\Leftrightarrow (\alpha.f)(a) \in X, && \text{(extension definition)} \end{aligned}$$

that is, by image definition, $\alpha.f$ is a mapping from A to B such that $(\alpha.f)(A) \subset X$.

3. By extension definition, the extended nullary operation on B^A selects the constant mapping $a \mapsto e$. Since X is e–closed the image of A through $a \mapsto e$, $\{e\}$ is contained in X. □

Some operations derived from an Abelian group

A set G equipped with an internal binary operation $+$ and a nullary operation $\boldsymbol{o}$ which satisfy the following properties (Cagnac et al.,1963, p. 34) for any x, y, and z in G,

$$(x + y) + z = x + (y + z) \qquad \text{(associativity)}$$

$$x + o = o + x = x \qquad \text{(unit element)}$$

$$\exists x' \in G, \quad x' + x = x + x' = \boldsymbol{o} \qquad \text{(opposite existence)}$$

is a *group*. This algebraic system is denoted $(G, +, \boldsymbol{o})$. The *opposite* element x' of an element x in G is unique and is denoted by $-x$. We denote by $x - y$ the expression given by

$$x - y \triangleq x + (-y).$$

If the operation $+$ is commutative then the group is called an *Abelian group*.

An example of Abelian group is $(\mathbf{Z}^2, +, o)$ where $\mathbf{Z}^2$ is the set of integer pairs, $+$ is usual addition and o is the nullary operation which selects the pair (0, 0).

Let X be a subset of $(G, +, \boldsymbol{o})$, the *transpose of* X is the subset of G, denoted by X^t and defined by,

$$X^t \triangleq \{x \in G : -x \in X\}.$$

Let X be a subset of $(G, +, \boldsymbol{o})$ and let u be an element of G, the *translate of* X *by* u is the subset of G, denoted by $X + u$ and defined by,

$$X + u \triangleq \{x \in G : x - u \in X\}.$$

Let f be a mapping from G to K and let u be an element of G, the *translate of* f *by* u is the mapping from G to K, denoted by $f + u$ and defined by, for any x in G,

$$(f + u)(x) \triangleq f(x - u).$$

Exercise 4.42 (transpose versus translate) – Prove, that for any subset W of G, and any x and y in G,

$$y \in W + x \Leftrightarrow x \in W^t + y$$ □

We now show how to combine two subsets of an Abelian Group.

Let A and B be two subsets of G. The *Minkowski sum of A and B* is the subset of G, denoted by $A \oplus B$ and defined by,

$$A \oplus B \triangleq \{x \in G : \exists a \in A \text{ and } \exists b \in B, x = a + b\}.$$

Let A and B be two subsets of G. The *Minkowski difference of A and B* is the subset of G, denoted by $A \ominus B$ and defined by,

$$A \ominus B \triangleq \{x \in G : \forall b \in B, \exists a \in A, x = a - b\}.$$

In general we don't have

$$C = A \ominus B \Leftrightarrow C \oplus B = A,$$

but we have (see, for example Banon & Barrera, 1998), for any subsets A, B and C of G,

$$C \subset A \ominus B \Leftrightarrow C \oplus B \subset A.$$

The operation $(A, B) \mapsto A \oplus B$ from $G \times G$ to G is called *Minkowski addition* and the operation $(A, B) \mapsto A \ominus B$ from $G \times G$ to G is called *Minkowski subtraction*

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Index

E

F

G

H

I

K

L

M

N

O

P

R

S